

2-NILPOTENT REAL SECTION CONJECTURE

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ABSTRACT. We show a 2-nilpotent section conjecture over \mathbb{R} : for a geometrically connected curve X over \mathbb{R} such that each irreducible component of its normalization has \mathbb{R} -points, $\pi_0(X(\mathbb{R}))$ is determined by the maximal 2-nilpotent quotient of the fundamental group with its Galois action, as the kernel of an obstruction of Jordan Ellenberg. This implies that for X smooth and proper, $X(\mathbb{R})^\pm$ is determined by the maximal 2-nilpotent quotient of $\text{Gal}(\mathbb{C}(X))$ with its $\text{Gal}(\mathbb{R})$ action, where $X(\mathbb{R})^\pm$ denotes the set of real points equipped with a real tangent direction, showing a 2-nilpotent birational real section conjecture.

1. INTRODUCTION

Grothendieck's section conjecture predicts that the rational points of hyperbolic curves over finitely generated fields are determined by their étale fundamental groups. Let X denote a geometrically connected, finite type scheme over a characteristic 0 field k , equipped with a geometric point $b : \text{Spec } \Omega \rightarrow X$. Let \bar{k} denote the algebraic closure of k in Ω . There is a canonical lift of b to $X_{\bar{k}} = X \times_{\text{Spec } k} \text{Spec } \bar{k}$ and an exact sequence of étale fundamental groups

$$(1) \quad 1 \rightarrow \pi_1(X_{\bar{k}}, b) \rightarrow \pi_1(X, b) \rightarrow G_k \rightarrow 1,$$

where $G_k = \text{Gal}(\bar{k}/k)$ is the absolute Galois group of k , and all fundamental groups are based at the geometric points naturally associated to b [SGAI, IX Thm 6.1]. A rational point $x : \text{Spec } k \rightarrow X$ induces a map $\pi_1(x) : G_k \rightarrow \pi_1(X, x)$, where $\pi_1(X, x)$ denotes the étale fundamental group of X based at the geometric point $\text{Spec } \bar{k} \rightarrow \text{Spec } k \rightarrow X$ associated to x . View x and b as geometric points of $X_{\bar{k}}$ and choose a path between them, giving a path between x and b in X and an isomorphism $\pi_1(X, x) \cong \pi_1(X, b)$ respecting the projections to G_k . Composing $\pi_1(x)$ with this isomorphism $\pi_1(X, x) \cong \pi_1(X, b)$ produces a section $s : G_k \rightarrow \pi_1(X, b)$ of (1), and a different choice of path will change the section to $g \mapsto \lambda s(g) \lambda^{-1}$ for some λ in $\pi_1(X_{\bar{k}}, b)$. Sections obtained from s in this way are said to be conjugate. Let $\mathcal{S}_{\pi_1(X/k)}$ denote the conjugacy classes of sections of (1), and let $X(k)$ denote the set of k -points of X . Given X and b as above, let κ be the map

$$\kappa : X(k) \rightarrow \mathcal{S}_{\pi_1(X/k)}$$

just constructed. For X a smooth, proper curve of genus > 1 over a finitely generated field, Grothendieck's section conjecture, which is unknown, is that κ is a bijection.

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For $k = \mathbb{R}$, the map κ factors through $\pi_0(X(\mathbb{R}))$, and the real section conjecture, saying that

$$\kappa : \pi_0(X(\mathbb{R})) \rightarrow \mathcal{S}_{\pi_1(X/\mathbb{R})}$$

is a bijection is proven, but non-trivial [Moc03] [Sul05] [Mil84] [Car91] [Pál11]. This paper proves a 2-nilpotent real section conjecture, determining $\pi_0(X(\mathbb{R}))$ from the maximal 2-nilpotent quotient of $\pi_1(X_{\mathbb{C}})$ with its $G_{\mathbb{R}}$ -action.

For a (profinite) group π , let $\pi = [\pi]_1 > [\pi]_2 > [\pi]_3 > \dots$ denote the lower central series of π , i.e. $[\pi]_{n+1} = [[\pi]_n, \pi]$ (respectively $[\pi]_{n+1} = \overline{[[\pi]_n, \pi]}$) is (the closure of) the subgroup generated by commutators of elements of $[\pi]_n$ and π . Pushing out (1) by the quotient $\pi_1(X_{\bar{k}}, b) \rightarrow \pi_1(X_{\bar{k}}, b)/[\pi_1(X_{\bar{k}}, b)]_n$ yields an exact sequence

$$(2) \quad 1 \rightarrow \pi_1(X_{\bar{k}}, b)/[\pi_1(X_{\bar{k}}, b)]_n \rightarrow \pi_1(X, b)/[\pi_1(X_{\bar{k}}, b)]_n \rightarrow G_k \rightarrow 1.$$

Let κ^{ab} be the map taking a k -point x of X to the section of (2) for $n = 2$ determined by $\kappa(x)$.

Define *curve* to mean a pure dimension 1, finite type scheme over a field. A curve X over k will be said to be *based* if it is equipped with a choice of a geometric point whose image is a k -point of X or which is associated to a k -tangent vector based at a smooth point of a compactification of X as described in 2.1. The complex analytic space $X(\mathbb{C})$ associated to a based curve X over \mathbb{R} has a distinguished point or tangent vector based at a smooth point of a compactification, allowing us to apply the topological or orbifold fundamental group functors to $X(\mathbb{C})$ or $X(\mathbb{C})/G_{\mathbb{R}}$, giving maps κ and κ^{ab} as above. This is also discussed in 2.2.

In the following theorem, X is a based curve over \mathbb{R} , π denotes either the étale or topological fundamental group of $X_{\mathbb{C}}$ or $X(\mathbb{C})$ respectively, and $\pi_1(X)$ denotes either the étale or orbifold fundamental group of X or $X(\mathbb{C})/G_{\mathbb{R}}$ respectively.

1.1. Theorem. — *Let X be a geometrically connected, based curve over \mathbb{R} , such that each irreducible component of its normalization has \mathbb{R} -points. Then κ^{ab} is a natural bijection from $\pi_0(X(\mathbb{R}))$ to conjugacy classes of sections of*

$$1 \rightarrow \pi/[\pi]_2 \rightarrow \pi_1(X)/[\pi]_2 \rightarrow G_{\mathbb{R}} \rightarrow 1$$

which lift to sections of

$$1 \rightarrow \pi/[\pi]_3 \rightarrow \pi_1(X)/[\pi]_3 \rightarrow G_{\mathbb{R}} \rightarrow 1.$$

Note that the assumption that X is based gives (1) a splitting, and that this implies that Theorem 1.1 says that the 2-nilpotent quotient $\pi/[\pi]_3$ of π with its $G_{\mathbb{R}}$ -action determines the connected components of $X(\mathbb{R})$. The real section conjecture shows that π with its $G_{\mathbb{R}}$ -action determines the connected components of $X(\mathbb{R})$ when the topological space $X(\mathbb{C})$ is a $K(\pi, 1)$, which is the case precisely when no component of the normalization of $X_{\mathbb{C}}$ is \mathbb{P}^1 – see Remark 3.15. The proof of Theorem 1.1 given below is independent of the real section conjecture, although assuming it, one would be saved the trouble of proving Proposition 3.10.

For X smooth and proper, Theorem 1.1 applied to smaller and smaller Zariski opens of X shows that $X(\mathbb{R})$ is determined by the maximal 2-nilpotent quotient of the absolute Galois group of the function field of $X_{\mathbb{C}}$ with its $G_{\mathbb{R}}$ -action. More specifically, for a geometrically integral curve X over k , and a field extension L/k , let $L(X)$ denote the function field of X_L . For a smooth, proper, connected, based curve X over \mathbb{R} , let $X(\mathbb{R})^{\pm}$ denote the set of real points of X equipped with a real tangent direction, i.e. a vector in the tangent space of the smooth 1-manifold $X(\mathbb{R})$ up to scaling by elements of $\mathbb{R}_{>0}$. The notation $X(\mathbb{R})^{\pm}$ is meant to indicate that after orienting $X(\mathbb{R})$, the two tangent directions associated to each element of $X(\mathbb{R})$ consist of the direction distinguished by the orientation and its negative. For any Zariski open U of X , there is a map $X(\mathbb{R})^{\pm} \rightarrow \pi_0(U(\mathbb{R}))$ given by taking a tangent direction to the connected component it is pointing towards. Note that the resulting map $X(\mathbb{R})^{\pm} \rightarrow \varprojlim_U \pi_0(U(\mathbb{R}))$ is a bijection. It follows that a corollary of the 2-nilpotent real section conjecture is that $X(\mathbb{R})^{\pm}$ is determined by $G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_3 \rightarrow G_{\mathbb{R}}$.

1.2. Corollary. — *Let X be a smooth, proper, connected curve over \mathbb{R} equipped with a chosen element of $X(\mathbb{R})^{\pm} \neq \emptyset$. There is a natural bijection between $X(\mathbb{R})^{\pm}$ and the conjugacy classes of sections of*

$$1 \rightarrow G_{\mathbb{C}(X)}/[G_{\mathbb{C}(X)}]_2 \rightarrow G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_2 \rightarrow G_{\mathbb{R}} \rightarrow 1$$

which lift to sections of

$$1 \rightarrow G_{\mathbb{C}(X)}/[G_{\mathbb{C}(X)}]_3 \rightarrow G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_3 \rightarrow G_{\mathbb{R}} \rightarrow 1.$$

Corollary 1.2 is shown as Corollary 5.1 in section 5.

Let $G = \mathbb{Z}/2$ and EG denote a contractible topological space with a free action of G . For a sufficiently well-behaved topological space Y with a G -action, e.g. Y a G -CW complex, let $\text{Map}(EG, Y)$ denote the function space of continuous maps $EG \rightarrow Y$ equipped with the G action given by $gf = gfg^{-1}$. The homotopy fixed points of G on Y are defined $Y^{hG} = \text{Map}(EG, Y)^G$ and there is a canonical map $Y^G \rightarrow Y^{hG}$ from the fixed points to the homotopy fixed points induced by the G -equivariant map from EG to the point.

Let $\mathcal{S}_{\pi_1(Y/G)}$ denote the conjugacy classes of sections of

$$(3) \quad 1 \rightarrow \pi \rightarrow \pi_1(Y) \rightarrow G \rightarrow 1$$

where π denotes the topological fundamental group of Y , based at some point not included in the notation, and $\pi_1(Y)$ denotes the orbifold fundamental group, which can be identified with the topological fundamental group of $EG \times_G Y$ or with the group of automorphisms of the universal cover of Y lying over an automorphism of Y induced by an element of G . There is a natural map $\pi_0(Y^{hG}) \rightarrow \mathcal{S}_{\pi_1(Y/G)}$, which is a bijection if Y is a $K(\pi, 1)$.

For X a geometrically connected, finite type scheme over \mathbb{R} , the map κ for the étale fundamental group is the composition

$$(4) \quad \pi_0(X(\mathbb{C})^{G_{\mathbb{R}}}) \rightarrow \pi_0(X(\mathbb{C})^{hG_{\mathbb{R}}}) \rightarrow \mathcal{S}_{\pi_1(X(\mathbb{C})/G_{\mathbb{R}})} \rightarrow \mathcal{S}_{\pi_1(X/\mathbb{R})}$$

where the last map is induced by the canonical isomorphism from the profinite completion of (3) to (1) [SGAI, XII Cor 5.2], and the map κ for the topological fundamental group

is the composition of the first two maps of (4). For $X(\mathbb{C})$ a $K(\pi, 1)$, as in the section conjecture, the second map is a bijection.

The Sullivan conjecture [Sul05], proven by Miller [Mil84], Dwyer-Miller-Neisendorfer [DMN89], Carlsson [Car91], and Lannes [Lan92], shows that the first map of (4) is a bijection. Precisely, it says that the natural map from the p -completion of the fixed points to the homotopy fixed points of the p -completion is a weak equivalence for a finite p -group G acting on a finite G -CW complex. In particular, applying π_0 to $Y^G \rightarrow Y^{hG}$, as in the first map, is a bijection for $G = \mathbb{Z}/2$ acting on a finite G -CW complex Y [Car91, Theorem B (a)]. So if one overlooks the map $\mathcal{S}_{\pi_1(X(\mathbb{C})/G_{\mathbb{R}})} \rightarrow \mathcal{S}_{\pi_1(X/\mathbb{R})}$ coming from the map $\pi \rightarrow \pi^\wedge$ from π to its profinite completion, or if one considers κ for the topological fundamental group, the real section conjecture is π_0 of Sullivan's conjecture applied to a $K(\pi, 1)$. Also see [Pál11] for a nice proof of the real section conjecture in the étale and topological case which does not appeal to Sullivan's conjecture.

It is not true in general that Sullivan's conjecture holds for the infinite symmetric product of a finite G -CW complex, nor that

$$(5) \quad \pi_0((\text{Sym}^\infty Y)^G) \rightarrow \pi_0((\text{Sym}^\infty Y)^{hG})$$

is a bijection. The 2-nilpotent real section conjecture is proven by showing that for $Y = X(\mathbb{C})$ with X a real based algebraic curve such that each irreducible component of its normalization has \mathbb{R} -points, the map (5) is a bijection. This is the content of Proposition 3.1. Literally, Proposition 3.1 and Proposition 3.7 show that

$$\pi_0((\text{Sym}^\infty X(\mathbb{C}))^{G_{\mathbb{R}}}) \rightarrow \pi_0((\text{Sym}^\infty X(\mathbb{C}))^{hG_{\mathbb{R}}}) \rightarrow \mathcal{S}_{\pi_1((\text{Sym}^\infty X(\mathbb{C})/G_{\mathbb{R}})}$$

is a bijection, but it is straight-forward to see from the Dold-Thom theorem and the spectral sequence $H^i(G_{\mathbb{R}}, \pi_j(\text{Sym}^\infty X(\mathbb{C}))) \Rightarrow \pi_{j-i}(\text{Sym}^\infty X(\mathbb{C}))^{hG}$ [BK72, IX §4] that the second map is injective. Proposition 3.1 is shown by reducing to the case of a smooth curve and using the generalized Jacobian for which the map from fixed points to homotopy fixed points is a bijection on π_0 – see Proposition 3.10.

