Cohen–Macaulayness of Tensor Product

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Abstract

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. Suppose that M and N are finitely generated modules over R such that M has finite projective dimension and such that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$. The main result of this note gives a condition on M which is necessary and sufficient for the tensor product of M and N to be a Cohen–Macaulay module over R, provided N is itself a Cohen–Macaulay module.

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0. Introduction

Throughout this note $(R, \mathfrak{m})$ is a commutative Noetherian local ring with non-zero identity and the maximal ideal $\mathfrak{m}$. By M and N we always mean non-zero finitely generated R–modules. The projective dimension of module M is denoted by $\mathrm{proj.dim}\, M$.

The well known notion "grade of M", grade M, has been introduced by Rees, see [**7**], as the least integer $t \geq 0$ such that $\mathrm{Ext}_R^t(M, R) \neq 0$. In [**9**] we have defined the "grade of M and N", grade (M, N), as the least integer $t \geq 0$ such that $\mathrm{Ext}_R^t(M, N) \neq 0$. One of the main results of this note is:

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Let N be a Cohen–Macaulay R–module and let M be an R–module with finite projective dimension. If $\mathrm{Tor}_i^R(M,N)=0$ for all $i>0$ then $M\otimes_R N$ is Cohen–Macaulay if and only if $\mathrm{grade}\,(M,N)=\mathrm{proj.dim}\,M$.

This theorem can be considered as a generalization of the well–known statement:

(**T1**) Let R be a Cohen–Macaulay local ring and let M be a finite R–module with finite projective dimension. Then M is Cohen–Macaulay if and only if $\mathrm{grade}\,M=\mathrm{proj.dim}\,M$.

On the other hand the following statement from Yoshida can be concluded from our result:

Yoshida [**10**; Prop (2.4)] "Suppose that $\mathrm{grade}\,M=\mathrm{proj.dim}\,M(<\infty)$ and that N is a maximal Cohen–Macaulay R–module (that is $\mathrm{depth}\,N=\dim N=\dim R$). Then $M\otimes_R N$ is Cohen–Macaulay and $\dim M\otimes_R N=\dim M$."

In another theorem of the first section we improve a theorem from Kawasaki:

Kawasaki [**5**; Theorem 3.3(i)] "Let R be a Cohen–Macaulay local ring and let K be a canonical module of R. Let M be a finite R–module of finite projective dimension. Then $M\otimes_R K$ is Cohen–Macaulay if and only if M is Cohen–Macaulay."

The following statement generalizes Kawasaki's theorem:

Let R be a Cohen–Macaulay local ring and let K be a canonical module of R. If M is an R–module with finite Gorenstein dimension, then $M\otimes_R K$ is Cohen–Macaulay if and only if M is Cohen–Macaulay.

In the above statement the Gorenstein dimension is an invariant for finite modules which was introduced by Auslander, in [**1**]. It is a finer invariant than projective dimension in the sense that $\mathrm{G\text{–}dim}\,M\le\mathrm{proj.dim}\,M$ for every finite non-zero R–module M and equality holds when $\mathrm{proj.dim}\,M<\infty$. There exist modules with finite Gorenstein dimension which have infinite projective dimension.

In the second section we consider Serre's condition. We say M satisfies Serre's condition (S_n), for a non–negative integer n, when for every $\mathfrak{p}\in\mathrm{Supp}\,M$ the following inequality holds:

$$\mathrm{depth}\,M_{\mathfrak{p}}\ge\min(n,\dim M_{\mathfrak{p}}).$$

Obviously every Cohen–Macaulay module satisfies (S_n) for all non-negative integers n.

The main result of section 2 is:

Let M and N be R–modules such that $\mathrm{Tor}_i^R(M,N)=0$ for all $i>0$. If projective dimension of M is finite and $M\otimes_R N$ satisfies (S_n), then so does N. This result generalizes [**10**; Prop.(4.1)].

