1: Geometry and Distance

The arena for multivariable calculus is the plane and space. Geometry is still a frontier: we have explored the micro-cosmos with microscopes, the macro-cosmos with telescopes but only recently started to conquer our meso-scale with 3D scanning, 3D printing, mapping and simulation.

A point in the plane has two coordinates $P = (x, y)$. A point in space is determined by three coordinates $P = (x, y, z)$. The signs of the coordinates determine 4 quadrants in the plane or 8 octants in space. These regions intersect at the origin $O = (0, 0)$ or $O = (0, 0, 0)$ and are bound by coordinate axes $\{y = 0\}$ and $\{x = 0\}$ or coordinate planes $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

In two dimensions, the $x$-coordinate usually directs to the "east" and the $y$-coordinate points "north". In three dimensions, the usual coordinate system has the $xy$-plane as the "ground" and the $z$-coordinate axes pointing "up".

1. $P = (2, -3)$ is in the forth quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -5)$ is located on the negative $z$ axis. The point $P = (1, 2, -3)$ is below the $xy$-plane. Can you spot the point $Q$ on the $xy$-plane which is closest to $P$?

2. **Problem.** Find the midpoint $M$ of $P = (1, 2, 5)$ and $Q = (-5, 4, 7)$. Answer. The midpoint is the average of each coordinate $M = (P + Q)/2 = (-2, 3, 6)$.

3. In computer graphics of photography, the $xy$-plane contains the retina or film plate. The $z$-coordinate measures the distance towards the viewer. In this photographic coordinate system, your eyes and chin define the plane $z = 0$ and the nose points in the positive $z$ direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0), (-1, 0, 0)$, then the tip of your nose might have the coordinates $(0, -1, 1)$.

The Euclidean distance between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

Note that this is a definition not a result. It is motivated by Pythagoras theorem but we will prove the later.

4. **Problem:** Find the distance $d(P, Q)$ between $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

Remarks.

1) Distances can be introduced more abstractly: take any nonnegative function $d(P, Q) = d(Q, P)$ which satisfies the triangle inequality $d(P, Q) + d(Q, R) \geq d(P, R)$ and has the property that $d(P, Q) = 0$ if and only if $P = Q$. A set $X$ with such a distance function $d$ is called a metric
space. Examples of distances are the Manhattan distance \( d_m(P, Q) = |x - a| + |y - b| \), the quartic distance \( d_4(P, Q) = ((x - a)^4 + (y - b)^4) \) or the Fermat distance \( d_f(x, y) = d(x, y) \) if \( y > 0 \) and \( d_f(x, y) = 1.33d(x, y) \) if \( y < 0 \). In the last example, the constant 1.33 is the refractive index in a model where the upper half plane is filled with air and the lower half plane with water. Shortest paths in this metric are no more lines: light rays get bent at the water surface. Each of these distances \( d, d_m, d_4, d_f \) equip the plane with a different metric space.

2) It is symmetry which distinguishes the Euclidean distance as the most natural one. The Euclidean distance is determined by the property \( d((1, 0, 0), (0, 0, 0)) = 1 \) together with the requirement of rotational and translational and scaling symmetry \( d(\lambda P, \lambda Q) = \lambda d(P, Q) \).

3) We usually work with a right handed coordinate system, where the \( x, y, z \) axes can be matched with the thumb, pointing and middle finger of the right hand. The photographers coordinate system is an example of a left handed coordinate system. The \( x, y, z \) axes are matched with the thumb and pointing finger and middle finger of the left hand. Nature is not oblivious to parity. Some fundamental laws in particle physics are different when they are observed in a mirror. Coordinate systems with different parity can not be rotated into each other.

4) When dealing with geometric problems in the plane, we usually leave the \( z \) coordinate away and have \( d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2} \), where \( P = (x, y), Q = (a, b) \).

Points, curves, surfaces and solids are geometric objects which can be described with functions of several variables. An example of a curve is a line, an example of a surface is a plane, an example of a solid is the interior of a sphere. We focus next on spheres or circles.

A circle of radius \( r \geq 0 \) centered at \( P = (a, b) \) is the collection of points in the plane which have distance \( r \) from \( P \).

A sphere of radius \( \rho \geq 0 \) centered at \( P = (a, b, c) \) is the collection of points in space which have distance \( \rho \) from \( P \). The equation of a sphere is \((x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2\).

An ellipse is the collection of points \( P \) in the plane for which the sum \( d(P, A) + d(P, B) \) of the distances to two points \( A, B \) is a fixed constant \( l \) larger than \( d(A, B) \). This allows to draw the ellipse with a string of length \( l \) attached at \( A, B \). When 0 is the midpoint of \( A, B \), an algebraic description is the set of points which satisfy the equation \( x^2/a^2 + y^2/b^2 = 1 \).

5) Problem: Is the point \((3, 4, 5)\) outside or inside the sphere \((x - 2)^2 + (y - 6)^2 + (z - 2)^2 = 16\)? Answer: The distance of the point to the center of the sphere is \( \sqrt{1 + 4 + 9} = 16 \). Since this is smaller than 4 the radius of the sphere, the point is inside.

6) Problem: Find an algebraic expression for the set of all points for which the sum of the distances to \( A = (1, 0) \) and \( B = (-1, 0) \) is equal to 3. Answer: Square the equation \( \sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2} = 3 \), separate the remaining single square root on one side and square again. Simplification gives \( 20x^2 + 36y^2 = 45 \) which is equivalent to \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), where \( a, b \) can be computed as follows: because \( P = (a, 0) \) satisfies this equation, \( d(P, A) + d(P, B) = (a - 1) + (a + 1) = 3 \) so that \( a = 3/2 \). Similarly, the point \( Q = (0, b) \) satisfying it gives \( d(Q, A) + d(P, B) = 2\sqrt{b^2 + 1} = 3 \) or \( b = \sqrt{5}/2 \).

Here is a verification with the computer algebra system Mathematica. Writing \( L = d(P, A) \) and \( M = d(P, B) \) we simplify the equation \( L^2 + M^2 = 3^2 \). The part without square root is
\[
((L + M)^2 + (L - M)^2)/2 - 3^2. \text{ The remaining square root is } ((L + M)^2 - (L - M)^2)/2. 
\]
Now square both and set them equal to see the equation \(20x^2 + 36y^2 = 45\).

The completion of the square of an equation \(x^2 + bx + c = 0\) is the idea to add \((b/2)^2 - c\) on both sides to get \((x + b/2)^2 = (b/2)^2 - c\). Solving for \(x\) gives the solution \(x = -b/2 \pm \sqrt{(b/2)^2 - c}\).

7 The equation \(2x^2 - 10x + 12 = 0\) is equivalent to \(x^2 + 5x = -6\). Adding \((5/2)^2\) on both sides gives \((x + 5/2)^2 = 1/4\) so that \(x = 2\) or \(x = 3\).

8 The equation \(x^2 + 5x + y^2 - 2y + z^2 = -1\) is after completion of the square \((x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1\) or \((x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2\). We see a sphere center \((5/2, 1, 0)\) and radius \(5/2\).

The method is due to Al-Khwarizmi who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work "Compendium on Calculation by Completion and Reduction" was dedicated to the Caliph al Ma’mun, who had established research center called "House of Wisdom" in Baghdad. 1 In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, René Descartes promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the Cartesian coordinate system. His ideas profoundly changed mathematics. Ideas do not grow in a vacuum. Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. 2

What happens in higher dimensions? A point in four dimensional space for example is labeled with four coordinates \((t, x, y, z)\). How many hyper chambers is space are obtained when using coordinate hyperplanes \(t = 0, x = 0, y = 0, z = 0\) as walls? Answer: There are 16 hyper-regions and each of them contains one of the 16 points \((x, y, z, w)\), where \(x, y, z, w\) are either +1 or −1.