The connection between (5) and Theorem 1.1 is that $\pi_0((\text{Sym}^\infty X(\mathbb{C}))^{hG_{\mathbb{R}}})$ is naturally identified with the conjugacy classes of sections of

$$1 \rightarrow \pi/[\pi]_2 \rightarrow \pi_1(X)/[\pi]_2 \rightarrow G_{\mathbb{R}} \rightarrow 1.$$

The obstruction to lifting such a section to the two-nilpotent sequence is a quadratic form studied by Jordan Ellenberg as an obstruction to rational points of a curve's Jacobian lying in the image of an Abel-Jacobi map [Ell00], and also studied by Zarhin [Zar74]. Theorem 1.1 was guessed by Jordan Ellenberg in the proper, smooth case, as he told me after I had observed it held in several examples, and it can naturally be expressed in terms of his ideas in [Ell00]: a smooth based curve embeds into its generalized Jacobian by its Abel-Jacobi map. Those rational points y of the Jacobian which are the image of a point of the curve satisfy the condition that $\kappa(y)$ lifts to a section of (1) where X denotes the curve. Filtering $\pi_1(X_{\mathbb{C}})$ by its lower central series provides a series of obstructions, the first of which is the quadratic form, here denoted δ_2 and discussed in §4.

We give a topological interpretation of Ellenberg’s point of view in Section 6, constructing a diagram of finite $G_{\mathbb{R}}$ -CW complexes

(6)

$$\begin{array}{ccc} & & \text{Alb}_2 \\ & \nearrow & \downarrow q \\ X(\mathbb{C}) & \longrightarrow & \text{Alb}_1 \end{array}$$

for an arbitrary geometrically connected curve X over \mathbb{R} with a chosen real base point, such that Alb_2 is a $K(\pi_1(X(\mathbb{C}))/[\pi_1(X(\mathbb{C}))]_3, 1)$, Alb_1 is a $K(\pi_1(X(\mathbb{C}))/[\pi_1(X(\mathbb{C}))]_2, 1)$, q is a fiber bundle, and all maps induce the obvious quotient maps on topological fundamental groups. Sullivan’s conjecture gives an equivalence between Theorem 1.1 and the statement that the connected components of the real points of the curve are those of the abelian approximation Alb_1 which can be lifted to the 2-nilpotent approximation Alb_2 . See Theorem 6.5.

Relation to other work: Grothendieck’s section conjecture is part of his anabelian conjectures predicting that certain schemes are determined by their étale fundamental groups. Birational variants of the anabelian conjectures replace $\pi_1^{\text{ét}}$ by the absolute Galois group of the function field. There has been considerable work describing varieties using small quotients of their fundamental groups or the Galois groups of their function fields. Pop has shown a meta-abelian birational section conjecture over p -adic fields [Pop10a]. Bogomolov and Tschinkel developed an approach to recognize the function field of certain varieties using the 2-nilpotent quotient of the absolute Galois group. Work of Bogomolov, Pop, and Tschinkel shows that it is possible to recover the function field of certain varieties of dimension ≥ 2 over algebraically closed fields from the 2-nilpotent quotient of the absolute Galois group [Bog91b] [Bog91a] [BT02] [BT08] [Pop10b] [Pop12] – see [BT12] and [Pop11] for more discussion. There is also interesting work limiting when such minimalistic anabelian results can hold. Yuichiro Hoshi has found examples where any section of a pro- p homotopy exact sequence of the Jacobian lifts to a section of a pro- p homotopy exact sequence of the curve [Hos10].

Acknowledgments: I wish to thank Jordan Ellenberg for sharing [Ell00] with me and for suggesting that [GH81] could be used to prove the main result of this paper in the smooth, proper case. I also wish to thank Gunnar Carlsson for recognizing the connection between this problem and Sullivan’s conjecture. It is a pleasure to express my deepest gratitude and admiration to both Carlsson and Ellenberg. I thank an anonymous reviewer of a previous version of this paper for Lemma 4.2 and other helpful suggestions. I would also like to thank Florian Pop for suggesting that Theorem 1.1 could be used to show a birational result, as well as for many helpful comments.

2. PRELIMINARIES

2.1. *Review of tangential base points.* It is convenient, and necessary for section 5, to allow fundamental groups of curves to be based at tangent vectors at smooth points of compactifications. For the topological fundamental group, this is accomplished by noting that the fundamental group can be based at an inverse system of simply connected

subsets and that a tangent vector determines such a system [Del89, Intro p 85 and §15]. For the étale fundamental group, a tangential base point can be defined as follows [Del89, §15] [Nak99]. Let X and \bar{X} be geometrically connected curves over a characteristic 0 field k , with \bar{X} proper and $X \subset \bar{X}$ an open immersion. Let $x \in \bar{X}(k)$ be a smooth point, so in particular, x could be a smooth point in $(\bar{X} - X)(k)$.

A local parameter z at x gives rise to an isomorphism $\widehat{\mathcal{O}}_{\bar{X},x} \xleftarrow{\cong} k[[z]]$, where $\widehat{\mathcal{O}}_{\bar{X},x}$ denotes the completion of the local ring of x . The field of Puiseux series

$$\bar{k}((z^{\mathbb{Q}})) := \bigcup_{n \in \mathbb{Z}_{>0}} \bar{k}((z^{1/n}))$$

is algebraically closed, and the composition

$$\mathrm{Spec} \bar{k}((z^{\mathbb{Q}})) \rightarrow \mathrm{Spec} k[[z]] \cong \mathrm{Spec} \widehat{\mathcal{O}}_{\bar{X},x} \rightarrow \bar{X}$$

factors through the generic point of the component of \bar{X} containing x , thus defining a geometric point of X . This geometric point is said to be associated to the tangent vector

$$\mathrm{Spec} k[[z]]/\langle z^2 \rangle \rightarrow \bar{X},$$

given by the local parameter z . A *path* between geometric points is a natural transformation between the associated fiber functors. For any two geometric points associated to the same tangent vector, there is a G_k -invariant path between their canonical lifts to geometric points of $X_{\bar{k}}$, justifying the nomenclature.

2.2. κ for based spaces. For a topological space Y with a G -action, equipped with a G -fixed base point, there is a map

$$\kappa : \pi_0(Y^G) \rightarrow H^1(G, \pi)$$

consistent with κ as defined in the introduction, where π denotes the fundamental group of the based space Y , defined by sending $y \in Y^G$ to the twisted homomorphism

$$(7) \quad g \mapsto \gamma^{-1}(g\gamma)$$

for all $g \in G$, where γ is a path from the base point to y , and composition in the fundamental group is written so that $\gamma^{-1}(g\gamma)$ is the loop starting at the base point obtained by first traversing $g\gamma$ and then traversing γ^{-1} . It is straight-forward to check that the cohomology class of (7) is independent of the choice of γ and that points in the same connected component of Y^G determine the same cohomology class. The same definition of κ holds when the G -fixed base point is replaced by an inverse system of simply connected G -stable sets.

For a group or profinite group π , let $\pi^{\mathrm{ab}} := \pi/[\pi]_2$ denote the abelianization of π . The map κ^{ab} is κ followed by $H^1(G, \pi) \rightarrow H^1(G, \pi^{\mathrm{ab}})$ induced from the quotient $\pi \rightarrow \pi^{\mathrm{ab}}$.

Let X denote a geometrically connected, finite type scheme over a characteristic 0 field k , equipped with a geometric point $b : \mathrm{Spec} \Omega \rightarrow X$ and assume that the image of b is a k -point. Let \bar{k} denote the algebraic closure of k in Ω . For any $x \in X(k)$, the torsor of paths from the lift of b to $X_{\bar{k}}$ to the lift $x : \mathrm{Spec} \bar{k} \rightarrow X_{\bar{k}}$ of x has a canonical G_k -action classified by the element of $H^1(G_k, \pi_1(X_{\bar{k}}, b))$ represented by the cocycle (7), where γ is any chosen path from b to x , and g is in G_k . Sending x to this element gives a well-defined map

$$X(k) \rightarrow H^1(G_k, \pi_1(X_{\bar{k}}, b))$$

consistent with the map κ from the introduction. When $k = \mathbb{R}$, this map factors through $\pi_0(X(\mathbb{R}))$, resulting in the map

$$\kappa : \pi_0(X(\mathbb{R})) \rightarrow H^1(G_{\mathbb{R}}, \pi_1(X_{\mathbb{C}}, b)).$$

For X a based curve, the torsor of paths from the lift of the base point to $X_{\bar{k}}$ to the lift of a k -point likewise has a G_k -action, and κ is defined by the same formula.

3. ABELIAN APPROXIMATION

Let \mathcal{I} denote the forgetful functor from vector spaces over $\mathbb{Z}/2$ to pointed sets, sending a vector space to its underlying set, pointed by the identity. Let \mathcal{V} denote the left adjoint to \mathcal{I} , called the *free vector space* on the pointed set. The *unit* is the canonical map of pointed sets $(S, s_0) \rightarrow \mathcal{I}\mathcal{V}(S, s_0)$.

Note that for a based curve X over \mathbb{R} , the set $\pi_0(X(\mathbb{R}))$ is naturally pointed, as the base point either lies in a particular connected component or points towards one.

In the following proposition, π can denote either the étale or topological fundamental group of $X_{\mathbb{C}}$ or $X(\mathbb{C})$ respectively.

3.1. Proposition. — *Let X be a geometrically connected, based curve over \mathbb{R} , such that each irreducible component of its normalization has \mathbb{R} -points. Then*

$$\kappa^{\text{ab}} : \pi_0(X(\mathbb{R})) \rightarrow H^1(G_{\mathbb{R}}, \pi^{\text{ab}})$$

is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(X(\mathbb{R}))$.

We first prove Proposition 3.1 when X is smooth, so make this assumption now: let X be a geometrically connected, smooth, based curve over \mathbb{R} .

Let $\text{Jac}^n X$ denote the degree n generalized Jacobian of X , i.e. the degree n Picard scheme of X if X is proper, and the degree n Picard scheme of the one point compactification X^+ of X for X non-proper; see appendix A. There is a canonical map $\text{Sym}^n X \rightarrow \text{Jac}^n X$, where $\text{Sym}^n X$ denotes the n^{th} symmetric product of X by composing [SGA4_{III}, XVII 6.3.8.2] and the map (25) of the appendix. For a coherent sheaf \mathcal{E} , let $V(\mathcal{E}) = \underline{\text{Spec}} \text{Sym} \mathcal{E}$ and $\mathbb{P}(\mathcal{E}) = \underline{\text{Proj}} \text{Sym} \mathcal{E}$, as in [EGAII, 1.7.8] and [EGAII, 3.1.3].

3.2. Proposition. — *For n sufficiently large,*

$$\text{Sym}^n X \rightarrow \text{Jac}^n X$$

is given by $V(\mathcal{E}) \rightarrow \text{Jac}^n X$ (respectively $\mathbb{P}(\mathcal{E}) \rightarrow \text{Jac}^n X$) for a non-zero locally free sheaf \mathcal{E} on $\text{Jac}^n X$ and X non-proper (respectively proper).

Proof. For X proper, note that since X is based, we may choose an \mathbb{R} -point b of X . By [BLR90, 8.1 Prop. 4], there is a unique universal line bundle \mathcal{P} over $X \times \text{Pic}_X$ whose restriction to $b \times \text{Pic}_X$ is trivial. By [SGA4_{III}, 6.3.9] and [BLR90, 8.2 Prop 7], $\text{Sym}^n X \rightarrow \text{Jac}^n X$ is given by $\mathbb{P}(\mathcal{E})$ for \mathcal{E} a $\mathcal{O}_{\text{Jac}^n X}$ -module of finite presentation. By Lemma A.1 in the

appendix, \mathcal{P} is cohomologically flat in dimension 0 when restricted to $X \times \text{Pic}_X^n$ for large n . By [BLR90, 8.2 Prop 7], this implies that $p_*\mathcal{P}$ is locally free and dual to \mathcal{E} which is also locally free, where $p : X \times \text{Pic}_X^n \rightarrow \text{Pic}_X^n$ denotes the projection.

For X non-proper, a proof is provided as Proposition A.5 in the appendix, although the result is also well-known in this case. \square

3.3. Corollary. — *When the real points of $\text{Sym}^n X$ and $\text{Jac}^n X$ are given the analytic topology, $\text{Sym}^n X(\mathbb{R}) \rightarrow \text{Jac}^n X(\mathbb{R})$ is a vector bundle (respectively projectivized vector bundle) for n sufficiently large and X non-proper (respectively proper).*

3.4. Remark. Corollary 3.3 is an alternate way to see [GH81, Prop 3.2 (2) $n(W^d) = n(S^d X)$ for large d] and [Bro96, 2.7.4 $\mathcal{O}(D') \simeq \mathcal{O}(D)$].

For notational simplicity, when X is proper, let $X^+ = X$.

Choose $b' \in X(\mathbb{R})$ representing the distinguished element of $\pi_0(X(\mathbb{R}))$. The point $b' : \text{Spec } \mathbb{R} \rightarrow X$ determines maps $\text{Sym}^n X \rightarrow \text{Sym}^n X \times X \rightarrow \text{Sym}^{n+1} X$ for $n \in \mathbb{Z}_{\geq 1}$. Since X is smooth, b' is a smooth point of X^+ , and therefore determines a degree 1 invertible sheaf \mathcal{L} of rational functions with at worst a pole at b' . The sheaf \mathcal{L} determines isomorphisms $\text{Jac}^n X \rightarrow \text{Jac}^{n+1} X$ corresponding to the tensor product of the pullback of \mathcal{L} with a/the universal line bundle on $X^+ \times \text{Jac}^n X$. These maps are compatible with the maps $\text{Sym}^n X \rightarrow \text{Sym}^{n+1} X$ in the sense that the diagram

$$\begin{array}{ccc} \text{Sym}^n X & \longrightarrow & \text{Sym}^{n+1} X \\ \downarrow & & \downarrow \\ \text{Jac}^n X & \longrightarrow & \text{Jac}^{n+1} X \end{array}$$

commutes. Composing $\text{Sym}^n X \rightarrow \text{Jac}^n X$ with the isomorphism $\text{Jac}^n X \rightarrow \text{Jac} X$ obtained from the inverses of $\text{Jac}^m X \rightarrow \text{Jac}^{m+1} X$ for $m = n - 1, n - 2, \dots, 0$, we obtain maps

$$(8) \quad \text{Sym}^n X \rightarrow \text{Jac} X$$

which commute with the maps $\text{Sym}^n X \rightarrow \text{Sym}^{n+1} X$.