1. Cohen–Macaulayness.

Definition 1.1 We define

$$\mathrm{grade}\,(M,N)=\inf\{i\,|\,\mathrm{Ext}_R^i(M,N)\neq 0\}$$

Since M is finite, using [**3**; 1.2.10] we have that

$$\begin{aligned}\mathrm{grade}\,(M,N) &= \inf\{\mathrm{depth}\,N_{\mathfrak{p}}\,|\,\mathfrak{p}\in\mathrm{Supp}\,M\}\\ &= \inf\{\mathrm{depth}\,N_{\mathfrak{p}}\,|\,\mathfrak{p}\in\mathrm{Supp}\,M\cap\mathrm{Supp}\,N\}.\end{aligned}$$

The second equality holds because the depth of the zero module is defined to be infinite.

Proposition 1.2 [**9**; Theorem 2.1] the following inequalities hold:

(a) $\mathrm{depth}\,N-\dim M\leq\mathrm{grade}\,(M,N)$

(b) If $\mathrm{Supp}\,M\subseteq\mathrm{Supp}\,N$ then $\mathrm{grade}\,(M,N)\leq\dim N-\dim M$

For a finite R–module M of finite projective dimension, the invariant $\mathrm{imp}\,M$, imperfection of M, is defined to be $\mathrm{proj.dim}\,M-\mathrm{grade}\,M$. This is, using Auslander–Buchsbaum equality, equal to $\mathrm{depth}\,R-\mathrm{depth}\,M-\mathrm{grade}\,M$.

Definition 1.3 For finite R–modules M and N (which may have infinite projective dimensions) we define $\mathrm{imp}\,(M,N)=\mathrm{depth}\,N-\mathrm{depth}\,M-\mathrm{grade}\,(M,N)$ (This may be negative).

It is clear that if $\mathrm{proj.dim}\,M<\infty$, then $\mathrm{imp}\,M=\mathrm{imp}\,(M,R)$.
By $\mathrm{cmd}\,M$ we mean the difference $\dim M-\mathrm{depth}\,M$.

Proposition 1.4 The following inequalities hold:

(a) $\operatorname{imp}(M,N) \leq \operatorname{cmd} M$

(b) If $\operatorname{Supp} M \subseteq \operatorname{Supp} N$, then $\operatorname{cmd} M \leq \operatorname{imp}(M,N) + \operatorname{cmd} N$.

Proof. This is clear from Proposition 1.2 and the definition.

Corollary 1.5 Let N be a Cohen–Macaulay R–module and $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. Then $\operatorname{cmd} M = \operatorname{imp}(M,N)$; in particular the module M is a Cohen–Macaulay module if and only if $\operatorname{imp}(M,N) = 0$. □

(T1) says that over a Cohen–Macaulay local ring R, the R–module M with finite projective dimension is Cohen–Macaulay if $\operatorname{Ext}^i_R(M,R) = 0$ for $i \neq \operatorname{proj.dim} M$. The following corollary is a generalization of (T1).

Corollary 1.6 Let N be a Cohen–Macaulay R–module with $\operatorname{depth} N = \operatorname{depth} R$. Let M have finite projective dimension and $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. Then M is Cohen–Macaulay if and only if $\operatorname{Ext}^i_R(M,N) = 0$ for $i \neq \operatorname{proj.dim} M$.

Proof. Note that $\operatorname{proj.dim} M = \sup\{i | \operatorname{Ext}^i_R(M,N) \neq 0 \text{ for any } N\}$, cf. [**6**] and so it is always greater than or equal to $\operatorname{grade}(M,N)$.

$$\begin{aligned} \operatorname{imp}(M,N) &= \operatorname{depth} N - \operatorname{depth} M - \operatorname{grade}(M,N) \\ &= \operatorname{depth} R - \operatorname{depth} M - \operatorname{grade}(M,N) \\ &= \operatorname{proj.dim} M - \operatorname{grade}(M,N) \end{aligned}$$

Now the claim is clear from Corollary 1.5 . □

Recall that a finite R–module M with finite projective dimension is called perfect if $\operatorname{proj.dim} M = \operatorname{grade} M$.