1The book "The mathematics of Egypt, Mesopotamia, China, India and Islam, a Sourcebook, Ed Victor Katz, page 542 contains translations of some of this work.
2An entertaining read is "Descartes secret notebook" by Amir Aczel which deals with an other discovery of Descartes.
Homework

1 Describe and sketch the set of points \( P = (x, y, z) \) in three-dimensional space \( \mathbb{R}^3 \) represented by
   a) \((z + 2)^2 + (y - 3)^2 - 121 = 0\)
   b) \(40x + 30y - 50z = 600\)
   c) \(x^4y^8z^4 = 0\)
   d) \(y^2 = z\)

2 a) Find the distances of \( P = (8, 15, 0) \) to each of the 3 coordinate axes.

   b) Find the distances of \( P = (9, 11, -8) \) to each of the 3 coordinate planes.

   The figure shows two rectangles. One has the area \(8 \cdot 8 = 64\). The other has the area \(65 = 13 \cdot 5\). But these triangles are made up by matching pieces. Measure distances to see what is going on.

3 Find the center and radius of the sphere \(x^2 + 4x + y^2 - 16y + z^2 + 10z + 57 = 0\). Describe the traces of this surface, its intersection with each of the coordinate planes.

4 The isosceles triangles built at each corner of the 3:4:5 triangle have 1:1, 2:1 and 3:1 ”height to halfbase” ratios. Use this to find the circumference of the inner triangle XYZ. (For the magic about the 3:4:5 triangle see Grattan-Guinness book ”Routes of Learning”.)
2: Vectors and Dot Product

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a vector $\vec{v} = \langle x - a, y - b, z - c \rangle$. It connects $P$ with $Q$ and we write also $\vec{v} = PQ$. The real numbers $p, q, r$ in a vector $\vec{v} = \langle p, q, r \rangle$ are called the components of $\vec{v}$.

Vectors can be drawn everywhere in space but two vectors with the same components are considered equal. Vectors can be translated into each other if and only if their components are the same. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = \langle p, q, r \rangle$ heads to the point $(p, q, r)$. One can therefore identify points $P = (a, b, c)$ with vectors $\vec{v} = \langle a, b, c \rangle$ attached to the origin. For clarity, we often draw an arrow on top of vectors and if $\vec{v} = PQ$ then $P$ is the ”tail” and $Q$ is the ”head” of the vector. To distinguish vectors from points, it is custom to use different brackets and write $\langle 2, 3, 4 \rangle$ for vectors and $\langle 2, 3, 4 \rangle$ for points.

The sum of two vectors is $\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$. The scalar multiple $\lambda \vec{u} = \lambda \langle u_1, u_2 \rangle = \langle \lambda u_1, \lambda u_2 \rangle$. The difference $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

The vectors $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ are called standard basis vectors in the plane. In space, one has the basis vectors $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$.

Every vector $\vec{v} = \langle p, q \rangle$ in the plane can be written as a combination $\vec{v} = p \vec{i} + q \vec{j}$ of standard basis vectors and every vector $\vec{v} = \langle p, q, r \rangle$ in space can be written as $\vec{v} = p \vec{i} + q \vec{j} + r \vec{k}$. Vectors are abundant in applications. They appear in mechanics: if $\vec{r}(t) = \langle f(t), g(t) \rangle$ is a point in the plane which depends on time $t$, then $\vec{v} = \langle f'(t), g'(t) \rangle$ will be called the velocity vector at $\vec{r}(t)$. Here $f'(t), g'(t)$ are the derivatives. In physics, we often want to determine forces acting on objects. Forces are represented as vectors. In particular, electromagnetic or gravitational fields or velocity fields in fluids are described by vectors. Vectors appear also in computer science: the scalable vector graphics is a standard for the web for describing two-dimensional graphics. In quantum computation, rather than working with bits, one deals with qbits, which are vectors. Finally, color can be written as a vector $\vec{v} = \langle r, g, b \rangle$, where $r$ is red, $g$ is green and $b$ is blue component of the color vector. An other coordinate system for color is $\vec{v} = \langle c, m, y \rangle = \langle 1 - r, 1 - g, 1 - b \rangle$, where $c$ is cyan, $m$ is magenta and $y$ is yellow. Vectors appear in probability theory and statistics. On a finite probability space, a random variable is a vector.

The addition and scalar multiplication of vectors satisfy the laws you know from arithmetic. commutativity $\vec{u} + \vec{v} = \vec{v} + \vec{u}$, associativity $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ and $r \ast (s \ast \vec{v}) = (r \ast s) \ast \vec{v}$ as well as distributivity $(r + s) \vec{v} = \vec{v}(r + s)$ and $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$, where $\ast$ denotes multiplication with a scalar.

The length $|\vec{v}|$ of a vector $\vec{v} = PQ$ is defined as the distance $d(P, Q)$ from $P$ to $Q$. A vector of length 1 is called a unit vector. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.
$|\langle 3, 4 \rangle| = 5$ and $|\langle 3, 4, 12 \rangle| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$ and $\langle 3/5, 4/5 \rangle$ and $\langle 3/13, 4/13, 12/13 \rangle$. The only vector of length 0 is the zero vector $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

**Remarks.**

a) Different notations for the dot product are used in different mathematical fields. While pure mathematicians write $\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle$, one can see $\langle \vec{v} | \vec{w} \rangle$ in quantum mechanics or $v_i w^i$ or more generally $g_{ij} v^i w^j$ in general relativity. The dot product is also called **scalar product** or **inner product**.

b) Any product $g(v, w)$ which is linear in $v$ and $w$ and satisfies the symmetry $g(v, w) = g(w, v)$ and $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v, w) = 3v^1 w^1 + 2v^2 w^2 + v^3 w^3$.

The dot product determines distance and distance determines the dot product.

**Proof:** Write $v = \vec{v}$. Using the dot product one can express the length of $v$ as $|v| = \sqrt{\vec{v} \cdot \vec{v}}$. On the other hand, from $(v + w) \cdot (v + w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

$$v \cdot w = (|v + w|^2 - |v|^2 - |w|^2)/2$$

The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

**Proof.** We can assume $|w| = 1$ by rescaling the equation. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw) \cdot (v - aw)$ to get $0 \leq (v - (v \cdot w)w) \cdot (v - (v \cdot w)w) = |v|^2 + (v \cdot w)^2 - 2(v \cdot w)^2 = |v|^2 - (v \cdot w)^2$ which means $(v \cdot w)^2 \leq |v|^2$.

Having established this, it is possible to give a clear definition of what an **angle** is, without referring to geometric pictures:

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. 

![Diagram of vectors and angles](image-url)
Al Kashi’s theorem: If $a, b, c$ are the side lengths of a triangle $ABC$ and $\alpha$ is the angle opposite to $c$, then $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \overrightarrow{AB}, \vec{w} = \overrightarrow{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, we know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab \cos(\alpha)$.

The angle definition works in any space with a dot product. In statistics you work with vectors of $n$ components. They are called data or random variables and $\cos(\alpha)$ is called the correlation between two random variables $\vec{v}, \vec{w}$ of zero expectation $E[\vec{v}] = (v_1 + \cdots + v_n)/n$. The dot product $v_1 w_1 + \cdots + v_n w_n$ is then the covariance, the length $|v|$ is the standard deviation and denoted by $\sigma(v)$. The formula $\text{Cov}(v, w) = \text{Cov}[v, w]/(\sigma(v)\sigma(w))$ for the correlation is the familiar angle formula we have seen. Geometry in $n$ dimensions can be useful in other fields.