For $n = 1$, the map (8) is the Abel-Jacobi map $\alpha : X \rightarrow \text{Jac} X$ corresponding to the invertible sheaf \mathcal{L} . Since $\text{Jac} X$ is an abelian group scheme, α determines a map $\text{Sym}^n X \rightarrow \text{Jac} X$, which recovers (8) for any n .

As the maps (8) commute with $X = \text{Sym}^1 X \rightarrow \text{Sym}^n X$, the diagram

$$\begin{array}{ccc} & & \text{Sym}^n X \\ & \nearrow & \downarrow \\ X & \xrightarrow{\alpha} & \text{Jac} X \end{array}$$

commutes. Corollary 3.3 implies that:

3.5. Corollary. — For n sufficiently large, $\pi_0(\mathrm{Sym}^n X(\mathbb{R})) \rightarrow \pi_0(\mathrm{Jac} X(\mathbb{R}))$ is a bijection.

Let $X(\mathbb{C})$, $\mathrm{Sym}^n X(\mathbb{C})$ and $\mathrm{Jac} X(\mathbb{C})$ denote the complex points of X , $\mathrm{Sym}^n X$ and $\mathrm{Jac} X$ respectively equipped with the analytic topology. The natural map from the n^{th} symmetric power of $X(\mathbb{C})$ to $\mathrm{Sym}^n X(\mathbb{C})$ is a homeomorphism, so both may be denoted $\mathrm{Sym}^n X(\mathbb{C})$. The maps $\mathrm{Sym}^n X \rightarrow \mathrm{Sym}^{n+1} X$ produce maps $\mathrm{Sym}^n X(\mathbb{C}) \rightarrow \mathrm{Sym}^{n+1} X(\mathbb{C})$ described as taking an n -tuple $x_1 \times x_2 \times \dots \times x_n$ of \mathbb{C} -points of X to $x_1 \times x_2 \times \dots \times x_n \times b'$. Let $\mathrm{Sym}^\infty X(\mathbb{C}) = \varinjlim_n \mathrm{Sym}^n X(\mathbb{C})$ be the infinite symmetric product of the topological space $X(\mathbb{C})$. The action of $G_{\mathbb{R}}$ on $\mathrm{Sym}^n X(\mathbb{C})$ extends to $\mathrm{Sym}^\infty X(\mathbb{C})$, and the natural map from $\varinjlim_n \mathrm{Sym}^n X(\mathbb{R})$ to the fixed points of $\mathrm{Sym}^\infty X(\mathbb{C})$ is an isomorphism, where as above $\mathrm{Sym}^n X(\mathbb{R})$ denotes the \mathbb{R} -points of $\mathrm{Sym}^n X$ equipped with the analytic topology. We use the notation $\mathrm{Sym}^\infty X(\mathbb{R})$ for the fixed points of $\mathrm{Sym}^\infty X(\mathbb{C})$. The maps (8) determine the commutative diagram

$$(9) \quad \begin{array}{ccc} & & \mathrm{Sym}^\infty X(\mathbb{C}) \\ & \nearrow & \downarrow \\ X(\mathbb{C}) & \xrightarrow{\alpha(\mathbb{C})} & \mathrm{Jac} X(\mathbb{C}) \end{array}$$

and Corollary 3.5 shows:

3.6. Proposition. — The map $\pi_0(\mathrm{Sym}^\infty X(\mathbb{R})) \rightarrow \pi_0(\mathrm{Jac} X(\mathbb{R}))$ resulting from (9) is a bijection.

The monoid $\pi_0(\mathrm{Sym}^\infty X(\mathbb{R}))$ is computed by the following proposition, whose proof is essentially contained in Proposition 3.2 of [GH81], which is credited to Shimura.

3.7. Proposition. — Let \mathcal{X} be a path connected, Hausdorff, topological space with an action of $G = \mathbb{Z}/2$ equipped with a G -fixed base point. Assume that the path components of \mathcal{X}^G are closed in \mathcal{X}^G . Then the monoid structure on $\mathrm{Sym}^\infty \mathcal{X}$ gives $\pi_0((\mathrm{Sym}^\infty \mathcal{X})^G)$ a $\mathbb{Z}/2$ vector space structure, and $\pi_0(-)^G$ applied to $\mathcal{X} \rightarrow \mathrm{Sym}^\infty \mathcal{X}$ is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(\mathcal{X}^G)$.

Proof. Let p in \mathcal{X} be the base point, and let τ be the generator of $\mathbb{Z}/2$. Points of $\mathrm{Sym}^\infty \mathcal{X}$ are determined by finite unordered lists x_1, x_2, \dots, x_n of elements of \mathcal{X} where two unordered lists correspond to the same point of $\mathrm{Sym}^\infty \mathcal{X}$ if they are equal after removing all p 's. G -fixed points are lists formed from $x_i \in \mathcal{X}^G$ and pairs $x, \tau x$ for $x \in \mathcal{X}$. Let $[p]$ in $\pi_0(\mathcal{X}^G)$ denote the path component containing p . For α in $\pi_0(\mathcal{X}^G)$, let $X_\alpha \subset \mathcal{X}^G$ denote the corresponding path component. Let Y be the subset of $\mathcal{X} \times \mathcal{X}$ of pairs $(x, \tau x)$. Products of copies of Y and the X_α determine path connected subsets of $(\mathrm{Sym}^\infty \mathcal{X})^G$ whose union is all of $(\mathrm{Sym}^\infty \mathcal{X})^G$. Note that such a path connected subset arising from a product with two factors of the same X_α intersects the path connected subset arising from the same product with the two factors of X_α replaced by Y . Note also that the path connected subset of $(\mathrm{Sym}^\infty \mathcal{X})^G$ arising from a product containing factors of $X_{[p]}$ intersects the path connected subset determined by the product with those factors removed. Thus the monoid structure on $\mathrm{Sym}^\infty \mathcal{X}$ gives $\pi_0((\mathrm{Sym}^\infty \mathcal{X})^G)$ a $\mathbb{Z}/2$ vector space structure such that the elements $\{X_\alpha : \alpha \in \pi_0(\mathcal{X}^G) - [p]\}$ span.

To show these elements are linearly independent, it suffices to show that the homomorphism $h_\alpha : (\text{Sym}^\infty \mathcal{X})^\mathbb{G} \rightarrow \mathbb{Z}/2$ taking a finite unordered list to the number of elements contained in X_α is continuous. Since $(\text{Sym}^\infty \mathcal{X})^\mathbb{G}$ is canonically isomorphic to $\varinjlim (\text{Sym}^n \mathcal{X})^\mathbb{G}$, it is sufficient to show that the restriction of h_α to $(\text{Sym}^n \mathcal{X})^\mathbb{G}$ is continuous. Since \mathcal{X} is Hausdorff, $\mathcal{X}^\mathbb{G}$ is closed in \mathcal{X} , and thus for all α' in $\pi_0(\mathcal{X}^\mathbb{G})$, $X_{\alpha'}$ is closed in \mathcal{X} . Thus products of copies of Y and the X_α determine finitely many closed subsets of $(\text{Sym}^n \mathcal{X})^\mathbb{G}$. Taking unions of these closed subsets as appropriate gives a decomposition of $(\text{Sym}^n \mathcal{X})^\mathbb{G}$ into closed and open subsets on which h_α is constant.

□

Applying Proposition 3.7 to $\mathcal{X} = X(\mathbb{C})$ shows:

3.8. Corollary. — *The map $\pi_0(X(\mathbb{R})) \rightarrow \pi_0(\text{Sym}^\infty X(\mathbb{R}))$ resulting from (9) is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(X(\mathbb{R}))$.*

Combining Corollary 3.8 with Proposition 3.6 shows:

3.9. Proposition. — *Let X be a smooth curve over \mathbb{R} such that $X(\mathbb{R}) \neq \emptyset$. Let $\alpha : X \hookrightarrow \text{Jac } X$ be the Abel-Jacobi map corresponding to a base point $b' \in X(\mathbb{R})$. Then $\pi_0(X(\mathbb{R})) \rightarrow \pi_0(\text{Jac } X(\mathbb{R}))$ is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(X(\mathbb{R}))$.*

Let π_J denote either the étale fundamental group $\pi_1^{\text{ét}}(\text{Jac } X_{\mathbb{C}}, 0)$ of $\text{Jac } X_{\mathbb{C}}$ based at the identity or the topological fundamental group $\pi_1^{\text{top}}(\text{Jac } X(\mathbb{C}), 0)$. Let $\kappa_J : \pi_0(\text{Jac } X(\mathbb{R})) \rightarrow H^1(G_{\mathbb{R}}, \pi_J)$ denote the corresponding map κ as in 2.2.

3.10. Proposition. — *The map $\kappa_J : \pi_0(\text{Jac } X(\mathbb{R})) \rightarrow H^1(G_{\mathbb{R}}, \pi_J)$ is a $\mathbb{Z}/2$ vector space isomorphism.*

The content of the proof of Proposition 3.10 is identical in the étale and topological setting using [VW11, §3.3, 4.1]. We give the étale proof with the modification for the topological case in parentheses.

Proof. Let $\widetilde{\text{Jac } X_{\mathbb{C}}}$ (resp. $\widetilde{\text{Jac } X(\mathbb{C})}$) denote the universal covering space of $\text{Jac } X_{\mathbb{C}}$ (resp. $\text{Jac } X(\mathbb{C})$), which is automatically an abelian group. The canonical exact sequence

$$0 \rightarrow \pi_J \rightarrow \widetilde{\text{Jac } X_{\mathbb{C}}} \rightarrow \text{Jac } X_{\mathbb{C}} \rightarrow 0$$

gives a short exact sequence of abelian groups with $G_{\mathbb{R}}$ action

$$0 \rightarrow \pi_J \rightarrow \widetilde{\text{Jac } X_{\mathbb{C}}(\mathbb{C})} \rightarrow \text{Jac } X(\mathbb{C}) \rightarrow 0$$

$$(\text{resp. } 0 \rightarrow \pi_J \rightarrow \widetilde{\text{Jac } X(\mathbb{C})} \rightarrow \text{Jac } X(\mathbb{C}) \rightarrow 0).$$

For notational simplicity, let

$$0 \rightarrow \pi_J \rightarrow \widetilde{\text{Jac}} \rightarrow \text{Jac} \rightarrow 0$$

denote the previous exact sequence, in either the étale or topological case. Applying Tate cohomology gives the exact sequence

$$(10) \quad \dots \hat{H}^0(G_{\mathbb{R}}, \widetilde{Jac}) \rightarrow \hat{H}^0(G_{\mathbb{R}}, Jac) \rightarrow \hat{H}^1(G_{\mathbb{R}}, \pi_J) \rightarrow \hat{H}^1(G_{\mathbb{R}}, \widetilde{Jac}) \dots$$

Multiplication by 2 is a profinite-étale covering space $Jac X_{\mathbb{C}} \xrightarrow{[2]} Jac X_{\mathbb{C}}$ (resp. covering space $Jac X(\mathbb{C}) \xrightarrow{[2]} Jac X(\mathbb{C})$), whence $\widetilde{Jac X_{\mathbb{C}}} \rightarrow Jac X_{\mathbb{C}} \xrightarrow{[2]} Jac X_{\mathbb{C}}$ and $\widetilde{Jac X(\mathbb{C})} \rightarrow Jac X(\mathbb{C}) \xrightarrow{[2]} Jac X(\mathbb{C})$ are simply connected covering spaces of $Jac X_{\mathbb{C}}$. Thus there is a unique lift of [2] to an isomorphism $\widetilde{Jac X_{\mathbb{C}}} \rightarrow \widetilde{Jac X_{\mathbb{C}}}$ (resp. $\widetilde{Jac X(\mathbb{C})} \rightarrow \widetilde{Jac X(\mathbb{C})}$) sending the identity to itself, and this lift is multiplication by 2. See [VW11, Prop. 3.1, Thm 3.1]. Thus multiplication by 2 is an isomorphism on \widetilde{Jac} , and $\hat{H}^i(G_{\mathbb{R}}, \widetilde{Jac}) = 0$ for all i . Thus $\hat{H}^0(G, Jac) \rightarrow \hat{H}^1(G, \pi_J)$ from (10) is an isomorphism.

Let τ denote the generator of $G_{\mathbb{R}}$. The canonical map $Jac X(\mathbb{R}) \rightarrow \hat{H}^0(G_{\mathbb{R}}, Jac)$ is a surjective homomorphism of abelian groups, with kernel $(1 + \tau) Jac$. Since $1 + \tau$ determines a continuous map $Jac X(\mathbb{C}) \rightarrow Jac X(\mathbb{R})$ and $Jac X(\mathbb{C})$ is connected, $(1 + \tau) Jac$ is contained in the connected component of the identity $Jac X(\mathbb{R})^0$ of $Jac X(\mathbb{R})$. Since $Jac X(\mathbb{R})^0$ is a connected abelian real Lie group, its universal cover is given by its exponential map $\mathbb{R}^d \rightarrow Jac X(\mathbb{R})^0$, where $d = \dim Jac X(\mathbb{R})^0$. Since \mathbb{R}^d divisible, it follows that $Jac X(\mathbb{R})^0$ is divisible, which implies that $(1 + \tau) Jac = Jac X(\mathbb{R})^0$. Thus $Jac X(\mathbb{R}) \rightarrow \hat{H}^0(G_{\mathbb{R}}, Jac)$ determines an isomorphism $\pi_0 Jac X(\mathbb{R}) \rightarrow \hat{H}^0(G_{\mathbb{R}}, Jac)$ of abelian groups.

Composing this isomorphism $\pi_0(Jac X(\mathbb{R})) \rightarrow \hat{H}^0(G_{\mathbb{R}}, Jac)$ with the isomorphism

$$\hat{H}^0(G_{\mathbb{R}}, Jac) \rightarrow \hat{H}^1(G_{\mathbb{R}}, \pi_J)$$

from (10) yields κ_J , completing the proof. □

3.11. *Remark.* Proposition 3.10 also follows from the real section conjecture. The above proof is included in order to give a proof of the 2-nilpotent real section conjecture which is independent of the real section conjecture or Sullivan's conjecture.