Definition 1.7 Let M and N be R–modules with $\operatorname{proj.dim} M < \infty$. We say that M is N-perfect if $\operatorname{proj.dim} M = \operatorname{grade}(M,N)$.

In the proof of the following statements we use the well–known result:

(**T2**) Let M and N be finite R–modules with $\operatorname{proj.dim} M < \infty$. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, then we have the equality $\operatorname{depth} M \otimes_R N = \operatorname{depth} N - \operatorname{proj.dim} M$.

Theorem 1.8 Let N be a Cohen–Macaulay R–module and let M be an R–module with finite projective dimension. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, then $M \otimes_R N$ is Cohen–Macaulay if and only if M is N–perfect.

Proof. We claim that M is N–perfect if and only if $\operatorname{imp}(M \otimes_R N, N) = 0$, and then the assertion will be clear from Corollary 1.5 . We know that $\operatorname{depth} M \otimes_R N = \operatorname{depth} N - \operatorname{proj.dim} M$. On the other hand

$$\begin{aligned} \operatorname{grade}(M \otimes_R N, N) &= \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \otimes_R N\} \\ &= \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid p \in \operatorname{Supp} M \cap \operatorname{Supp} N\} \\ &= \operatorname{grade}(M, N). \end{aligned}$$

Then we have the equality $\operatorname{imp}(M \otimes_R N, N) = \operatorname{proj.dim} M - \operatorname{grade}(M, N)$, which proves our claim. □

Now [**10**; 2.4] can be deduced from the above theorem, for when N is maximal Cohen–Macaulay and $\operatorname{proj.dim} M < \infty$ by [**10**; 2.2] we have that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$. For every $\mathfrak{p} \in \operatorname{Supp} N$, the $R_{\mathfrak{p}}$–module $N_{\mathfrak{p}}$ is maximal Cohen–Macaulay module and, then $\operatorname{depth} N_{\mathfrak{p}} = \dim R_{\mathfrak{p}} \geq \operatorname{depth} R_{\mathfrak{p}}$ and hence we have inequalities

$$\operatorname{grade} M \leq \operatorname{grade}(M, N) \leq \operatorname{proj.dim} M.$$

This means that every perfect module is N–perfect.

Definition 1.9 A finite R–module N is said to be of Gorenstein dimension zero, $\operatorname{G-dim} N = 0$, if and only if

(a) $\operatorname{Ext}_R^i(N, R) = 0$ for $i > 0$.

(b) $\operatorname{Ext}_R^i(\operatorname{Hom}_R(N, R), R) = 0$ for $i > 0$.

(c) The canonical map $N \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(N, R), R)$ is an isomorphism.

For a non–negative integer n, the R–module N is said to be of Gorenstein dimension at most n, if and only if there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \dots \longrightarrow G_0 \longrightarrow N \longrightarrow 0$$

where $\operatorname{G-dim} G_i = 0$ for $0 \leq i \leq n$. If such a sequence does not exist then $\operatorname{G-dim} N = \infty$.

Lemma 1.10 [**2**; 3.7, 3.14, and 4.12] If $\operatorname{G-dim} M < \infty$ then the following hold:

(a) $\operatorname{G-dim} M + \operatorname{depth} M = \operatorname{depth} R$.

(b) $\operatorname{G-dim} M = \sup\{t \mid \operatorname{Ext}^t_R(M, R) \neq 0\}$.

(c) $\operatorname{Tor}^R_i(M, P) = 0$ for all $i > \operatorname{G-dim} M$ and all modules P with finite projective dimension.

The following theorem improves Kawasaki's result [**5**; 3.3(i)].