The triangle inequality tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.

Proof: $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| \leq (|\vec{u}| + |\vec{v}|)^2$.

Two vectors are called orthogonal or perpendicular if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$.

Having given precise definitions of all objects, we can now prove the Pythagoras theorem:

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$.

Remarks:
1) You have seen here something powerful: results like the theorems of Pythagoras (570-495BC) and Al Khashi (1380-1429) were derived from scratch on a space $V$ equipped with a dot product. The dot product appeared much later in mathematics (Hamilton 1843, Grassman 1844, Sylvester 1851, Cayley 1858). While we have used geometry as an intuition, the structure was built algebraically without any unjustified assumptions. This is mathematics: if we have a space $V$ in which addition $\vec{v} + \vec{w}$ and scalar multiplication $\lambda \vec{v}$ is given and in which a dot product is defined, then all the just derived results apply. We have not used results of Al Khashi or Pythagoras but we have derived them and additionally obtained a clear definition what an angle is.
2) The derivation you have seen works in any dimension. Why do we care about higher dimensions? As already mentioned, a compelling motivation is statistics. Given 12 data points like the average monthly temperatures in a year, we deal with a 12 dimensional space. Geometry is useful to describe data. Pythagoras theorem is the property that the variance of two uncorrelated random variables adds up with the formula $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$.
3) A far reaching generalization of the geometry you have just seen is obtained if the dot product $g(v, w)$ is allowed to depend on the place, where the two vectors are attached. This produces Riemannian geometry and allows to work with spaces which are intrinsically curved. This mathematics is important in general relativity which describes gravity in a geometric way and which is one of the pillars of modern physics. But it appears in daily life too. If you look close at an object on a hot asphalt street, the object can appear distorted or flickers. The dot product and so the angles depends on the temperature of the air. Light rays no more move on straight
lines but gets bent. In extreme cases, when the curvature of light rays is larger than the curvature of the earth, it leads to Fata morgana effects: one can see objects which are located beyond the horizon.

4) Why don’t we define vectors as algebraic objects \( \langle 1, 2, 3 \rangle \)? The reason is that in applications of physics or geometry, we want to work with affine vectors, vectors which are attached at points. Forces for example act on points of a body, vector fields are families of vectors attached to points of space. Considering vectors with the same components as equal gives then the vector space in which we do the algebra. One could define a vector space axiomatically and then build from this affine vectors but it is a bit too abstract and not much is actually gained for the goals we have in mind. An even more modern point of view replaces affine vectors with members of a tangent bundle. But this is only necessary if one deals with spaces which are not flat. Even more general is to allow the space attached at each point to be a more general space like a ”group” called fibres. So called ”fibre bundles” are the framework of mathematical concepts which describe elementary particles or even space itself. Attaching a circle for example at each point leads to electromagnetism attaching classes of two dimensional matrices leads to the weak force and attaching certain three dimensional matrices leads to the strong force. Allowing this to happen in a curved framework incorporates gravity. One of the main challenges is to include quantum mechanics into that picture. Fundamental physics contains the question ”What is space”?

The vector \( P(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}|} \vec{w} \) is called the projection of \( \vec{v} \) onto \( \vec{w} \). The scalar projection \( \frac{\vec{v} \cdot \vec{w}}{||\vec{w}|} \) is a signed length of the vector projection. Its absolute value is the length of the projection of \( \vec{v} \) onto \( \vec{w} \). The vector \( \vec{b} = \vec{v} - P(\vec{v}) \) is a vector orthogonal to the \( \vec{w} \)-direction.

2) For example, with \( \vec{v} = \langle 0, -1, 1 \rangle \), \( \vec{w} = \langle 1, -1, 0 \rangle \), \( P(\vec{v}) = \langle 1/2, -1/2, 0 \rangle \). Its length is \( 1/\sqrt{2} \).

3) Projections are important in physics. For example, if you apply a wind force \( \vec{F} \) to a car which drives in the direction \( \vec{w} \) and \( P \) denotes the projection on \( \vec{w} \) then \( P(\vec{F}) \) is the force which accelerates or slows down the car.

The projection allows to visualize the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if \( v \) points more towards to \( w \), it is negative if \( v \) points away from it. In the next lecture we use the projection to compute distances between various objects.
Homework

1. Find a **unit vector** parallel to \( \vec{u} + \vec{v} - 2\vec{w} \) if \( \vec{u} = \langle 6, 7, 3 \rangle \) and \( \vec{v} = \langle 2, 2, 3 \rangle \) and \( \vec{w} = \langle -2, -1, 1 \rangle \).

   An **Euler brick** is a cuboid of dimensions \( a, b, c \) such that all face diagonals are integers.
   a) Verify that \( \vec{v} = \langle a, b, c \rangle = \langle 240, 117, 44 \rangle \) is a vector which leads to an Euler brick.
   It had been found by Halcke in 1719.
   b) (*) Verify that \( \langle a, b, c \rangle = \langle u(4v^2 - w^2), v(4u^2 - w^2), 4uvw \rangle \) leads to an Euler brick if \( u^2 + v^2 = w^2 \).
   (Sounderson 1740) If also the space diagonal \( \sqrt{a^2 + b^2 + c^2} \) is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.

2. **Colors** are encoded by vectors \( \vec{v} = \langle \text{red}, \text{brightgreen}, \text{blue} \rangle \).
   The red, green and blue components of \( \vec{v} \) are all real numbers in the interval \([0, 1]\).
   a) Determine the angle between the colors magenta and cyan.
   b) What is the vector projection of the magenta-orange mixture \( \vec{x} = (\vec{v} + \vec{w})/2 \) onto green \( \vec{y} \)?
4 Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex. You can leave the answer in the form \( \cos(\alpha) = \ldots \).

5 Assume \( \vec{v} = (-4, 2, 2) \) and \( \vec{w} = (4, 0, 3) \).
   a) Find the vector projection of \( \vec{v} \) onto \( \vec{w} \).
   b) Find the scalar projection (component) of \( \vec{v} \) on \( \vec{w} \).
3: Cross product

The cross product of two vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ in the plane is the scalar $v_1w_2 - v_2w_1$.

To remember this, write it as a determinant of a matrix which is an array of numbers: take the product of the diagonal entries and subtract the product of the side diagonal.

$$\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

The cross product of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle.$$

To remember it we write the product as a "determinant":

$$\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = i \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - j \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + k \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

which is $\vec{i}(v_2w_3 - v_3w_2) - \vec{j}(v_1w_3 - v_3w_1) + \vec{k}(v_1w_2 - v_2w_1)$.

1. The cross product of $\langle 1, 2 \rangle$ and $\langle 4, 5 \rangle$ is $5 - 8 = -3$.
2. The cross product of $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 1 \rangle$ is $\langle -13, 11, -3 \rangle$.

The cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. The product is anticommutative.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.
The \textbf{sin} formula: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

\textbf{Proof:} We verify first the \textbf{Lagrange’s identity} $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ which is also called \textbf{Cauchy-Binet} formula by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the \textbf{area of the parallelogram} spanned by $\vec{v}$ and $\vec{w}$.