Proof of Proposition 3.1 for X smooth. Keep the notation from above that X is a smooth, geometrically connected, based curve over \mathbb{R} , π denotes either the topological fundamental group of $X(\mathbb{C})$ or the étale fundamental group of $X_{\mathbb{C}}$, and $b' \in X(\mathbb{R})$ represents the distinguished element of $\pi_0(X(\mathbb{R}))$. Let $Jac X$ denote the generalized Jacobian of X , and $\alpha : X \hookrightarrow Jac X$ denote the Abel-Jacobi map corresponding to b' . Since b' is in the distinguished path component of $X(\mathbb{R})$, there is a $G_{\mathbb{R}}$ -invariant path connecting b' to the base point. (In the étale case, b' will also denote a geometric point with image b' .) This path yields a $G_{\mathbb{R}}$ -equivariant isomorphism $\pi \rightarrow \pi'$, where π' denotes the appropriate fundamental group based at b' , and a commutative diagram

$$\begin{array}{ccc} \pi_0(X(\mathbb{R})) & \xrightarrow{=} & \pi_0(X(\mathbb{R})) \\ \downarrow \kappa & & \downarrow \kappa' \\ H^1(G_{\mathbb{R}}, \pi) & \xrightarrow{\cong} & H^1(G_{\mathbb{R}}, \pi') \end{array}$$

where κ' denotes the map of 2.2 with respect to the base point b' and κ is based at b as in the statement of Proposition 3.1. Combining with the functoriality of κ , we see that the diagram

$$\begin{array}{ccc} \pi_0(X(\mathbb{R})) & \xrightarrow{\pi_0(\alpha(\mathbb{R}))} & \pi_0(\text{Jac } X(\mathbb{R})) \\ \downarrow \kappa & & \downarrow \kappa_J \\ H^1(G_{\mathbb{R}}, \pi) & \xrightarrow{\pi_1(\alpha)_*} & H^1(G_{\mathbb{R}}, \pi_J) \end{array}$$

commutes. The Abel-Jacobi map α induces $\pi_1(\alpha) : \pi \rightarrow \pi_J$ which is the abelianization in either the étale or topological case [Moc10, Prop A.8 (iii)]. Thus $\kappa^{\text{ab}} = \kappa_J \circ \pi_0(\alpha(\mathbb{R}))$. By Proposition 3.9, $\pi_0(\alpha(\mathbb{R}))$ is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$. By Proposition 3.10, κ_J is an isomorphism. Thus κ^{ab} is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$, showing Proposition 3.1 for X smooth. \square

3.12. Hypothesis. — For notational convenience in the next two lemmas, we will say that a topological space \mathcal{X} with an action of $G = \mathbb{Z}/2$ satisfies hypothesis 3.12 if \mathcal{X} is Hausdorff, path connected and equipped with a G -fixed base point such that $\kappa^{\text{ab}} : \pi_0(\mathcal{X}^G) \rightarrow H^1(G, \pi_1(\mathcal{X})^{\text{ab}})$ is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(\mathcal{X}^G)$. We will say that a finite set of points D of \mathcal{X} satisfies hypothesis 3.12 if $GD = D$ and there is an open neighborhood of D in \mathcal{X} that deformation retracts onto D .

3.13. Lemma. — *Let \mathcal{X} be a topological space with an action of $G = \mathbb{Z}/2$ and let D be a finite set of points of \mathcal{X} both satisfying hypothesis 3.12. Let \sim be a G invariant equivalence relation on D and let \mathcal{Y} be the quotient of \mathcal{X} formed by identifying equivalent elements of D . Then $\kappa^{\text{ab}} : \pi_0(\mathcal{Y}^G) \rightarrow H^1(G, \pi_1(\mathcal{Y})^{\text{ab}})$ is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(\mathcal{Y}^G)$.*

Proof. Let τ denote the generator of G .

Suppose that D is order 4 and has the form $D = \{x_1, x_2, \tau x_1, \tau x_2\}$. Suppose that \sim is the G invariant equivalence relation on D generated by $x_1 \sim x_2$. Since $\mathcal{X}^G \rightarrow \mathcal{Y}^G$ is a homeomorphism, to verify the lemma in this case, it suffices to show that $H^1(G, \pi_1(\mathcal{X})) \rightarrow H^1(G, \pi_1(\mathcal{Y}))$ is an isomorphism. There is a pushout square

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ D' & \longrightarrow & \mathcal{Y} \end{array}$$

where $D' \subset \mathcal{Y}$ is the image of D . This gives the exact sequence

$$(11) \quad 0 \rightarrow H_1(\mathcal{X}) \rightarrow H_1(\mathcal{Y}) \rightarrow \mathbb{Z}(x_1 - x_2) \oplus \mathbb{Z}(\tau x_1 - \tau x_2) \rightarrow 0,$$

where $\mathbb{Z}(x_1 - x_2) \oplus \mathbb{Z}(\tau x_1 - \tau x_2) = \text{Ker } H_0(D) \rightarrow H_0(\mathcal{X}) \oplus H_0(D')$, and H_i denotes singular homology with coefficients in \mathbb{Z} . As a $\mathbb{Z}[G]$ module, there is an isomorphism $\mathbb{Z}(x_1 - x_2) \oplus \mathbb{Z}(\tau x_1 - \tau x_2) \cong \mathbb{Z}[G]$, and in particular (11) splits. Applying $H^*(G, -)$ to (11) shows that $H^1(G, \pi_1(\mathcal{X})) \rightarrow H^1(G, \pi_1(\mathcal{Y}))$ is an isomorphism as desired.

Suppose that D is order 2 and has the form $D = \{x, \tau x\}$ with $x \sim \tau x$. The cofiber sequence $D \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$ gives the exact sequence

$$0 \rightarrow H_1(\mathcal{X}) \rightarrow H_1(\mathcal{Y}) \rightarrow \mathbb{Z}(x - \tau x) \rightarrow 0.$$

Applying $H^*(G, -)$ gives the left exact sequence

$$0 \rightarrow H^1(G, \pi_1(\mathcal{X})) \rightarrow H^1(G, \pi_1(\mathcal{Y})) \rightarrow \mathbb{Z}/2(x - \tau x).$$

Note that x and τx determine the same point y of \mathcal{Y} and κ_y^{ab} maps to the generator of $\mathbb{Z}/2(x - \tau x)$. This gives the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(G, \pi_1(\mathcal{X})) & \longrightarrow & H^1(G, \pi_1(\mathcal{Y})) & \longrightarrow & \mathbb{Z}/2(x - \tau x) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{V}\pi_0((\mathcal{X})^G) & \longrightarrow & \mathcal{V}\pi_0((\mathcal{Y})^G) & \longrightarrow & \mathbb{Z}/2(y) \longrightarrow 0 \end{array}$$

Since the left vertical maps is an isomorphism by hypothesis, it follows that

$$\mathcal{V}\pi_0((\mathcal{Y})^G) \rightarrow H^1(G, \pi_1(\mathcal{Y}))$$

is an isomorphism, showing the lemma in this case.

Suppose that D is order 2 and has the form $D = \{x_1, x_2\}$ with $x_1 \sim x_2$ and $x_1, x_2 \in X^G$. The cofiber sequence $D \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$ gives the exact sequence

$$0 \rightarrow H_1(\mathcal{X}) \rightarrow H_1(\mathcal{Y}) \rightarrow \mathbb{Z}(x_1 - x_2) \rightarrow 0.$$

Applying $H^*(G, -)$ gives the right exact sequence

$$\mathbb{Z}/2(x_1 - x_2) \rightarrow H^1(G, \pi_1(\mathcal{X})) \rightarrow H^1(G, \pi_1(\mathcal{Y})) \rightarrow 0,$$

where the generator of $\mathbb{Z}/2(x_1 - x_2)$ is mapped to $\kappa_{x_1}^{\text{ab}} - \kappa_{x_2}^{\text{ab}}$. By hypothesis, $\kappa_{x_1}^{\text{ab}}$ induces an isomorphism $\mathcal{V}\pi_0((\mathcal{X})^G) \rightarrow H^1(G, \pi_1(\mathcal{X}))$, giving an isomorphism

$$\mathcal{V}(\pi_0((\mathcal{X})^G)/(x_1 \sim x_2)) \rightarrow H^1(G, \pi_1(\mathcal{Y})),$$

where $\pi_0((\mathcal{X})^G)/(x_1 \sim x_2)$ denotes the pointed set $\pi_0((\mathcal{X})^G)$ with x_1 identified to x_2 . Since $\pi_0((\mathcal{X})^G)/(x_1 \sim x_2)$ is identified with $\pi_0((\mathcal{Y})^G)$ via $\mathcal{X} \rightarrow \mathcal{Y}$, we have that the map

$$\mathcal{V}\pi_0((\mathcal{Y})^G) \rightarrow H^1(G, \pi_1(\mathcal{Y}))$$

induced by κ_y^{ab} is an isomorphism by functoriality of κ .

By induction on the order of D , it suffices to consider the three cases shown above. \square

3.14. Lemma. — Let \mathcal{X}_i be a topological space with an action of $G = \mathbb{Z}/2$ and let D_i be a finite non-empty set of points of \mathcal{X}_i for $i = 1, 2$, all satisfying hypothesis 3.12. Suppose given a G equivariant bijection $f : D_1 \rightarrow D_2$ and let \mathcal{Y} be the quotient of $\mathcal{X}_1 \amalg \mathcal{X}_2$ obtained by identifying points of D_1 with their images under f . Then $\kappa^{\text{ab}} : \pi_0(\mathcal{Y}^G) \rightarrow H^1(G, \pi_1(\mathcal{Y})^{\text{ab}})$ is the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(\mathcal{Y}^G)$ for any choice of G fixed base point.

Proof. Let τ denote the generator of G . Let H_i denote singular homology with coefficients in \mathbb{Z} .

Suppose D_1 has order 1, and let x denote the point of D_1 . Note that $\tau x = x$. We may assume that $x, f(x)$, and the image of x in \mathcal{Y} are the chosen base points of $\mathcal{X}_1, \mathcal{X}_2$, and \mathcal{Y} respectively. Then

$$\begin{aligned}\mathcal{V}\pi_0(\mathcal{Y}^G) &= \mathcal{V}\pi_0(\mathcal{X}_1^G) \oplus \mathcal{V}\pi_0(\mathcal{X}_2^G), \\ H_1(\mathcal{Y}) &= H_1(\mathcal{X}_1) \oplus H_1(\mathcal{X}_2),\end{aligned}$$

and $\kappa_{\mathcal{Y}}^{\text{ab}}$ can be identified with the direct sum of $\kappa_{\mathcal{X}_1}^{\text{ab}}$ and $\kappa_{\mathcal{X}_2}^{\text{ab}}$, showing the lemma in this case.

Suppose D_1 has order 2 and is of the form $D_1 = \{x, \tau x\}$. Let b_i be the base point of \mathcal{X}_i , and let γ denote a path from b_1 to b_2 . Note that the path $\tau\gamma^{-1} \circ \gamma$ from b_1 to itself determines a non-trivial element of $H_1(\mathcal{Y})$. This element splits the Mayer-Vietoris sequence

$$0 \rightarrow H_1(\mathcal{X}_1) \oplus H_1(\mathcal{X}_2) \rightarrow H_1(\mathcal{Y}) \rightarrow H_0(D_1) \rightarrow 0$$

giving an isomorphism

$$H_1(\mathcal{Y}) \cong H_1(\mathcal{X}_1) \oplus H_1(\mathcal{X}_2) \oplus \mathbb{Z}(\tau\gamma^{-1} \circ \gamma).$$

Let b_1 be the base point of \mathcal{Y} , giving an isomorphism

$$\mathcal{V}\pi_0(\mathcal{Y}^G) \cong \mathcal{V}\pi_0(\mathcal{X}_1^G) \oplus \mathcal{V}\pi_0(\mathcal{X}_2^G) \oplus \mathbb{Z}/2(b_2).$$

Under these identifications $\kappa_{\mathcal{Y}}^{\text{ab}}$ is the direct sum of $\kappa_{\mathcal{X}_1}^{\text{ab}}, \kappa_{\mathcal{X}_2}^{\text{ab}}$, and the isomorphism

$$\mathbb{Z}/2(b_2) \rightarrow H^1(G, \mathbb{Z}(\tau\gamma^{-1} \circ \gamma)) = \mathbb{Z}/2,$$

showing the lemma in this case.

By Lemma 3.13, it suffices to consider the two cases shown above. □

Proof of Proposition 3.1. For a free abelian group \mathcal{L} with an action of $\mathbb{Z}/2$, the map on group cohomology $H^i(\mathbb{Z}/2, \mathcal{L}) \rightarrow H^i(\mathbb{Z}/2, \mathcal{L}^\wedge)$ induced by the profinite completion $\mathcal{L} \rightarrow \mathcal{L}^\wedge$ is an isomorphism. (This can be checked with the cyclic resolution

$$(12) \quad \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\tau} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\tau} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\tau} \mathbb{Z}[\mathbb{Z}/2] \longrightarrow \mathbb{Z} \longrightarrow 0$$

of \mathbb{Z} as a $\mathbb{Z}/2$ module. Note that $\mathcal{L}^\wedge \cong \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}^\wedge$ and that \mathbb{Z}^\wedge is torsion free, whence flat over \mathbb{Z} .) We may therefore reduce to showing Proposition 3.1 for π the topological fundamental group.