Theorem 1.11 Let R be a Cohen–Macaulay local ring and let K be a canonical module of R. If M is an R–module with finite Gorenstein dimension, then $M \otimes_R K$ is Cohen–Macaulay if and only if M is Cohen–Macaulay.

Proof. Proposition [**4**; 2.5] says that $\operatorname{Tor}^R_i(M, K) = 0$ for $i > 0$, and then since injective dimension of K is finite we have that $\operatorname{depth} M \otimes_R K = \operatorname{depth} K - \operatorname{G-dim} M$, cf. [**8**; 2.13]. Then $\operatorname{imp}(M \otimes_R K, K) = \operatorname{G-dim} K - \operatorname{grade}(M \otimes_R K, K)$. Since $\operatorname{Supp} K = \operatorname{Spec} R$ we have that

$$\begin{aligned} \operatorname{grade}(M \otimes_R K, K) &= \inf\{\operatorname{depth} K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \otimes K\} \\ &= \inf\{\operatorname{depth} K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M\} \end{aligned}$$

But since $\operatorname{depth} K_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp} K = \operatorname{Spec} R$ we have that $\operatorname{grade}(M \otimes_R K, K) = \operatorname{grade} M$. The claim of the theorem is now clear from Corollary 1.5 and the fact that over a Cohen–Macaulay local ring R, the R–module M with $\operatorname{G-dim} M < \infty$ is Cohen–Macaulay if and only if $\operatorname{grade} M = \operatorname{G-dim} M$, cf. [**9**]. $\square$

2. Serre Conditions.

First recall that for a non–negative integer n, we say that a finite R–module M satisfies Serre's condition (S_n) if $\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim M_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Supp} M$ or equivalently if $M_{\mathfrak{p}}$ is a Cohen–Macaulay $R_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \operatorname{Supp} M$ such that $\operatorname{depth} M_{\mathfrak{p}} < n$.

We also recall the intersection theorem:

(**T3**) Let M and N be finite R–modules with $\operatorname{proj.dim} M < \infty$. We have the inequality $\dim N \leq \operatorname{proj.dim} M + \dim (M \otimes_R N)$.

Theorem 2.1 Let N be a finite R–module which satisfies (S_n). Let M be an N-perfect R–module with $t = \operatorname{proj.dim} M \leq n$, such that $\operatorname{Tor}_i^R(M, N) = 0$ for $i > 0$. Then $M \otimes_R N$ satisfies (S_{n-t}).

Proof. For every $\mathfrak{p} \in \operatorname{Supp}(M \otimes_R N)$ it is clear that

$$\operatorname{grade}(M, N) \leq \operatorname{grade}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \operatorname{proj.dim} M_{\mathfrak{p}} \leq \operatorname{proj.dim} M = t.$$

Since M is N–perfect, $M_{\mathfrak{p}}$ is $N_{\mathfrak{p}}$–perfect with $\operatorname{proj.dim} M_{\mathfrak{p}} = t$. From Proposition 1.2 we have that $t = \operatorname{grade}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \operatorname{grade}((M \otimes_R N)_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \dim N_{\mathfrak{p}} - \dim (M_R \otimes N)_{\mathfrak{p}}$.

On the other hand from the fact that N satisfies (S_n) we have the following inequality:

$$\begin{aligned} \operatorname{depth}(M \otimes_R N)_{\mathfrak{p}} &= \operatorname{depth}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}) \\ &= \operatorname{depth} N_{\mathfrak{p}} - \operatorname{proj.dim} M_{\mathfrak{p}} \\ &= \operatorname{depth} N_{\mathfrak{p}} - t \\ &= \min(n, \dim N_{\mathfrak{p}}) - t \\ &= \min(n - t, \dim N_{\mathfrak{p}} - t) \end{aligned}$$

Now the assertion holds. □

Corollary 2.2 If R satisfies (S_n), then every perfect R–module with projective dimension t (less than or equal to n) satisfies (S_{n-t}).