Note that this was the \textbf{definition} of area so that nothing needs to be proven. To see that the definition fits with our common intuition we have about area, note that $|\vec{w}| \sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. The area formula also proves the sin-formula because the area does not depend on which pair of sides to a triangle we take. Area also is linear in each of the vectors $v$ and $w$. If we make $v$ twice as long, then the area gets twice as large.

$\vec{v} \times \vec{w}$ is zero exactly if $\vec{v}$ and $\vec{w}$ are \textbf{parallel}, that is if $\vec{v} = \lambda \vec{w}$ for some real $\lambda$.

\textbf{Proof.} Use the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that $v$ and $-v$ are considered parallel even so it is sometimes called \textbf{anti-parallel}.

The \textbf{trigonometric sin-formula}: if $a, b, c$ are the side lengths of a triangle and $\alpha, \beta, \gamma$ are the angles opposite to $a, b, c$ then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma)$.

\textbf{Proof.} We express the area of the triangle in three different ways:

$$ab \sin(\gamma) = bc \sin(\alpha) = ac \sin(\beta).$$

Divide the first equation by $\sin(\gamma) \sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha) \sin(\beta)$ to get the second identity.

3 If $\vec{v} = \langle a, 0, 0 \rangle$ and $\vec{w} = \langle b \cos(\alpha), b \sin(\alpha), 0 \rangle$, then $\vec{v} \times \vec{w} = \langle 0, 0, ab \sin(\alpha) \rangle$ which has length $|ab \sin(\alpha)|$.

The scalar $\langle \vec{u}, \vec{v}, \vec{w} \rangle = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the \textbf{triple scalar product} of $\vec{u}, \vec{v}, \vec{w}$.
The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$.

The **orientation** of three vectors is defined as the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors define a right-handed coordinate system.

Again, there was no need to prove anything because we defined volume and orientation. Let us still see why this fits with with our intuition. The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$.

The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n})|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$, which is the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system.

Since the triple scalar product is linear with respect to each vector, we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with twice the volume.

4 **Problem**: Find the volume of a **cuboid** of width $a$, length $b$, and height $c$. **Answer**: The cuboid is a parallelepiped spanned by $\langle a, 0, 0 \rangle$, $\langle 0, b, 0 \rangle$, and $\langle 0, 0, c \rangle$. The triple scalar product is $abc$.

5 **Problem**: Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that the solid is spanned by the vectors $\vec{u} = (1, 2, 1), \vec{v} = (3, 2, 1)$, and $\vec{w} = (0, 3, 1)$. We get $\vec{v} \times \vec{w} = (-1, -3, 9)$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

6 **Problem**: A 3D scanner is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (0, 1, 1), Q = (1, 1, 0)$ and $R = (1, 2, 3)$. Find the area of the triangle. **Solution**: We have to find the length of the cross product of $\vec{PQ}$ and $\vec{PR}$ which is $\langle 1, -3, 1 \rangle$. The length is $\sqrt{11}$.

7 **Problem**: There is an other point $A = (1, 1, 1)$ detected. On which side of the triangle is it located if the cross product of $\vec{PQ}$ and $\vec{PR}$ is considered the direction ”up”. **Solution**: The cross product is $\vec{n} = \langle 1, -3, 1 \rangle$. We have to see whether the vector $\vec{PA} = \langle 1, 0, 0 \rangle$ points into the direction of $\vec{n}$ or not. To see that, we have to form the dot product. It is 1 so that indeed, $A$ is ”above” the triangle. Note that a triangle in space a priori does not have an orientation. We have to tell, what direction is ”up”. That is the reason that file formats for 3D printing like STL contain the data for three points in space as well as a vector, telling the direction.

**Homework**

1 a) Find the volume of the parallelepiped for which the base parallelogram is given by the points $(2, 1, 1), (2, 0, 1), (0, 3, 1), (0, 2, 1)$
and which has an edge connecting (2, 1, 1) with (4, 5, 6).
b) Find the area of the base and use a) to get the height of the parallelepiped.

2 a) Assume \( \vec{u} + \vec{v} + \vec{w} = \vec{0} \). Verify that \( \vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u} \).
b) Find \( (\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w}) \) if \( \vec{u}, \vec{v}, \vec{w} \) are unit vectors which are orthogonal to each other and \( \vec{u} \times \vec{v} = \vec{w} \).

3 To find the equation \( ax + by + cz = d \) for the plane which contains the point \( P = (1, 2, 3) \) as well as the line which passes through \( Q = (3, 4, 4) \) and \( R = (1, 1, 2) \), we find a vector \( \langle a, b, c \rangle \) normal to the plane and fix \( d \) so that \( P \) is in the plane.

4 (*) Verify the Lagrange formula

\[
\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})
\]

for general vectors \( \vec{a}, \vec{b}, \vec{c} \) in space. The formula can be remembered as ”BAC minus CAB”.

5 Assume you know that the triple scalar product \( [\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) \) between \( \vec{u}, \vec{v}, \vec{w} \) is equal to 3. Find the values of \( [\vec{v}, \vec{u}, \vec{w}] \) and \( [\vec{u} + \vec{v}, \vec{v}, \vec{w}] \).
4: Lines and Planes

A point \( P = (p, q, r) \) and a vector \( \vec{v} = (a, b, c) \) define the line
\[
L = \{(p, q, r) + t(a, b, c), t \in \mathbb{R}\}.
\]

The line consists of all points obtained by adding a multiple of the vector \( \vec{v} \) to the vector \( \vec{OP} = (p, q, r) \). The line contains the point \( P \) as well as a suitably translated copy of \( \vec{v} \). Every vector contained in the line is necessarily parallel to \( \vec{v} \). We think about the parameter \( t \) as "time". At time \( t = 0 \), we are at the point \( P \), whereas at time \( t = 1 \) we are at \( \vec{OP} + \vec{v} \).

If \( t \) is restricted to values in a parameter interval \( [s, u] \), then \( L = \{(p, q, r) + t(a, b, c), s \leq t \leq u\} \) is a line segment which connects \( \vec{r}(s) \) with \( \vec{r}(u) \).

1 To get the line through \( P = (1, 1, 2) \) and \( Q = (2, 4, 6) \), we form the vector \( \vec{v} = \vec{PQ} = (1, 3, 4) \) and get \( L = \{(x, y, z) = (1, 1, 2) + t(1, 3, 4); \} \). This can be written also as \( \vec{r}(t) = (1 + t, 1 + 3t, 2 + 4t) \). If we write \( \langle x, y, z \rangle = (1, 1, 2) + t(1, 3, 4) \) as a collection of equations \( x = 1 + 2t, y = 1 + 3t, z = 2 + 4t \) and solve the first equation for \( t \) gives the symmetric equation
\[
L = \{(x, y, z) | (x - 1)/2 = (y - 1)/3 = (z - 2)/4\}.
\]

The line \( \vec{r} = \vec{OP} + t\vec{v} \) defined by \( P = (p, q, r) \) and vector \( \vec{v} = (a, b, c) \) with nonzero \( a, b, c \) satisfies the symmetric equations
\[
\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c}.
\]

Proof. Each of these expressions is equal to \( t \). These symmetric equations have to be modified a bit one or two of the numbers \( a, b, c \) are zero. If \( a = 0 \), replace the first equation with \( x = p \), if \( b = 0 \) replace the second equation with \( y = q \) and if \( c = 0 \) replace third equation with \( z = r \).

2 Find the symmetric equations for the line through the two points \( P = (0, 1, 1) \) and \( Q = (2, 3, 4) \), we first form the parametric equations \( \langle x, y, z \rangle = (0, 1, 1) + t(2, 2, 3) \) or \( x = 2t, y = 1 + 2t, z = 1 + 3t \). Solving each equation for \( t \) gives the symmetric equation \( x/2 = (y - 1)/2 = (z - 1)/3 \).