We may assume that X is reduced. Every irreducible component of the normalization of X is a smooth curve with real points. By the above, Proposition 3.1 holds for each irreducible component of the normalization, after choosing a base point. By Lemma 3.13, Proposition 3.1 holds for each irreducible component of X . By Lemma 3.14, Proposition 3.1 holds for X . □

3.15. *Remark.* The hypothesis of Proposition 3.1 and the resulting hypothesis of Theorem 1.1 is different from the hypothesis of the real section conjecture, which is that the topological space $X(\mathbb{C})$ be a $K(\pi, 1)$. The reason for this difference was discussed in the introduction – see the discussion near (5). The difference itself is made explicit as follows. The normalization $\tilde{X} \rightarrow X$ induces a continuous map $\pi : \tilde{X}(\mathbb{C}) \rightarrow X(\mathbb{C})$ where $\tilde{X}(\mathbb{C})$

and $X(\mathbb{C})$ are given the analytic topology. The map π factors through the quotient map $q : \tilde{X}(\mathbb{C}) \rightarrow Y$ where Y is obtained by identifying points of $\tilde{X}(\mathbb{C})$ with equal images under π . The homeomorphism $Y \rightarrow X(\mathbb{C})$ shows that $X(\mathbb{C})$ is homotopy equivalent to the wedge of the connected components of $\tilde{X}(\mathbb{C})$ and a certain number of circles S^1 . Since a wedge of $K(\pi, 1)$'s is a $K(\pi, 1)$, one sees that $X(\mathbb{C})$ is a $K(\pi, 1)$ if and only if none of the connected components of $\tilde{X}(\mathbb{C})$ are $\mathbb{P}_{\mathbb{C}}^1$.

4. 2-NILPOTENT OBSTRUCTION

For a profinite group π , let $\pi = [\pi]_1 > [\pi]_2 > [\pi]_3 > \dots$ denote the lower central series of π , so $[\pi]_{n+1} = \overline{[[\pi]_n, \pi]}$ is the closure of the subgroup generated by commutators of elements of $[\pi]_n$ and π .

For an extension of profinite groups

$$(13) \quad 1 \rightarrow \pi \rightarrow \tilde{\pi} \rightarrow G \rightarrow 1,$$

the conjugacy class of a section $s : G \rightarrow \tilde{\pi}$ refers to the set of sections of the form

$$g \mapsto \gamma s(g) \gamma^{-1}$$

where γ is in π .

Pushing out (13) by $\pi \rightarrow \pi/[\pi]_n$ gives the extension

$$(14) \quad 1 \rightarrow \pi/[\pi]_n \rightarrow \tilde{\pi}/[\pi]_n \rightarrow G \rightarrow 1.$$

Given a section $s : G \rightarrow \tilde{\pi}/[\pi]_2$ of (14) for $n = 2$, there exists a section $\tilde{s} : G \rightarrow \tilde{\pi}/[\pi]_3$ of (14) for $n = 3$ such that the composition $G \xrightarrow{\tilde{s}} \tilde{\pi}/[\pi]_3 \rightarrow \tilde{\pi}/[\pi]_2$ is in the conjugacy class of s if and only if the extension

$$1 \rightarrow [\pi]_2/[\pi]_3 \rightarrow (\pi/[\pi]_3) \times_{\pi/[\pi]_2} G \rightarrow G \rightarrow 1$$

obtained by pulling back

$$(15) \quad 1 \rightarrow [\pi]_2/[\pi]_3 \rightarrow \pi/[\pi]_3 \rightarrow \pi/[\pi]_2 \rightarrow 1$$

along s splits. When (13) is equipped with a splitting, the extensions (14) inherit splittings, which induce bijections between $H^1(G, \pi/[\pi]_n)$ and the conjugacy classes of sections of (14). Then for s as above, there exists \tilde{s} if and only if the class of s vanishes under

$$\delta_2 : H^1(G, \pi/[\pi]_2) \rightarrow H^2(G, [\pi]_2/[\pi]_3)$$

where δ_2 is the boundary map in continuous group cohomology from the extension (15).

δ_2 is quadratic with associated bilinear form given by the following cup product [Zar74, p 242]: let

$$[-, -] : \pi/[\pi]_2 \otimes \pi/[\pi]_2 \rightarrow [\pi]_2/[\pi]_3$$

be given by $[\gamma, \delta] = \tilde{\gamma} \tilde{\delta} \tilde{\gamma}^{-1} \tilde{\delta}^{-1}$ where $\tilde{\gamma}, \tilde{\delta}$ in $\pi/[\pi]_3$ map to γ, δ , respectively, in $\pi/[\pi]_2$. (Since the choices of $\tilde{\gamma}$ differ by an element of the center, this map is well-defined on $\pi/[\pi]_2 \times \pi/[\pi]_2$. Bilinearity follows from [MKS04, Thm 5.1 p. 290].) The cup product

$$H^1(G, \pi/[\pi]_2) \otimes H^1(G, \pi/[\pi]_2) \rightarrow H^2(G, \pi/[\pi]_2 \otimes \pi/[\pi]_2)$$

can be pushed forward by $[-, -]$ to give a map

$$(16) \quad H^1(G, \pi/[\pi]_2) \otimes H^1(G, \pi/[\pi]_2) \rightarrow H^2(G, [\pi]_2/[\pi]_3).$$

4.1. Proposition (Zarkhin). — For all x, y in $H^1(G, \pi/[\pi]_2)$,

$$\delta_2(x + y) = \delta_2(x) + \delta_2(y) + [-, -]_* x \cup y.$$

For π the étale fundamental group of a smooth, based, algebraic curve over a field and G the absolute Galois group, Jordan Ellenberg introduced and studied δ_2 as an obstruction to rational points of the Jacobian lying on the curve [Ell00].

For an abelian group \mathcal{L} , let $\mathcal{L} \wedge \mathcal{L}$ denote the quotient of $\mathcal{L} \otimes \mathcal{L}$ by the relation $\ell \otimes \ell = 0$ for all ℓ in \mathcal{L} . When $G = \mathbb{Z}/2$, the pairing

$$\cup : H^1(G, \mathcal{L}) \otimes H^1(G, \mathcal{L}) \rightarrow H^2(G, \mathcal{L} \wedge \mathcal{L})$$

induced by the cup product satisfies

$$\ell \cup \ell = 0$$

by a straight forward computation, giving a natural map

$$H^1(G, \mathcal{L}) \wedge H^1(G, \mathcal{L}) \rightarrow H^2(G, \mathcal{L} \wedge \mathcal{L}).$$

4.2. Lemma. — Let \mathcal{L} be an abelian group or profinite abelian group with no 2-torsion and an action of $G = \mathbb{Z}/2$. Then the natural map

$$H^1(G, \mathcal{L}) \wedge H^1(G, \mathcal{L}) \rightarrow H^2(G, \mathcal{L} \wedge \mathcal{L})$$

induced by the cup product is injective.

Proof. Let τ denote the generator of G . Since G has order 2, $H^1(G, \mathcal{L})$ is a \mathbb{F}_2 vector space. The short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{2} \mathcal{L} \longrightarrow \mathcal{L}/2\mathcal{L} \longrightarrow 0$$

shows that

$$H^1(G, \mathcal{L}) \hookrightarrow H^1(G, \mathcal{L}/2\mathcal{L})$$

is injective. Similarly,

$$H^2(G, \mathcal{L} \wedge \mathcal{L}) \hookrightarrow H^2(G, \mathcal{L}/2 \wedge \mathcal{L}/2)$$

is injective. Thus, it suffices to show that

$$H^1(G, W) \wedge H^1(G, W) \rightarrow H^2(G, W \wedge W)$$

is injective for any \mathbb{F}_2 vector space W with a G action. This map is described by

$$[w_1] \wedge [w_2] \mapsto [w_1 \wedge \tau w_2]$$

where w_i is in the kernel of $\tau + 1 : W \rightarrow W$, and $[w_i]$ denotes the corresponding cohomology class via the cyclic G resolution of \mathbb{Z} (12) [Bro94, V §1 pg. 108].

Note that $(1 + \tau)(w_1 \wedge w_2) = w_1 \wedge w_2 + \tau w_1 \wedge \tau w_2 = (w_1 + \tau w_1) \wedge w_2 + \tau w_1 \wedge (w_2 + \tau w_2)$. Therefore, we have a map $H^2(G, W \wedge W) \rightarrow (W/I) \wedge (W/I)$ where I denotes the image of $\tau + 1 : W \rightarrow W$.

Let K denote the kernel of $\tau + 1 : W \rightarrow W$. Since the automorphisms $\tau + 1$ and $\tau - 1$ of W are equal, $H^1(G, W) \cong K/I$. The inclusion $K \hookrightarrow W$ induces an injection $K/I \wedge K/I \hookrightarrow W/I \wedge W/I$ because W is a vector space.

The commutative diagram

$$\begin{array}{ccc} H^1(G, W) \wedge H^1(G, W) & \xrightarrow{\cup} & H^2(G, W \wedge W) \\ \downarrow \cong & & \downarrow \\ K/I \wedge K/I & \longrightarrow & (W/I) \wedge (W/I) \end{array}$$

gives the desired injectivity. □

Note that $[-, -]$ factors through $\pi/[\pi]_2 \wedge \pi/[\pi]_2$ giving a map

$$[-, -] : \pi/[\pi]_2 \wedge \pi/[\pi]_2 \rightarrow [\pi]_2/[\pi]_3$$

which will also be denoted by $[-, -]$.

4.3. Lemma. — *Let X be a geometrically connected, based curve over \mathbb{R} and let π denote the étale or topological fundamental group of $X_{\mathbb{C}}, X(\mathbb{C})$ respectively. Then $[-, -]_* : H^2(G_{\mathbb{R}}, \pi/[\pi]_2 \wedge \pi/[\pi]_2) \rightarrow H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ is injective.*

Proof. For a free abelian group \mathcal{L} with an action of $\mathbb{Z}/2$, the map on group cohomology $H^i(\mathbb{Z}/2, \mathcal{L}) \rightarrow H^i(\mathbb{Z}/2, \mathcal{L}^{\wedge})$ induced by the profinite completion $\mathcal{L} \rightarrow \mathcal{L}^{\wedge}$ is an isomorphism (cf. proof of Proposition 3.1). Since $\pi/[\pi]_2$ and $[\pi]_2/[\pi]_3$ are free abelian groups for π the topological fundamental group of $X(\mathbb{C})$, it suffices to prove the lemma in this case.

By [Dwy75, Lemma 1.3], there is a right exact sequence

$$(17) \quad H_2(\pi) \rightarrow H_2(\pi/[\pi]_2) \rightarrow [\pi]_2/[\pi]_3 \rightarrow 1$$

for any group π , where H_2 denotes group homology with integer coefficients. When $\pi/[\pi]_2$ is a free \mathbb{Z} -module, $H_2(\pi/[\pi]_2)$ is canonically identified with $\pi/[\pi]_2 \wedge \pi/[\pi]_2$ and the surjection of (17) is $[-, -]$.

Since complex conjugation is orientation reversing, the canonical surjection $H_2(X(\mathbb{C})) \rightarrow H_2(\pi)$ [Bro94, Thm 5.2 p. 41] shows that $G_{\mathbb{R}}$ acts on $H_2(\pi)$ by multiplication by -1 . Thus $G_{\mathbb{R}}$ acts on the kernel K of $[-, -] : \pi/[\pi]_2 \wedge \pi/[\pi]_2 \rightarrow [\pi]_2/[\pi]_3$ by multiplication by -1 . Since $\pi/[\pi]_2 \wedge \pi/[\pi]_2$ is free, so is K . Thus $H_2(G_{\mathbb{R}}, K) = 0$, giving the desired injection. □

Let X be a geometrically connected, based curve over \mathbb{R} , such that each irreducible component of its normalization has \mathbb{R} -points, and use

$$1 \rightarrow \pi_1^{\text{ét}}(X_{\mathbb{C}}) \rightarrow \pi_1^{\text{ét}}(X) \rightarrow G_{\mathbb{R}} \rightarrow 1$$

or

$$1 \rightarrow \pi_1^{\text{top}}(X(\mathbb{C})) \rightarrow \pi_1^{\text{orb}}(X(\mathbb{C})) \rightarrow G_{\mathbb{R}} \rightarrow 1$$

for the extension (13), giving δ_2 as above.

4.4. Theorem. — $\text{Ker } \delta_2 = \text{Image } \kappa^{\text{ab}}$.

Proof. For any point p of $X(\mathbb{R})$, the image $\kappa(p)$ of p under κ determines a section of (13), whence a section of (14) for $n = 3$. Thus $\delta_2(\kappa^{\text{ab}}(p)) = 0$, giving the containment $\text{Image } \kappa^{\text{ab}} \subset \text{Ker } \delta_2$.

By Proposition 3.1, an arbitrary element of $H^1(G_{\mathbb{R}}, \pi^{\text{ab}})$ is of the form $x_1 + x_2 + \dots + x_n$ where the x_i are images under κ^{ab} of n distinct elements of $\pi_0(X(\mathbb{R}))$ none of which are the connected component distinguished by the base point. By Proposition 4.1,

$$\delta_2(x_1 + x_2 + \dots + x_n) = \sum_{i=1}^n \delta_2(x_i) + \sum_{1 \leq i < j \leq n} [-, -]_* x_i \cup x_j.$$

Since the x_i are images of elements of $X(\mathbb{R})$, $\delta_2(x_i) = 0$ for all i . If $n > 1$, then $\sum_{1 \leq i < j \leq n} x_i \wedge x_j$ is a non-zero element of $H^1(G_{\mathbb{R}}, \pi^{\text{ab}}) \wedge H^1(G_{\mathbb{R}}, \pi^{\text{ab}})$ by Proposition 3.1. By Lemmas 4.2 and 4.3, it follows that $\sum_{1 \leq i < j \leq n} [-, -]_* x_i \cup x_j$ is a non-zero element of $H^1(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ if $n > 1$. Thus $n = 1$ for any element of $\text{Ker } \delta_2$, showing $\text{Ker } \delta_2 \subset \text{Image } \kappa^{\text{ab}}$. \square

Since κ^{ab} is injective by Proposition 3.1, Theorem 4.4 implies Theorem 1.1.

5. BIRATIONAL 2-NILPOTENT REAL SECTION CONJECTURE

In this section, π_1 will always denote the étale fundamental group.

Let X be a smooth, proper, geometrically connected curve over \mathbb{R} such that $X(\mathbb{R}) \neq \emptyset$. Let $X(\mathbb{R})^{\pm}$ denote the set of real points of X equipped with a real tangent direction.