It is well known that if a local ring admits a finite Cohen–Macaulay module with finite projective dimension, then the ring itself is Cohen–Macaulay.

In [**10**; 4.1] Yoshida has proved a more general statement, by replacing "being Cohen–Macaulay" with "satisfying Serre's condition (S_n)".

Our next two theorems improve those results by similar proofs. The Theorem 2.3 is a special case of the Theorem 2.4 , and the proof of it is only included because it is so simple.

Theorem 2.3 Let M and N be R–modules such that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i > 0$. If $\mathrm{proj.dim}\, M < \infty$ and $M \otimes_R N$ is Cohen–Macaulay then so is N.

Proof. The intersection theorem (T3) gives the inequality:

$$\dim N \leq \dim M \otimes_R N + \mathrm{proj.dim}\, M$$

On the other hand (T2) gives the equality

$$\mathrm{depth}\, N = \mathrm{depth}\, M \otimes_R N + \mathrm{proj.dim}\, M.$$

Since $\dim N \geq \mathrm{depth}\, N$, the assertion is clear. □

Theorem 2.4 Let M and N be R–modules such that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i > 0$. If $\mathrm{proj.dim}\, M < \infty$ and $M \otimes_R N$ satisfies (S_n), then so does N.

Proof. Choose $\mathfrak{p} \in \mathrm{Supp}\, N$. There are two cases:
The first case is when $\mathfrak{p} \in \mathrm{Supp}\, M$ and then $\mathfrak{p} \in \mathrm{Supp}\, M \otimes_R N$.

If $\mathrm{depth}\,(M \otimes_R N)_{\mathfrak{p}} < n$ then $(M \otimes_R N)_{\mathfrak{p}}$ is Cohen–Macaulay and by the Theorem 2.3 so is $N_{\mathfrak{p}}$.

If $\mathrm{depth}\,(M \otimes_R N)_{\mathfrak{p}} \geq n$, then $\mathrm{depth}\, N_{\mathfrak{p}} \geq n$ because by (T2) $\mathrm{depth}\, N_{\mathfrak{p}} = \mathrm{depth}\,(M \otimes_R N)_{\mathfrak{p}} + \mathrm{proj.dim}\, M_{\mathfrak{p}}$. The second case is when $\mathfrak{p} \notin \mathrm{Supp}\, M$. Let $\mathfrak{q}$ be a minimal prime over the ideal $(\mathrm{Ann} M + \mathfrak{p})$. From (T3) we have the inequality

$$\dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} \leq \mathrm{proj.dim}\, M_{\mathfrak{q}} + \dim M_{\mathfrak{q}}/\mathfrak{p}M_{\mathfrak{q}} = \mathrm{proj.dim}\, M_{\mathfrak{q}}.$$

Since $\mathfrak{p}R_{\mathfrak{q}} \in \mathrm{Supp}\, R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ we have that

$$\begin{aligned} \mathrm{depth}\, N_{\mathfrak{p}} &\geq \mathrm{grade}\,(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}, N_{\mathfrak{q}}) \\ &\geq \mathrm{depth}\, N_{\mathfrak{q}} - \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} \text{ (proposition1.2)} \\ &\geq \mathrm{depth}\, N_{\mathfrak{q}} - \mathrm{proj.dim}\, M_{\mathfrak{q}} \\ &= \mathrm{depth}\, M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} N_{\mathfrak{q}} \end{aligned}$$

If $\operatorname{depth} M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} N_{\mathfrak{q}} < n$, then $M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} N_{\mathfrak{q}}$ is Cohen–Macaulay and from Theorem 2.3 we will have that $N_{\mathfrak{q}}$ is Cohen–Macaulay, then so is $N_{\mathfrak{p}} \cong (N_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$.

If $\operatorname{depth} M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} N_{\mathfrak{q}} \geq n$ then the above inequality guarantees that $\operatorname{depth} N_{\mathfrak{p}} \geq n$. □

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