3 Problem: Find the symmetric equation for the \( z \) axes. Answer: This is a situation where \( a = b = 0 \) and \( c = 1 \). The symmetric equations are simply \( x = 0, y = 0 \). If two of the numbers \( a, b, c \) are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.
A point $P$ and two vectors $\vec{v}, \vec{w}$ define a plane $\Sigma = \{ O\vec{P} + t\vec{v} + s\vec{w}, \text{where } t, s \text{ are real numbers } \}$.

4 An example is $\Sigma = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 2, 4, 6 \rangle + s\langle 1, 0, -1 \rangle \}$. This is called the parametric description of a plane.

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$.

The equation for a plane containing $\vec{v}$ and $\vec{w}$ and a point $P$ is

$$ax + by + cz = d,$$

where $\langle a, b, c \rangle = \vec{v} \times \vec{w}$ and $d$ is obtained by plugging in $P$.

5 Problem: Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.

Answer: The plane contains the two vectors $\vec{v} = \langle 1, 2, 0 \rangle$ and $\vec{w} = \langle 2, 2, 2 \rangle$. We have $\vec{n} = \langle 4, -2, -2 \rangle$ and the equation is $4x - 2y - 2z = d$. The constant $d$ is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

The angle between the two planes $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two vectors $\vec{n} = \langle a, b, c \rangle$ and $\vec{m} = \langle e, f, g \rangle$.

6 Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. Answer: find the angle between $\vec{n} = \langle 1, 1, 0 \rangle$ and $\vec{m} = \langle 1, 1, 1 \rangle$. It is $\arccos(2/\sqrt{6})$.

Finally, let's look at some distance formulas.

1) If $P$ is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point $Q$, then

$$d(P, \Sigma) = \frac{|P\vec{Q} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between $P$ and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{|\langle -7, 0, -3 \rangle \cdot \langle 2, 4, 5 \rangle|}{|\langle 2, 4, 5 \rangle|} = \frac{29}{\sqrt{45}}.$$

2) If $P$ is a point in space and $L$ is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(P\vec{Q}) \times \vec{u}|}{|\vec{u}|}$$
is the distance between \( P \) and the line \( L \). Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from \( P = (2, 3, 1) \) to the line \( \vec{r}(t) = (1, 1, 2) + t(5, 0, 1) \), compute

\[
d(P, L) = \frac{|\langle -1, -2, 1 \rangle \times \langle 5, 0, 1 \rangle|}{\langle 5, 0, 1 \rangle} = \frac{|\langle -2, 6, 10 \rangle|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}.
\]

3) If \( L \) is the line \( \vec{r}(t) = Q + t\vec{u} \) and \( M \) is the line \( \vec{s}(t) = P + t\vec{v} \), then

\[
d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}
\]
is the distance between the two lines \( L \) and \( M \). Proof: the distance is the length of the vector projection of \( \vec{PQ} \) onto \( \vec{u} \times \vec{v} \) which is normal to both lines. For example, to compute the distance between \( \vec{r}(t) = (2, 1, 4) + t(-1, 1, 0) \) and \( M \) is the line \( \vec{s}(t) = (-1, 0, 2) + t(5, 1, 2) \) form the cross product of \( (-1, 1, 0) \) and \( (5, 1, 2) \) is \( (2, 2, -6) \). The distance between these two lines is

\[
d(L, M) = \frac{|(3, 1, 2) \cdot (2, 2, -6)|}{|\langle 2, 2, -6 \rangle|} = \frac{4}{\sqrt{44}}.
\]

4) To get the distance between two planes \( \vec{n} \cdot \vec{x} = d \) and \( \vec{n} \cdot \vec{x} = e \), then their distance is

\[
d(\Sigma, \Pi) = \frac{|e - d|}{|\vec{n}|}
\]

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, \( 5x + 4y + 3z = 8 \) and \( 10x + 8y + 6z = 2 \) have the distance

\[
\frac{|8 - 1|}{|\langle 5, 4, 3 \rangle|} = \frac{7}{\sqrt{50}}.
\]

Here is a distance problem which has a great deal of application and motivates the material of the upcoming week: The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites. Each GPS satellite sends periodically signals which are triggered by an atomic clock. While the distance to each satellite is not known, the difference from the distances to two satellites can be determined from the time delay of the two signals. This clever trick has the consequence that the receiver does not need to contain an atomic clock itself. To understand this better, we need to know about functions of three variables and surfaces.
Homework

1. Find the parametric and symmetric equation for the line which passes through the points $P = (3, 3, 4)$ and $Q = (4, 5, 6)$.

A regular tetrahedron has vertices at the points $P_1 = (0, 0, 3), P_2 = (0, \sqrt{8}, -1), P_3 = (-\sqrt{6}, -\sqrt{2}, -1)$ and $P_4 = (\sqrt{6}, -\sqrt{2}, -1)$. Find the distance between two edges which do not intersect.

2. Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1 + 4t, y = 1 - 4t, z = 8t$ and intersects this line in a point $Q$.

Given three spheres of radius 9 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side.

3. Given three spheres of radius 9 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side.

5. a) Find the distance between the point $P = (3, 3, 4)$ and the line $x = y = z$.

b) Parametrize the line $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ in a) and find the minimum of the function $f(t) = d(P, \vec{r}(t))^2$. Verify that the minimal value agrees with a).
Name:

The week 1 homework set is due July 1, 2014. Start to work early on the homework! At the end, please copy your HW answers to this cover sheet and staple this first page to the homework of lectures 1.1 and 1.2:

**Homework 1.1**

1

2

3

4

5

**Homework 1.2**

1

2

3

4

5
The first week homework is due July 1, 2014. Start working early on the homework! At the end, please copy your HW answers to this cover sheet and staple this first page to the homework of lectures 1.3 and 1.4:

**Homework 1.3**

1

2

3

4

5

**Homework 1.4**

1

2

3

4

5
I want to meet with all of you individually you during the first week.

My Office SC 432 is in the 4th floor of the Science center. It is close to the mathematics common room. Please fill in your name to the time slots. These are short 10 min meetings.