Choose a local parameter $z \in \mathbb{R}(X)$ at a point of $X(\mathbb{R})$, and let $b : \text{Spec } \mathbb{C}((z^{\mathbb{Q}})) \rightarrow X$ be the associated geometric point, as in 2.1. Note that z embeds the function field $\mathbb{R}(X)$ into $\mathbb{C}((z^{\mathbb{Q}}))$. Taking the algebraic closure Ω_z of $\mathbb{R}(X)$ in $\mathbb{C}((z^{\mathbb{Q}}))$ and considering the intermediary extension $\mathbb{R}(X) \subset \mathbb{C}(X)$ gives

$$(18) \quad 1 \rightarrow \text{Gal}(\Omega_z/\mathbb{C}(X)) \rightarrow \text{Gal}(\Omega_z/\mathbb{R}(X)) \rightarrow G_{\mathbb{R}} \rightarrow 1$$

which should be viewed as an analogue of the homotopy exact sequence (1). The coefficientwise action of $G_{\mathbb{R}}$ on $\mathbb{C}((z^{\mathbb{Q}}))$ defines a splitting of (18).

Given a second local parameter w and the associated geometric point

$$x : \text{Spec } \mathbb{C}((w^{\mathbb{Q}})) \rightarrow X,$$

choose an isomorphism $\Omega_z \cong \Omega_w$ which is the identity on the inclusion of $\mathbb{C}(X)$ in both fields, determining an isomorphism $\text{Gal}(\Omega_z/\mathbb{R}(X)) \cong \text{Gal}(\Omega_w/\mathbb{R}(X))$. The map $G_{\mathbb{R}} \rightarrow \text{Gal}(\Omega_w/\mathbb{R}(X))$ defined by the coefficientwise action of $G_{\mathbb{R}}$ on $\mathbb{C}((w^{\mathbb{Q}}))$ then gives a

second splitting of (18). It is not difficult to check that the conjugacy class of the section determining this splitting depends only on the tangent direction associated to x and is independent of the choice of isomorphism $\Omega_z \cong \Omega_w$.

It follows that given $b \in X(\mathbb{R})^\pm$ to be used as a base point, we obtain a map

$$\kappa : X(\mathbb{R})^\pm \rightarrow \mathcal{S}_{\text{Gal}(X/\mathbb{R})}$$

where $\mathcal{S}_{\text{Gal}(X/\mathbb{R})}$ denotes the set of conjugacy classes of sections of

$$1 \rightarrow G_{\mathbb{C}(X)} \rightarrow G_{\mathbb{R}(X)} \rightarrow G_{\mathbb{R}} \rightarrow 1.$$

The set of conjugacy classes of sections of the push-out sequence

$$(19) \quad 1 \rightarrow G_{\mathbb{C}(X)}/[G_{\mathbb{C}(X)}]_n \rightarrow G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_n \rightarrow G_{\mathbb{R}} \rightarrow 1$$

will be denoted by $\mathcal{S}_{\text{Gal}(X/\mathbb{R})}^n$, and for $m > k$ the set of conjugacy classes of those sections of (19) for $n = k$ which have a lift to a section of (19) for $n = m$ will be denoted $\mathcal{S}_{\text{Gal}(X/\mathbb{R})}^{k \leftarrow m}$.

Recall the notation that for a geometrically connected, based curve V over \mathbb{R} , the set of conjugacy classes of sections of the homotopy exact sequence

$$1 \rightarrow \pi_1(V_{\mathbb{C}}) \rightarrow \pi_1(V) \rightarrow G_{\mathbb{R}} \rightarrow 1$$

is denoted $\mathcal{S}_{\pi_1(V/\mathbb{R})}$. The conjugacy classes of sections of

$$(20) \quad 1 \rightarrow \pi_1(V_{\mathbb{C}})/[\pi_1(V_{\mathbb{C}})]_n \rightarrow \pi_1(V)/[\pi_1(V_{\mathbb{C}})]_n \rightarrow G_{\mathbb{R}} \rightarrow 1$$

is denoted $\mathcal{S}_{\pi_1(V/\mathbb{R})}^n$ and the conjugacy classes of those sections of (20) for $n = k$ which have a lift to a section of (20) for $n = m$ is denoted $\mathcal{S}_{\pi_1(V/\mathbb{R})}^{k \leftarrow m}$.

5.1. Corollary. — κ gives a natural bijection from $X(\mathbb{R})^\pm$ to $\mathcal{S}_{\text{Gal}(X/\mathbb{R})}^{2 \leftarrow 3}$.

Proof. For U a Zariski open of X , applying π_1 to the inclusion of the generic point gives a map $G_{\mathbb{R}(X)} \rightarrow \pi_1(U, b)$ [SGAI, V Proposition 8.1]. By functoriality of π_1 , for any $V \subset U \subset X$, we have compatible maps

$$\begin{aligned} \mathcal{S}_{\text{Gal}(X/\mathbb{R})}^n &\rightarrow \mathcal{S}_{\pi_1(V/\mathbb{R})}^n \rightarrow \mathcal{S}_{\pi_1(U/\mathbb{R})}^n \\ \mathcal{S}_{\text{Gal}(X/\mathbb{R})}^{n \leftarrow m} &\rightarrow \mathcal{S}_{\pi_1(V/\mathbb{R})}^{n \leftarrow m} \rightarrow \mathcal{S}_{\pi_1(U/\mathbb{R})}^{n \leftarrow m} \end{aligned}$$

for any n and $m > n$. These maps are compatible with κ in that the diagram

$$\begin{array}{ccc} X(\mathbb{R})^\pm & \xrightarrow{\quad} & \mathcal{S}_{\text{Gal}(X/\mathbb{R})}^{2 \leftarrow 3} \\ \downarrow & & \downarrow \\ \varprojlim_U \pi_0(U(\mathbb{R})) & \xrightarrow{\quad} & \varprojlim_U \mathcal{S}_{\pi_1(U/\mathbb{R})}^{2 \leftarrow 3} \end{array}$$

commutes, where $X(\mathbb{R})^\pm \rightarrow \pi_0(U(\mathbb{R}))$ sends an element of $X(\mathbb{R})^\pm$ to the connected component the tangent direction points to, and where U runs over the Zariski opens of X .

By Theorem 1.1, $\kappa : \pi_0(U(\mathbb{R})) \rightarrow \mathcal{S}_{\pi_1(U/\mathbb{R})}^{2 \leftarrow 3}$ is a bijection, showing that the bottom horizontal arrow is a bijection. The left vertical arrow is a bijection by inspection. It thus

suffices to show that $\mathcal{S}_{\text{Gal}(X/\mathbb{R})}^2 \rightarrow \varprojlim_{\mathcal{U}} \mathcal{S}_{\pi_1(\mathcal{U}/\mathbb{R})}^2$ is injective, which is equivalent to showing that $H^1(G_{\mathbb{R}}, G_{\mathbb{C}(X)}^{\text{ab}}) \rightarrow \varprojlim_{\mathcal{U}} H^1(G_{\mathbb{R}}, \pi_1(\mathcal{U}_{\mathbb{C}})^{\text{ab}})$ is injective [Bro94, IV 2.3].

By [SGAI, V Proposition 8.2] the natural map $G_{\mathbb{C}(X)} \rightarrow \varprojlim_{\mathcal{U}} \pi_1(\mathcal{U}_{\mathbb{C}}, \mathfrak{b})$ is an isomorphism. It follows that $G_{\mathbb{C}(X)}^{\text{ab}} \rightarrow \varprojlim_{\mathcal{U}} \pi_1(\mathcal{U}_{\mathbb{C}}, \mathfrak{b})^{\text{ab}}$ is an isomorphism. Since \varprojlim is exact as a functor on inverse systems of compact abelian groups, the natural map

$$H^1(G_{\mathbb{R}}, \varprojlim_{\mathcal{U}} \pi_1(\mathcal{U}_{\mathbb{C}})^{\text{ab}}) \rightarrow \varprojlim_{\mathcal{U}} H^1(G_{\mathbb{R}}, \pi_1(\mathcal{U}_{\mathbb{C}})^{\text{ab}})$$

is an isomorphism, showing the corollary. \square

6. 2-NILPOTENT TOPOLOGICAL APPROXIMATION

Let X be a geometrically connected curve over \mathbb{R} equipped with a base point in $X(\mathbb{R})$, and let π denote the topological fundamental group of $X(\mathbb{C})$.

6.1. Abelian approximation. The action of $\pi/[\pi]_2$ on itself by left translation gives an injective homomorphism

$$\pi/[\pi]_2 \rightarrow \text{Affine}(\pi/[\pi]_2),$$

where $\text{Affine}(\pi/[\pi]_2)$ denotes the group of invertible affine transformations of the free abelian group $\pi/[\pi]_2$. (The image of $\pi/[\pi]_2$ is contained in the subgroup of translations, but the notation is set up for the larger group.) Tensoring with \mathbb{R} gives an action of $\pi/[\pi]_2$ on $\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$, and taking the quotient gives a model for $K(\pi/[\pi]_2, 1)$ denoted

$$\text{Alb}_1 = (\pi/[\pi]_2) \backslash (\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}).$$

The Galois group $G_{\mathbb{R}}$ acts on $\pi/[\pi]_2$, giving a linear action on $\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$. For $g \in G_{\mathbb{R}}$ and $\gamma \in \pi/[\pi]_2$, we have an element $g\gamma$ of $\pi/[\pi]_2$, and the equality $g(\gamma v) = (g\gamma)(gv)$ for all $v \in \pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$. Thus Alb_1 inherits a $G_{\mathbb{R}}$ -action.

6.2. 2-Nilpotent approximation. Choose a set of generators x_1, \dots, x_n for π which are a basis for $\pi/[\pi]_2$. Let s be the set-theoretic section of the quotient map $q : \pi/[\pi]_3 \rightarrow \pi/[\pi]_2$ given by taking $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in $\pi/[\pi]_2$ to $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in $\pi/[\pi]_3$. The section s determines a bijection between $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ and $\pi/[\pi]_3$ by sending $v \oplus z \in \pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ to $s(v)z$. Via this bijection, the group law on $\pi/[\pi]_3$ gives a composition law on $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$, denoted \circ . Let $+$ denote addition on the free abelian group $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$. There is a bilinear pairing

$$\langle -, - \rangle : \pi/[\pi]_2 \oplus \pi/[\pi]_2 \rightarrow [\pi]_2/[\pi]_3$$

such that

$$v \circ w = v + w + \langle qv, qw \rangle,$$

for v and w in $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$. Here and later, we slightly abuse the notation q by letting q also denote the projection $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3 \rightarrow \pi/[\pi]_2$. Thus the action of $\pi/[\pi]_3$ on itself by left translation gives an injective homomorphism

$$\pi/[\pi]_3 \rightarrow \text{Affine}(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \subset \text{Affine}((\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}).$$

For example, x_2 is sent to the affine transformation

$$(a_1, a_2, \dots, a_n) \times z \mapsto (a_1, a_2 + 1, \dots, a_n) \times (z - a_1[x_1, x_2]).$$

Taking the quotient of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ by $\pi/[\pi]_3$ gives a model for $K(\pi/[\pi]_3, 1)$, denoted

$$\text{Alb}_2 = (\pi/[\pi]_3) \setminus ((\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}).$$

$G_{\mathbb{R}}$ acts on $\pi/[\pi]_3$. Give $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ the $G_{\mathbb{R}}$ -action inherited from its bijection with $\pi/[\pi]_3$. Since for all $z \in 0 \oplus [\pi]_2/[\pi]_3$ we have $z \circ w = w \circ z = w + z$, it follows that

$$g(v \circ w) = g((v + w) \circ \langle qv, qw \rangle) = g(v + w) \circ g\langle qv, qw \rangle = g(v + w) + g\langle qv, qw \rangle,$$

$$(gv) \circ (gw) = (gv) + (gw) + \langle qgv, qgw \rangle.$$

Since $g(v \circ w) = (gv) \circ (gw)$ for $g \in G_{\mathbb{R}}$, we have that

$$g(w + v) - g(w) - g(v) = \langle gqv, gqw \rangle - g\langle qv, qw \rangle$$

is bilinear. Let $B_g(v, w) = \langle gqv, gqw \rangle - g\langle qv, qw \rangle$ denote this bilinear form. It follows that

$$(21) \quad g(v) = \frac{1}{2}B_g(v, v) + L_g(v)$$

where L_g is a linear endomorphism of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$. In particular, the $G_{\mathbb{R}}$ action on $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ extends to $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$. As above, the equality $g(\gamma v) = (g\gamma)(gv)$ for all $v \in (\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$, $g \in G_{\mathbb{R}}$ and $\gamma \in \pi/[\pi]_3$, implies that $G_{\mathbb{R}}$ acts on Alb_2 .

The map $\text{Alb}_2 \rightarrow \text{Alb}_1$ induced by the projection q is a $G_{\mathbb{R}}$ -equivariant fiber bundle with fiber $K([\pi]_2/[\pi]_3, 1)$.

6.3. Mapping $X(\mathbb{C})$ to its approximations. Equip Alb_2 with the base point induced by the origin of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$, and say that a map between spaces with base points is pointed if the image of the base point of the domain is the base point of the codomain. Choose a pointed map $f : X(\mathbb{C}) \rightarrow \text{Alb}_2$ such that the induced map f_* on fundamental groups is the quotient. Let $\widetilde{X}(\mathbb{C}) \rightarrow X(\mathbb{C})$ and $\widetilde{\text{Alb}}_2 \rightarrow \text{Alb}_2$ denote the universal pointed covering maps of $X(\mathbb{C})$ and Alb_2 respectively. Note that $\widetilde{\text{Alb}}_2 \rightarrow \text{Alb}_2$ is $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Alb}_2$, up to unique pointed isomorphism, where the origin is the base point of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$. There is a unique lift of f to a pointed map $\tilde{f} : \widetilde{X}(\mathbb{C}) \rightarrow \widetilde{\text{Alb}}_2$. For all $\gamma \in \pi$, we can view γ as an automorphism of $\widetilde{X}(\mathbb{C})$ or $\widetilde{\text{Alb}}_2$, and the choice of f_* implies that $\gamma\tilde{f} = \tilde{f}\gamma$. Give $\widetilde{X}(\mathbb{C})$ and $\widetilde{\text{Alb}}_2$ the $G_{\mathbb{R}}$ -actions lifting those of $X(\mathbb{C})$ and Alb_2 and such that the base points are fixed under $G_{\mathbb{R}}$. This $G_{\mathbb{R}}$ -action on $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ is consistent with the one above. Let τ denote complex conjugation and note that $\tau f \tau^{-1} : X(\mathbb{C}) \rightarrow \text{Alb}_2$ is a pointed map such that the induced map on π_1 is the quotient. $\tau\tilde{f}\tau^{-1}$ is the unique lift of $\tau f \tau^{-1}$ to a pointed map between the pointed universal covering spaces, and $\gamma\tau\tilde{f}\tau^{-1} = \tau\tilde{f}\tau^{-1}\gamma$.