<table>
<thead>
<tr>
<th>Tuesday 6/24/2014</th>
<th>Tuesday 6/24/2014</th>
</tr>
</thead>
<tbody>
<tr>
<td>14:50</td>
<td>16:00</td>
</tr>
<tr>
<td>15:00</td>
<td>16:10</td>
</tr>
<tr>
<td>15:10</td>
<td>16:20</td>
</tr>
<tr>
<td>15:20</td>
<td>16:30</td>
</tr>
<tr>
<td>15:30</td>
<td>16:40</td>
</tr>
<tr>
<td>15:40</td>
<td>16:50</td>
</tr>
<tr>
<td>15:50</td>
<td>17:00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>14:50</td>
<td>16:00</td>
</tr>
<tr>
<td>15:00</td>
<td>16:10</td>
</tr>
<tr>
<td>15:10</td>
<td>16:20</td>
</tr>
<tr>
<td>15:20</td>
<td>16:30</td>
</tr>
<tr>
<td>15:30</td>
<td>16:40</td>
</tr>
<tr>
<td>15:40</td>
<td>16:50</td>
</tr>
<tr>
<td>15:50</td>
<td>17:00</td>
</tr>
<tr>
<td>-------</td>
<td>--------------------</td>
</tr>
<tr>
<td>14:50</td>
<td></td>
</tr>
<tr>
<td>15:00</td>
<td></td>
</tr>
<tr>
<td>15:10</td>
<td></td>
</tr>
<tr>
<td>15:20</td>
<td></td>
</tr>
<tr>
<td>15:30</td>
<td></td>
</tr>
<tr>
<td>15:40</td>
<td></td>
</tr>
<tr>
<td>15:50</td>
<td></td>
</tr>
<tr>
<td>16:00</td>
<td></td>
</tr>
<tr>
<td>16:10</td>
<td></td>
</tr>
<tr>
<td>16:20</td>
<td></td>
</tr>
<tr>
<td>16:30</td>
<td></td>
</tr>
<tr>
<td>16:40</td>
<td></td>
</tr>
<tr>
<td>16:50</td>
<td></td>
</tr>
<tr>
<td>17:00</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Friday 6/27/2014</th>
<th>Friday 6/27/2014</th>
</tr>
</thead>
<tbody>
<tr>
<td>14:50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17:00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Tuesday 7/1/2014</th>
<th>Tuesday 7/1/2014</th>
</tr>
</thead>
<tbody>
<tr>
<td>14:50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15:50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16:50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17:00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A function of two variables \( f(x, y) \) is a rule which assigns to two numbers \( x, y \) a third number \( f(x, y) \). For example, the function \( f(x, y) = x^3 y + 2x \) assigns to \((2, 3)\) the number \( 2^3 \cdot 3 + 4 = 28 \).

A function is usually defined for all points \((x, y)\) in the plane like for \( f(x, y) = x^2 + \sin(xy) \). Sometimes, it is required that we restrict the function to a domain \( D \) in the plane. An example is \( f(x, y) = \log|y| + \sqrt{x} \), where \((x, y)\) is only defined for \( x > 0 \) and \( y \neq 0 \). The range of a function \( f \) is the set of values which the function \( f \) can take. The function \( f(x, y) = 3 + x^2/(1 + x^2) \) for example takes all values \( 3 \leq z < 4 \) and the value \( z = 4 \) is not attained.

The graph of \( f(x, y) \) is the set \( \{(x, y, f(x, y)) \mid (x, y) \in D \} \) in space. Graphs are geometric objects which visualize a function visually.

The graph of \( f(x, y) = \sqrt{1 - x^2 - y^2} \) on the domain \( x^2 + y^2 < 1 \) is a half sphere.

<table>
<thead>
<tr>
<th>Example ( f(x, y) )</th>
<th>domain ( D ) of ( f )</th>
<th>range ( = f(D) ) of ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x, y) = \sin(3x + 3y) - \log(1 - x^2 - y^2) )</td>
<td>open unit disc ( x^2 + y^2 &lt; 1 )</td>
<td>([-1, \infty) )</td>
</tr>
<tr>
<td>( f(x, y) = f(x, y) = x^2 + y^2 - xy + \cos(xy) )</td>
<td>plane ( \mathbb{R}^2 )</td>
<td>the real line ( [0, 2] )</td>
</tr>
<tr>
<td>( f(x, y) = \sqrt{4 - x^2 - 2y^2} )</td>
<td>( x^2 + 2y^2 \leq 4 )</td>
<td>the real line ( (0, 2) )</td>
</tr>
<tr>
<td>( f(x, y) = 1/(x^2 + y^2 - 1) )</td>
<td>all except unit circle</td>
<td>positive real axis</td>
</tr>
<tr>
<td>( f(x, y) = 1/(x^2 + y^2)^2 )</td>
<td>all except origin</td>
<td></td>
</tr>
</tbody>
</table>

The set \( \{(x, y) \mid f(x, y) = c = \text{const} \} \) is called a contour curve or level curve of \( f \). For example, for \( f(x, y) = 4x^2 + 3y^2 \), the level curves \( f = c \) are ellipses if \( c > 0 \). Drawing several contour curves \( \{f(x, y) = c\} \) produces a contour map of the function \( f \).
Level curves allow to visualize functions of two variables $f(x, y)$ without leaving the plane. The picture to the right for example shows the level curves of the function $\sin(xy) - \sin(x^2 + v)$. Contour curves are encountered every day: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.

3 For $f(x, y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with their ”noses” at the point $(-1,0)$ and $(1,0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

4 The function $f(x, y) = 1 - 2x^2 - y^2$ has contour curves $f(x, y) = 1 - 2x^2 + y^2 = c$ which are ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.

5 For the function $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2 - y^2)e^{-x^2-y^2} = c$. We can draw the curves however with the help of a computer:

6 The surface $z = f(x, y) = \sin(\sqrt{x^2 + y^2})$ has concentric circles as contour curves.
In applications, functions appear which are not continuous. When plotting the rate of change of temperature of water in relation to pressure and volume for example, one experiences phase transitions, places where the function value can jump. Mathematicians have tamed singularities in a mathematical field called "catastrophe theory".

A function \( f(x, y) \) is called **continuous** at \((a, b)\) if we can find a finite value \( f(a, b) \) with \( \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) \). This means that for any sequence \((x_n, y_n)\) converging to \((a, b)\), also \( f(x_n, y_n) \to f(a, b) \). A function is **continuous** in a subset \( G \) of the plane if it is continuous at every point \((a, b)\) of \( G \).

Continuity means that if \((x, y)\) is close to \((a, b)\), then \( f(x, y) \) is close to \( f(a, b) \). Continuity for functions of more than two variables is defined in the same way. Continuity is not always easy to check. Fortunately however, we do not have to worry about it most of the time. Let's look at some examples:

7 **Example:** For \( f(x, y) = (xy)/(x^2 + y^2) \), we have \( \lim_{(x,y)\to(0,0)} f(x, x) = \lim_{x\to0} x^2/(2x^2) = 1/2 \) and \( \lim_{(x,0)\to(0,0)} f(0, x) = \lim_{(x,0)\to(0,0)} 0 = 0 \). The function is not continuous at \((0, 0)\).

8 For \( f(x, y) = (x^2y)/(x^2 + y^2) \), it is better to describe the function using polar coordinates: 
\[
f(r, \theta) = r^3 \cos^2(\theta) \sin(\theta)/r^2 = r \cos^2(\theta) \sin(\theta)\].
We see that \( f(r, \theta) \to 0 \) uniformly if \( r \to 0 \). The function is continuous if we extend it and postulate \( f(0, 0) = 0 \).

P.S. It is a modern standpoint to consider the function in this example to be **continuous**, as there is a canonical way on how we can define the value at the undefined point. Mathematicians can deal with interesting and powerful ideas: the sum \( 1+2+4+8+16+\ldots \) for example does not make any sense at first. But its value is \(-1\): the reason is that \( 1+a+a^2+a^3+\ldots = 1/(1-a) \) allows to breathe live into the meaning of the left hand side if \( a \) is equal to \( 2 \). The right hand side gives the answer: it is \(-1\). Such ideas are important in fundamental questions and pivotal to understand the secrets about prime numbers. The famous Riemann hypothesis for example claims that all the zeros of the function \( f(s) = 1+1/2^s+1/3^s+\ldots \) are located on the axes, where the real part is \( 1/2 \). Also here, the sum itself does not make sense for values \( s = 1/2 + it \) but we know how to interpret of it in a canonical way, and even evaluate it for \( s = 0 \). Back to the example: the function \( f(x, y) = (x^2y)/(x^2 + y^2) \) contains already all the information we need to define it at \((x, y) = 0\) even so the formula itself can not be evaluated at \((x, y) = (0, 0)\). Some calculus books would insist that the function is not continuous. We are more generous and consider it to be continuous. The just described shifts in the perception of the function concept happened pretty late, maybe with Fourier and then with Riemann. Before Fourier, one always insisted on a function to have an analytic expression.

Before Riemann, one would not look at the world hidden behind analytic continuation.

A function of three variables \( g(x, y, z) \) assigns to three variables \( x, y, z \) a real number \( g(x, y, z) \). The function \( f(x, y, z) = x^2 + y - z \) for example satisfies \( f(3, 2, 1) = 10 \).

We can visualize a function by **contour surfaces** \( g(x, y, z) = c \), where \( c \) is constant. The contour surface of \( g(x, y, z) = x^2 + y^2 + z^2 = c \) for example is a sphere if \( c > 0 \).

To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes \( x = 0, y = 0 \) or \( z = 0 \).
Many surfaces can be described as level surfaces. If this is the case, we call this an **implicit description** of a surface. Here are some examples:

9 The function \( g(x, y, z) = 2 + \sin(\pi x y z) \) could define a temperature distribution in space. We can no more draw the graph of \( g \) because that would be an object in 4 dimensions. We can however draw surfaces like \( g(x, y, z) = 0 \).

10 The level surfaces of \( g(x, y, z) = x^2 + y^2 + z^2 \) are spheres. The level surfaces of \( g(x, y, z) = 2x^2 + y^2 + 3z^2 \) are ellipsoids.

11 For \( g(x, y, z) = z - f(x, y) \), the level surface \( g = 0 \) agrees with the graph \( z = f(x, y) \) of \( f \). For example, for \( g(x, y, z) = z - x^2 - y^2 = 0 \), the graph \( z = x^2 + y^2 \) of the function \( f(x, y) = x^2 + y^2 \) is a paraboloid. Graphs are special surfaces. Most surfaces of the form \( g(x, y, z) = c \) can not be written as graphs. The sphere is an example. We would need two graphs to cover it.

12 The equation \( ax + by + cz = d \) is a plane. With \( \vec{n} = \langle a, b, c \rangle \) and \( \vec{x} = \langle x, y, z \rangle \), we can rewrite the equation \( \vec{n} \cdot \vec{x} = d \). If a point \( \vec{x}_0 \) is on the plane, then \( \vec{n} \cdot \vec{x}_0 = d \) so that \( \vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \). This means that every vector \( \vec{x} - \vec{x}_0 \) in the plane is orthogonal to \( \vec{n} \).

13 For \( f(x, y, z) = ax^2 + by^2 + cz^2 + dx y + exz + fyz + gx + hy + k z + m \) the surface \( f(x, y, z) = 0 \) is called a **quadric**. We will look at a few examples.

<table>
<thead>
<tr>
<th>Sphere</th>
<th>Paraboloid</th>
<th>Plane</th>
</tr>
</thead>
</table>

\[
(x-a)^2+(y-b)^2+(z-c)^2 = r^2 \quad (x - a)^2 + (y - b)^2 - c = z \quad ax + by + cz = d
\]
Higher order polynomial surfaces can be intriguingly beautiful and difficult to describe. If $f$ is a polynomial in several variables and $f(x, x, x)$ is a polynomial of degree $d$, then $f$ is called a degree $d$ polynomial surface. Degree 2 surfaces are quadrics, degree 3 surfaces cubics, degree 4 surfaces quartics, degree 5 surfaces quintics, degree 10 surfaces decics and so on.
Homework

1 a) Plot the graph of the function \( f(x, y) = x^3 - 3xy^2 \) called Monkey saddle. You find the graph already on this handout. Try by hand, possibly use Wolfram alpha or Mathematica or a graphing calculator to check your picture.

b) Draw the surface \( z^2 - 4z + x^2 - 2x - y = 0 \) by drawing traces, intercepts. Identify the surface.

2 a) Determine the domain and range of the logarithmic mean \( f(x, y) = \frac{(y - x)}{\log(y) - \log(x)} \), where \( \log \) the natural logarithm.

b) The function is not defined at \( x = y \) but one can define \( f(x, y) \) on the diagonal \( x = y \). Use Hôpital to show that the limit \( \lim_{x \to 2} f(x, 2) \) exists.

c) The function is also not defined at first if \( x = 0 \) or \( y = 0 \). Use Hôpital to show that the limit \( \lim_{x \to 0} f(x, 2) \) exists.

3 Use online tools or your computer to draw the level surface \( x^2 + y^2 + z^2 + x^2y^2z^2 = 20 \) and its traces.

4 a) Sketch the graph and contour map of the function \( f(x, y) = \sin(x^2 + y^2)/(2 + x^2 + y^2) \).

b) Sketch the graph and contour map of the function \( g(x, y) = 2|x| - 3|y| \).

5 a) Verify that the line \( \vec{r}(t) = \langle 1, 3, 2 \rangle + t\langle 1, 2, 1 \rangle \) is part of the hyperbolic paraboloid \( z^2 - x^2 - y = 0 \).

b) Verify that the line \( \vec{r}(t) = \langle 1 + t, 1 - t, \sqrt{2}t \rangle \) is part of the hyperboloid \( x^2 + y^2 - z^2 = 2 \).
Lecture 6: Parametrized surfaces

We have seen surfaces described as level surfaces of a function of three variables. There is a fundamentally different way to describe a surface: it is called a parametrization.

A parametrization of a surface is a vector-valued function

\[ \vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle , \]

where \( x(u, v), y(u, v), z(u, v) \) are three functions of two variables. The variables \( u, v \) are called parameters. They define a coordinate system on the surface.

Because two parameters \( u \) and \( v \) are involved, the map \( \vec{r} \) is also called \( uv \)-map. And like \( uv \)-light, it looks cool.

A parametrized surface is the image of the \( uv \)-map. The domain of the \( uv \)-map is called the parameter domain.

If we keep the first parameter \( u \) constant, then \( v \mapsto \vec{r}(u, v) \) is a curve on the surface. Similarly, if \( v \) is constant, then \( u \mapsto \vec{r}(u, v) \) traces a curve the surface. These curves are called grid curves.

A computer draws parametrized surfaces using grid curves. The world of parametric surfaces is intriguing and complex. You can explore this world with the help of a computer. You can survive the parametrization of surfaces topic by knowing four important examples. They are really important because they are cases we can understand well and which consequently will return again and again. They are also building blocks for more general surfaces.

1 Planes: Parametric: \( \vec{r}(s, t) = \vec{O}P + s\vec{v} + t\vec{w} \)

Implicit: \( ax + by + cz = d \). Parametric to implicit: find the normal vector \( \vec{n} = \vec{v} \times \vec{w} \).

Implicit to Parametric: find two vectors \( \vec{v}, \vec{w} \) normal to the vector \( \vec{n} \). For example, find three points \( P, Q, R \) on the surface and forming \( \vec{u} = \vec{P}Q, \vec{v} = \vec{PR} \).
2 **Spheres:**  
Parametric: \( \vec{r}(u, v) = (a, b, c) + (\rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v)) \).
Implicit: \((x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2\).
Parametric to implicit: reading off the radius.
Implicit to parametric: find the center \((a, b, c)\) and the radius \(r\) possibly by completing the square.

![Spheres](image)

3 **Graphs:**  
Parametric: \( \vec{r}(u, v) = (u, v, f(u, v)) \)
Implicit: \( z - f(x, y) = 0 \). Parametric to implicit: look up the function \(f(x, y)\) \(z = f(x, y)\)
Implicit to parametric: use \(x\) and \(y\) as variables.