Define $\tilde{g} : \widetilde{X(\mathbb{C})} \rightarrow (\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R} = \widetilde{\text{Alb}}_2$ by

$$\tilde{g} = \frac{1}{2}\tilde{f} + \frac{1}{2}\tau\tilde{f}\tau^{-1}.$$

By the above, γ acts by affine transformations on $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$, which implies that $\gamma\tilde{g} = \frac{1}{2}\gamma\tilde{f} + \frac{1}{2}\gamma\tau\tilde{f}\tau^{-1} = \frac{1}{2}\tilde{f}\gamma + \frac{1}{2}\tau\tilde{f}\tau^{-1}\gamma = \tilde{g}\gamma$. Thus, \tilde{g} induces a pointed map $g : X(\mathbb{C}) \rightarrow \text{Alb}_2$ such that g_* the quotient map on fundamental groups. Since τ acts linearly on the universal cover $\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$ of Alb_1 , we have that $q\tilde{g} : \widetilde{X(\mathbb{C})} \rightarrow \pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$ is $G_{\mathbb{R}}$ -equivariant. Thus for any \tilde{x} in $\widetilde{X(\mathbb{C})}$, the points \tilde{g} and $\tau\tilde{g}\tau^{-1}$ are contained in the same fiber of q . The calculation (21) implies that τ determines an affine transformation between the fibers of $q : \widetilde{\text{Alb}}_2 \rightarrow \widetilde{\text{Alb}}_1$ over $q\tilde{g}(\tilde{x})$ and $\tau q\tilde{g}(\tilde{x}) = q\tilde{g}(\tau\tilde{x})$.

Define $\tilde{h} : \widetilde{X(\mathbb{C})} \rightarrow \widetilde{\text{Alb}}_2$ by $\tilde{h} = \frac{1}{2}\tilde{g} + \frac{1}{2}\tau\tilde{g}\tau^{-1}$. As above, \tilde{h} induces a pointed map $h : X(\mathbb{C}) \rightarrow \text{Alb}_2$ such that h_* the quotient map on fundamental groups. Furthermore,

$$\tau\tilde{h}\tau^{-1} = \tau\left(\frac{1}{2}\tilde{g}\tau^{-1} + \frac{1}{2}\tau\tilde{g}\right) = \frac{1}{2}\tau\tilde{g}\tau^{-1} + \frac{1}{2}\tilde{g} = \tilde{h}$$

where the second to last equality follows because τ is affine on the fiber over $q\tilde{g}(\tau^{-1}\tilde{x})$ for all \tilde{x} in $\widetilde{X(\mathbb{C})}$. Thus \tilde{h} and h and $G_{\mathbb{R}}$ -equivariant. Use the notation α_2 for h , so $\alpha_2 = h$. Let $\alpha : X(\mathbb{C}) \rightarrow \text{Alb}_1$ denote the composition of α_2 with the projection. The notation is chosen to recall that for the Abel-Jacobi map.

6.4. *Remark.* $X(\mathbb{C})$, Alb_1 , and Alb_2 can be given the structure of finite $G_{\mathbb{R}}$ -CW complexes in the sense of [Bre67]; by the main theorem of [Ill78], a smooth compact manifold with a $\mathbb{Z}/2$ action can be given such a structure, showing the existence of finite $G_{\mathbb{R}}$ -CW complex structures for Alb_1 , Alb_2 , and the complex points of the smooth compactification of the normalization of X . Removing and or identifying finitely many points of a finite $G_{\mathbb{R}}$ -CW complex yields a finite $G_{\mathbb{R}}$ -CW complex, so $X(\mathbb{C})$ also has the structure of a finite $G_{\mathbb{R}}$ -CW complex.

We obtain the commutative diagram of $G_{\mathbb{R}}$ -equivariant maps between finite $G_{\mathbb{R}}$ -CW complexes

$$(22) \quad \begin{array}{ccc} & & \text{Alb}_2 \\ & \nearrow \alpha_2 & \downarrow \\ X(\mathbb{C}) & \xrightarrow{\alpha} & \text{Alb}_1 \end{array}$$

and view Alb_1 as an abelian approximation to X and Alb_2 as a 2-nilpotent approximation to X . For X smooth and proper, integration gives a natural map from $\pi/[\pi]_2$ to the \mathbb{C} -linear dual of the global holomorphic one-forms on X , denoted $H^0(X, \Omega)^*$. This map extends to a $G_{\mathbb{R}}$ -equivariant \mathbb{R} -linear isomorphism

$$\widetilde{\text{Alb}}_1 = \pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H^0(X, \Omega)^*$$

which identifies Alb_1 with the $G_{\mathbb{R}}$ -topological space underlying the complex points of the Albanese variety of X . Thus Alb_1 in (22) can be replaced with $\text{Jac } X(\mathbb{C})$. The notation Alb comes from the view point that a 2-nilpotent approximation to X is analogous to a higher Albanese variety. See [Hai87], [HZ87].

Theorem 1.1 can be rephrased as the statement that the connected components of real points of the curve are those of the Albanese which can be lifted to the 2-nilpotent approximation:

6.5. Theorem. — *Let X be a geometrically connected, based curve over \mathbb{R} , such that each irreducible component of its normalization has \mathbb{R} -points. Let Alb_1 , Alb_2 , and α be as constructed above to obtain (22). Then α induces a bijection from $\pi_0(X(\mathbb{R}))$ to the image of $\pi_0(\text{Alb}_2^{\text{G}\mathbb{R}}) \rightarrow \pi_0(\text{Alb}_1^{\text{G}\mathbb{R}})$.*

Proof. By Remark 6.4, Alb_1 is a finite $\text{G}_{\mathbb{R}}$ -CW complexes. By [Car91, Thm B(a)], the natural map $\pi_0(\text{Alb}_1^{\text{G}\mathbb{R}}) \rightarrow \pi_0(\text{Alb}_1^{\text{hG}\mathbb{R}})$ is a bijection. Since Alb_1 is a $K(\pi/[\pi]_2, 1)$, the natural map $\pi_0(\text{Alb}_1^{\text{hG}\mathbb{R}}) \rightarrow \mathcal{S}_{\pi_1(\text{Alb}_1/\text{G}\mathbb{R})}$ is a bijection. Under these bijections, α is identified with κ^{ab} . The same reasoning applied to Alb_2 identifies the image of $\pi_0(\text{Alb}_2^{\text{G}\mathbb{R}}) \rightarrow \pi_0(\text{Alb}_1^{\text{G}\mathbb{R}})$ with the image of $\mathcal{S}_{\pi_1(\text{Alb}_2/\text{G}\mathbb{R})} \rightarrow \mathcal{S}_{\pi_1(\text{Alb}_1/\text{G}\mathbb{R})}$. This shows that Theorem 6.5 and Theorem 1.1 are equivalent. \square

6.6. Example. Let $X = \mathbb{P}_{\mathbb{R}}^1 - \{0, 1, \infty\}$ equipped with a real base point b in $(0, 1)$, say $b = \frac{1}{2}$. The fundamental group π is freely generated by x_1 and x_2 , where x_1 is represented by the loop $t \mapsto e^{2\pi it}/2$ for $t \in [0, 1]$ and x_2 is the image of x_1 under the automorphism of X given by $z \mapsto 1 - z$.

The set $\{x_1, x_2, [x_1, x_2]\}$ is a basis for $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$, so points of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ can be labeled $(a_1, a_2, a_{12}) \in \mathbb{R}^3$ as an abbreviation for $a_1x_1 + a_2x_2 + a_{12}[x_1, x_2]$. The action of $\pi/[\pi]_3$ on $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ is given by $x_1(a_1, a_2, a_{12}) = (a_1 + 1, a_2, a_{12})$ and $x_2(a_1, a_2, a_{12}) = (a_1, a_2 + 1, a_{12} - a_1)$. Note that $[x_1, x_2](a_1, a_2, a_{12}) = (a_1, a_2, a_{12} + 1)$ as well. It follows that Alb_3 is the quotient of the unit cube in \mathbb{R}^3 given by identifying the $a_1 = 0$ face with the $a_1 = 1$ face via the translation x_1 , identifying the $a_{12} = 0$ face with the $a_{12} = 1$ face via the translation $[x_1, x_2]$, and identifying the $a_2 = 0$ face with the $a_2 = 1$ face via the translation-shear x_2 . The $\text{G}_{\mathbb{R}}$ -action on Alb_3 is given by $(a_1, a_2, a_{12}) \mapsto (-a_1, -a_2, a_{12})$.

Alb_2 is the torus given as the quotient of the unit square in $(\pi/[\pi]_2) \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the basis $\{x_1, x_2\}$ by the translations $(a_1, a_2) \mapsto (a_1 + 1, a_2)$ and $(a_1, a_2) \mapsto (a_1, a_2 + 1)$, and $\text{G}_{\mathbb{R}}$ acts by multiplication by -1 .

X deformation retracts $\text{G}_{\mathbb{R}}$ -equivariantly onto the union of the two circles which are the images of the representative loops for x_1 and x_2 described above. α_2 is the map taking the point $e^{2\pi it}/2$ in the image of x_1 to tx_1 in Alb_3 and similarly for the points in the image of x_2 .

Note that there are four $\text{G}_{\mathbb{R}}$ fixed points of Alb_2 given by the 2-torsion points

$$\{(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$$

of Alb_2 , and that the first three lift to fixed points of Alb_3 as they constitute the image of the fixed points of X in Alb_2 . The fourth point $(\frac{1}{2}, \frac{1}{2})$ does not lift to a fixed point of Alb_3 as $\text{G}_{\mathbb{R}}$ acts on the fiber above $(\frac{1}{2}, \frac{1}{2})$ by translation by $(0, 0, \frac{1}{2})$. The fact that $(\frac{1}{2}, \frac{1}{2})$ does not lift

can also be seen by applying Theorem 6.5. Example 6.6 is illustrated with figure 1. The fixed points are shown in red.

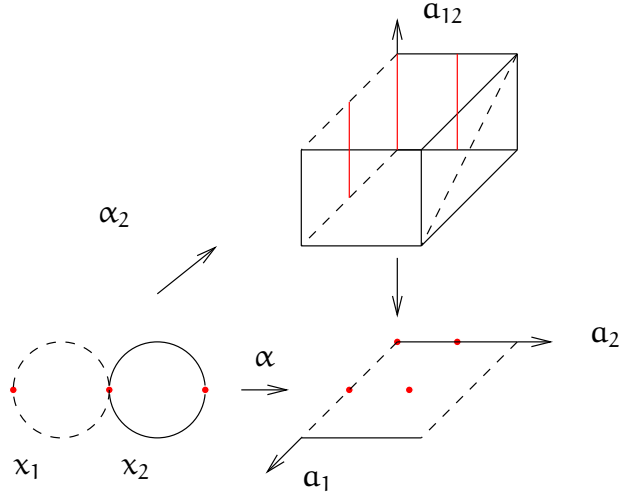


FIGURE 1. Approximations of $\mathbb{P}_{\mathbb{R}}^1 - \{0, 1, \infty\}$

APPENDIX A. GENERALIZED JACOBIANS

Let k be a field and let X be a geometrically integral curve over k such that X is not proper. There is an initial object $X \hookrightarrow X^+$ among all open immersions $X \hookrightarrow X'$ from X to a proper curve X' such that $X' - X$ is a single k -point. To see this: let Y denote the normalization of X , and Y^c denote the smooth compactification of Y . The curve $Y_{\bar{k}}^+$ can be described as the singular curve associated to the “modulus” $\sum_{p \in (Y^c - Y)(\bar{k})} p$ on $Y_{\bar{k}}^c$ by [Ser88, IV §1]. The $G_{\bar{k}}$ -action on the push-forward of the structure sheaf of $Y_{\bar{k}}^c$ to $Y_{\bar{k}}^+$ preserves the subsheaf given by the structure sheaf of $Y_{\bar{k}}^+$, and it follows that the effective descent datum for $Y_{\bar{k}}^c$ determines an effective descent datum on $Y_{\bar{k}}^+$. This descent datum determines a k -curve, and it can be checked that this curve is Y^+ . Let S_X denote the singular locus of X and let S_Y denote the inverse image of S_X in Y . The k -curve X^+ is described by gluing $Y^+ - S_Y$ to X along the open subscheme $X - S_X \cong Y - S_Y$. Call $X \hookrightarrow X^+$ the *one point compactification*. For $k = \mathbb{R}$, the complex points of $X_{\mathbb{C}}^+$ equipped with the analytic topology form the one point compactification of $X_{\mathbb{C}}(\mathbb{C})$ equipped with the analytic topology [Mun00, Thm 29.1].

Let Y be a geometrically integral, proper curve over k equipped with a k -point b . For instance, Y could be X^+ because ∞ has residue field k . The Picard functor of Y is represented by $\text{Pic}_Y = \coprod \text{Pic}_Y^n$, where Pic_Y^n represents the open and closed subfunctor of degree n line bundles. Pic_Y^n is quasi-projective, separated, connected and smooth [BLR90, 9.3 Thm 1]. Let $p : Y \times \text{Pic}_Y \rightarrow \text{Pic}_Y$ and $p' : Y \times \text{Pic}_Y \rightarrow Y$ denote the projections. By [BLR90, 8.1 Prop. 4], we have a unique universal line bundle \mathcal{P} over $Y \times \text{Pic}_Y$ whose restriction to $b \times \text{Pic}_Y$ is trivial.

A.1. Lemma. — For a coherent sheaf \mathcal{M} on Y , let \mathcal{G} be the coherent sheaf $(p')^* \mathcal{M} \otimes \mathcal{P}$ on $Y \times \text{Pic}_Y$. Then:

- (1) \mathcal{G} is flat over Pic_Y .
- (2) For large n , the restriction of $R^i p_* \mathcal{G}$ to Pic_Y^n is 0 for all $i > 0$, and locally free for $i = 0$.
- (3) For large n , the restriction of \mathcal{G} to $Y \times \text{Pic}_Y^n$ is cohomologically flat over Pic_Y^n in all dimensions.

Proof. (1): Near any point of $Y \times \text{Pic}_Y$, note that \mathcal{G} is isomorphic to $(p')^* \mathcal{M}$, which is flat because \mathcal{M} is flat over $\text{Spec } k$.

The $i > 0$ case of (2): Since Y is projective, there is a relatively very ample invertible sheaf \mathcal{L} for $p : Y \times \text{Pic}_Y \rightarrow \text{Pic}_Y$. Let $\mathcal{L}^{\otimes n}$ denote the n -fold tensor product of \mathcal{L} . Let \mathcal{G}^k denote the restriction of \mathcal{G} to $Y \times \text{Pic}_Y^k$. For a fixed k , there is N such that for $n > N$, we have $R^i p_*(\mathcal{G}^k \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ by [EGAIII₂, Thm 2.2.1]. The invertible sheaf $\mathcal{P} \otimes \mathcal{L}^{\otimes m}$ induces an isomorphism $t : \text{Pic}_Y^k \rightarrow \text{Pic}_Y^{k+md}$ such that $t^* \mathcal{G}^{k+md} = \mathcal{G}^k \otimes \mathcal{L}^{\otimes m}$, where d denotes the degree of \mathcal{L} . Since $t^* R^i p_*(\mathcal{G}^{k+md}) = R^i p_*(t^* \mathcal{G}^{k+md})$, we have that $R^i p_* \mathcal{G}^{k+md} = 0$ for a fixed k , and all m sufficiently large and $i > 0$. Taking $k = 0, 1, \dots, d-1$ shows that for m sufficiently large we have $R^i p_* \mathcal{G}^m = 0$ for all $i > 0$.

(3): By the $i > 0$ case of (2), the restriction to $Y \times \text{Pic}_Y^n$ of $R^i p_*(\mathcal{G})$ for n sufficiently large is locally free for all $i \geq 1$. By [EGAIII₂, Prop 7.8.5] it follows that this restriction of \mathcal{G} is cohomologically flat in dimensions $i \geq 1$. For a point z of Pic_Y , let \mathcal{G}_z denote the pullback of \mathcal{G} by the closed immersion $Y_{k(z)} = Y \times \text{Spec } k(z) \rightarrow Y \times \text{Pic}_Y^n$ corresponding to z . By [EGAIII₂, Prop 7.8.4], it follows that for a fixed $i \geq 1$, the function $z \mapsto d_i(z) = \dim_{k(z)} H^i(Y_{k(z)}, \mathcal{G}_z)$ on the points of Pic_Y^n for n sufficiently large is locally constant. Since the Euler characteristic of \mathcal{G}_z is locally constant [EGAIII₂, Thm 7.9.4], it follows that d_0 is locally constant when restricted to Pic_Y^n for n sufficiently large. Since Pic_Y^n is reduced, (3) follows from [EGAIII₂, Prop 7.8.4].

The $i = 0$ case of (2), follows from (3) and [EGAIII₂, 7.8.5]. □

Now suppose that X is a smooth, geometrically connected curve over k such that X is not proper.

The closed subscheme $\text{Spec } k \rightarrow X^+$ corresponding to ∞ gives the short exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_\infty \rightarrow \mathcal{O}_{X^+} \rightarrow \mathcal{O}_\infty \rightarrow 0.$$

Applying $(p')^*$ and tensoring with \mathcal{P} yields the short exact sequence

$$(23) \quad 0 \rightarrow \mathcal{P} \otimes (p')^* \mathcal{I}_\infty \rightarrow \mathcal{P} \rightarrow \mathcal{P} \otimes (p')^* \mathcal{O}_\infty \rightarrow 0$$

since \mathcal{P} is flat over Pic_{X^+} . Let $\iota : \text{Pic}_{X^+} \rightarrow X^+ \times \text{Pic}_{X^+}$ be the pullback of $\text{Spec } k \rightarrow X^+$ to Pic_{X^+} . Note that $\iota_* \mathcal{O}_{\text{Pic}_{X^+}} \cong (p')^* \mathcal{O}_\infty$, whence $\iota_* \iota^* \mathcal{P} \cong \mathcal{P} \otimes (p')^* \mathcal{O}_\infty$ by the projection formula.

By [BLR90, 8.1 Thm 7] and Lemma A.1 (1), there are coherent sheaves \mathcal{F} and \mathcal{N} on Pic_{X^+} with functorial isomorphisms

$$\begin{aligned} p_*(\mathcal{P} \otimes_{\mathcal{O}_{\text{Pic}_{X^+}}} \mathcal{M}) &\cong \mathcal{H}om(\mathcal{F}, \mathcal{M}) \\ p_*(\iota_* \iota^* \mathcal{P} \otimes_{\mathcal{O}_{\text{Pic}_{X^+}}} \mathcal{M}) &\cong \mathcal{H}om(\mathcal{N}, \mathcal{M}) \end{aligned}$$

for all quasi-coherent sheaves \mathcal{M} on Pic_{X^+} . For large n , the restrictions to $\text{Pic}_{X^+}^n$ of \mathcal{F} and \mathcal{N} are locally free and dual to $p_*\mathcal{P}$ and $p_*\iota_*\iota^*\mathcal{P} \cong \iota^*\mathcal{P}$ respectively by [BLR90, 8.1 Thm 7] and Lemma A.1 (3).

The surjection $\mathcal{P} \rightarrow \iota_*\iota^*\mathcal{P}$ in (23) gives a natural map $\mathcal{N} \rightarrow \mathcal{F}$. Applying p_* to (23) and restricting to $\text{Pic}_{X^+}^n$ for large n gives a surjection $p_*\mathcal{P} \rightarrow \iota^*\mathcal{P}$ by Lemma A.1 (2). It follows that we have a split short exact sequence of locally free sheaves

$$(24) \quad 0 \rightarrow \mathcal{N} \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0$$

on $\text{Pic}_{X^+}^n$ for large n . Note that since \mathcal{F} is locally free of high rank for large n , \mathcal{E} is non-zero for large n .

Let Div_{X^+} denote the functor taking a locally Noetherian scheme T over k to the closed subschemes D of $X^+ \times T$, flat over T , and with invertible ideal sheaf \mathcal{I} (see [BLR90, 8.2 p 212]). The association $D \mapsto \mathcal{I}^{-1}$ for $D \in \text{Div}_{X^+}(T)$ determines a map

$$(25) \quad \text{Div}_{X^+} \rightarrow \text{Pic}_{X^+}.$$

Let $\text{Div}_{X^+, \infty}$ be the functor taking T to the subset of $\text{Div}_{X^+}(T)$ consisting of those closed subschemes containing $\infty \times T$. Define $\text{Div}_{X^+}^n = \text{Div}_{X^+} \times_{\text{Pic}_{X^+}} \text{Pic}_{X^+}^n$ and similarly for $\text{Div}_{X^+, \infty}^n$.

For a coherent sheaf \mathcal{M} , let $V(\mathcal{M}) = \underline{\text{Spec}} \text{Sym } \mathcal{M}$ and $\mathbb{P}(\mathcal{M}) = \underline{\text{Proj}} \text{Sym } \mathcal{M}$, as in [EGAII, 1.7.8] and [EGAII, 3.1.3].

A.2. Proposition. — $\text{Div}_{X^+, \infty} \hookrightarrow \text{Div}_{X^+}$ is represented by the closed immersion $\mathbb{P}(\text{Coker}(\mathcal{N} \rightarrow \mathcal{F})) \hookrightarrow \mathbb{P}(\mathcal{F})$.

Proof. Let \mathcal{E} denote $\text{Coker}(\mathcal{N} \rightarrow \mathcal{F})$. For large n , this is consistent with the definition given in (24). It is sufficient to show the claim after restricting $\text{Div}_{X^+, \infty}$ and Div_{X^+} to functors over Pic_{X^+} . Let $T \rightarrow \text{Pic}_{X^+}$ be a scheme over Pic_{X^+} . Let \mathcal{P}_T denote the pullback of \mathcal{P} to $X^+ \times T$, and let \mathcal{N}_T , \mathcal{F}_T , and \mathcal{E}_T denote the pullbacks to T of \mathcal{N} , \mathcal{F} , and \mathcal{E} respectively. We will let p , p' (resp. ι) also denote their pullbacks to $X^+ \times T$ (resp. T). For a point $t : \text{Spec } k(t) \rightarrow T$ of T , let $X_{k(t)}^+$ be defined $X_{k(t)}^+ = (X^+ \times T) \times_T \text{Spec } k(t)$. For an invertible sheaf \mathcal{L} , let \mathcal{L}^{-1} be the dual invertible sheaf $\mathcal{L}^{-1} = \text{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.

A section $s : \mathcal{O}_{X^+ \times T} \rightarrow \mathcal{P}_T$ induces a map

$$s^{-1} : \mathcal{P}_T^{-1} \rightarrow \mathcal{O}_{X^+ \times T}^{-1} = \mathcal{O}_{X^+ \times T}$$

which is an injection such that the corresponding closed subscheme D is flat over T if and only if the restriction of s^{-1} to $X_{k(t)}^+$ is injective for all t by [EGAIV₃, Prop. 11.3.7]. Since $X_{k(t)}^+$ is reduced and irreducible, the restriction of s^{-1} to $X_{k(t)}^+$ is injective if and only if it is non-zero. By the definition of \mathcal{F} , the set of such s is in natural bijection with morphisms $\mathcal{F}_T \rightarrow \mathcal{O}_T$ which are non-zero on all stalks. By Nakayama's lemma, a morphism $\mathcal{F}_T \rightarrow \mathcal{O}_T$ is non-zero on all stalks if and only if it is surjective.

D contains $\infty \times T$ if and only if the image of s^{-1} is contained in the ideal sheaf $(p')^*\mathcal{I}_\infty$ of $\infty \times T$. This occurs if and only if the image of s is contained in $\mathcal{P}_T \otimes (p')^*\mathcal{I}_\infty$, which by

(23) occurs if and only if the composition of s with

$$\mathcal{P}_T \rightarrow \iota_* \iota^* \mathcal{P}_T$$

is 0. Thus, D contains $\infty \times T$ if and only if the morphism $\mathcal{F}_T \rightarrow \mathcal{O}_T$ corresponding to s is pulled back from a morphism $\mathcal{E}_T \rightarrow \mathcal{O}_T$.

It follows that the association $s \mapsto D$ induces a bijection between the set of elements D of $\text{Div}_{X^+, \infty}$ equipped with an isomorphism $\mathcal{I} \rightarrow \mathcal{P}_T^{-1}$, where \mathcal{I} denotes the ideal sheaf of D , and surjections $\mathcal{E}_T \rightarrow \mathcal{O}_T$. Let $\text{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\text{surj}} \subset \text{Hom}(\mathcal{E}_T, \mathcal{O}_T)$ denote the subsheaf of surjections. As the automorphisms of \mathcal{P}_T^{-1} are canonically isomorphic to the global sections $\Gamma(X^+ \times T, \mathcal{O}_{X^+ \times T}^*) = \Gamma(T, \mathcal{O}_T^*)$, the association $s \mapsto D$ induces a bijection between the set of elements D of $\text{Div}_{X^+, \infty}$ such that $\mathcal{I} \cong \mathcal{P}_T^{-1}$ and $\Gamma(T, \text{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\text{surj}}) / \Gamma(T, \mathcal{O}_T^*)$.

Since two invertible sheaves \mathcal{L}_1 and \mathcal{L}_2 on $X^+ \times T$ induce the same map $T \rightarrow \text{Pic}_{X^+}$ if and only if $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is pulled back from T [BLR90, 8.1 Prop 4], it follows that

$$\Gamma(T, \text{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\text{surj}} / \mathcal{O}_T^*) = (\text{Div}_{X^+} \times_{\text{Pic}_{X^+}} T)(T).$$

Note that $\Gamma(T, \text{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\text{surj}} / \mathcal{O}_T^*)$ is in natural bijection with the set of all equivalence classes of pairs of an invertible sheaf \mathcal{L} on T and a surjection $\varphi : \mathcal{E}_T \rightarrow \mathcal{L}$ where (\mathcal{L}, φ) and (\mathcal{L}', φ') are equivalent if there is an isomorphism $\theta : \mathcal{L} \rightarrow \mathcal{L}'$ such that $\varphi' = \theta \varphi$. By [EGAII, 4.2.3], it follows that the canonical closed immersion $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{F})$ represents $\text{Div}_{X^+, \infty} \rightarrow \text{Div}_{X^+}$. \square

A.3. Observation. — Let $\text{Div}_X^n = \text{Div}_{X^+}^n - \text{Div}_{X^+, \infty}^n$. It follows from Proposition A.2 and (24) that for large n , the vector bundle $V(\mathcal{E})$ represents Div_X^n .

Let $\text{Sym}^n X$ denote the n^{th} symmetric product of X , as in [BLR90, p. 252].

A.4. Proposition. — *There is a canonical isomorphism $\text{Sym}^n X = \text{Div}_X^n$.*

Proof. $\text{Sym}^n X$ represents the functor taking a locally Noetherian scheme T over k to the closed subschemes D of $X \times T$ with invertible ideal sheaf which are flat and finite over T of degree n by [SGA4_{III}, 6.3.9]. For any such closed subscheme D , the immersion $D \rightarrow X^+ \times T$ is proper because both $D \rightarrow T$ and the diagonal of $X^+ \times T \rightarrow T$ are proper morphisms. It follows that D is also in $\text{Div}_X^n(T)$. Similarly, if D is in $\text{Div}_X^n(T)$, we have that D is a closed subscheme of $X^+ \times T$ contained in the complement of $\infty \times T$, whence D is in $\text{Sym}^n X(T)$. Thus, we have an identification of $\text{Sym}^n X$ and Div_X^n . \square

Combining Proposition A.4 and Observation A.3 gives

A.5. Proposition. — *For X over k a non-proper, smooth, geometrically connected curve,*

$$\text{Sym}^n X \rightarrow \text{Pic}^n X^+$$

is given by $V(\mathcal{E})$ for a non-zero locally free sheaf \mathcal{E} on $\text{Pic}^n X^+$ for large n .

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