![Graphs](image)

4 **Surfaces of revolution:**  
Parametric: \( \vec{r}(u, v) = (g(v) \cos(u), g(v) \sin(u), v) \)
Implicit: \( \sqrt{x^2 + y^2} = r = g(z) \) can be written as \( x^2 + y^2 = g(z)^2 \).
Parametric to implicit: read off \(g(z)\) the distance to the \(z\)-axis.
Implicit to parametric: use the radius function \(g(z)\) and think about polar coordinates.

![Surfaces of revolution](image)
A point \((x, y)\) in the plane has the **polar coordinates**

\[
r = \sqrt{x^2 + y^2}, \quad \theta = \arctg(y/x).
\]

We have \((x, y) = (r \cos(\theta), r \sin(\theta))\).

The formula \(\theta = \arctg(y/x)\) defines the angle \(\theta\) only up to an addition of an integer multiple of \(\pi\). The points \((1, 2)\) and \((-1, -2)\) for example have the same \(\theta\) value. In order to get the correct \(\theta\) value one can take \(\arctan(y/x)\) in \((-\pi/2, \pi/2]\), where \(\pi/2\) is the limit when \(x \to 0^+\), then add \(\pi\) if \(x < 0\) or if \(x = 0\) and \(y < 0\).

The coordinate system obtained by representing points in space as

\[
(x, y, z) = (r \cos(\theta), r \sin(\theta), z)
\]

is called the **cylindrical coordinate system**.

Here are some level surfaces in cylindrical coordinates:

<table>
<thead>
<tr>
<th>Number</th>
<th>Surface Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(r = 1) is a <strong>cylinder</strong>, (r =</td>
</tr>
<tr>
<td>6</td>
<td>(r = 2 + \sin(z)) is an example of a <strong>surface of revolution</strong>.</td>
</tr>
</tbody>
</table>

**Spherical coordinates** use the distance \(\rho\) to the origin as well as two angles \(\theta\) and \(\phi\) called **Euler angles**. The first angle \(\theta\) is the angle we have used in polar coordinates. The second angle, \(\phi\), is the angle between the vector \(\overline{OP}\) and the \(z\)-axis. A point has the spherical coordinate

\[
(x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)).
\]

Here are two important figures. The distance to the \(z\) axes \(r = \rho \sin(\phi)\) and the height \(z = \rho \cos(\phi)\) can be read off by the left picture, the coordinates \(x = r \cos(\theta), y = r \sin(\theta)\) can be seen in the right picture.

Here are some level surfaces described in spherical coordinates:

<table>
<thead>
<tr>
<th>Number</th>
<th>Surface Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>(\rho = 1) is a <strong>sphere</strong>, the surface (\phi = \pi/2) is a <strong>single cone</strong>, (\rho = \phi) is an <strong>apple shaped surface</strong> and (\rho = 2 + \cos(3\theta) \sin(\phi)) is an example of a <strong>bumpy sphere</strong>.</td>
</tr>
</tbody>
</table>
Homework

1 Plot the surface with the parametrization

\[ \vec{r}(u, v) = \langle \cos^4(v) \cos(u), \cos^4(v) \sin(u), v \rangle , \]

where \( u \in [0, 2\pi] \) and \( v \in \mathbb{R} \). You can use technology, but it is possible also without.

2 Find a parametrization for the plane which contains the three points \( P = (5, 8, 2), Q = (3, 3, 2) \) and \( R = (2, 4, 6) \).

3 a) Find a parameterisations of the lower half of the ellipsoid \( 25x^2 + 16y^2 + z^2 = 1, z < 0 \) by using that the surface is a graph \( z = f(x, y) \).

b) Find a second parametrization but use angles \( \phi, \theta \) similarly as for the sphere.

4 Find a parametrisation of the \textbf{torus}, given as the set of points which have distance 2 from the circle \( \langle 5 \cos(\theta), 5 \sin(\theta), 0 \rangle \), where \( \theta \) is the angle occurring in cylindrical and spherical coordinates.

\textbf{Hint:} Keep \( u = t \) as a parameters and let \( r \) the distance of a point \( (x, y, z) \) to the \( z \)-axis. This distance is \( r = 5 + 2 \cos(\phi) \) if \( \phi \) is the angle you see on Figure 1. You can read off from the same picture also \( z = \sin(\phi) \). To finish the parametrization problem, translate back from cylindrical coordinates \( (r, \theta, z) = (5 + 2 \cos(\phi), \theta, 2 \sin(\phi)) \) to Cartesian coordinates \( (x, y, z) \).

5 a) What is the equation for the surface \( x^2 + y^2 = 3x + z^2 \) in cylindrical coordinates?

b) Describe in words or draw a sketch of the surface whose equation is \( \rho = |\sin(3\phi)| \) in spherical coordinates \( (\rho, \theta, \phi) \).
Lecture 7: Parametrized curves

A **parametrization** of a planar curve is a map $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. The parametrization of a space curve is $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. The image of the vector valued function $r$ is a **parametrized curve** in space.

We always think of the **parameter** $t$ as **time**. For a fixed time $t$, we have a vector $\langle x(t), y(t), z(t) \rangle$ in space. As $t$ varies, the end point of this vector moves along the curve. The parametrization contains **more information** about the curve than the curve alone. It tells for example, how fast we go along the curve.

1. The parametrization $\mathbf{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ describes a special curve in the plane. It is an example of a **Lissajous curve**.

2. If $x(t) = t$, $y(t) = f(t)$, the curve $\mathbf{r}(t) = \langle t, f(t) \rangle$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola.

3. With $x(t) = 2\cos(t), y(t) = 3\sin(t)$, then $\mathbf{r}(t)$ follows an **ellipse**. We can see this from $x(t)^2/4 + y(t)^2/9 = 1$. We can can overlay an other circular motion to get an epicycle $\mathbf{r}(t) = \langle 2\cos(t) + \cos(31t)/4, 3\sin(t) + \sin(31t)/4 \rangle$.

4. With $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$ we get the parametrization of a **space curve** $\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t \rangle$. It traces a **helix** which has a radius changing linearly.

5. If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in the previous example but the curve is traversed **faster**. The **parameterization** of the curve has changed.
If \( x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t \), then we have the same curve again but we traverse it in the **opposite direction**.

If \( P = (a, b, c) \) and \( Q = (u, v, w) \) are points in space, then \( \vec{r}(t) = \langle a + t(u - a), b + t(v - b), c + t(w - c) \rangle \) defined on \( t \in [0, 1] \) is a line segment connecting \( P \) with \( Q \). For example, \( \vec{r}(t) = \langle 1 + t, 1 - t, 2 + 3t \rangle \) connects the points \( P = (1, 1, 2) \) with \( Q = (2, 0, 1) \).

Sometimes it is possible to eliminate the time parameter \( t \) and write the curve using equations. We have seen the example of **symmetric equations** already. We need one equation to do so in two dimensions but two equations in three dimensions.

If a curve is written as the intersection of two surfaces \( f(x, y, z) = 0, g(x, y, z) = 0 \), this is called an implicit description of a curve.

The symmetric equations describing a line \( (x - x_0)/a = (y - y_0)/b = (z - z_0)/c \) can be seen as the intersection of two surfaces.

If \( f \) and \( g \) are polynomials, the set of points satisfying \( f(x, y, z) = 0, g(x, y, z) = 0 \) is an example of an **algebraic variety**. An example is the set of points in space satisfying \( x^2 - y^2 + z^3 = 0, x^5 - y + z^5 + xy = 3 \). An other example is the set of points satisfying \( x^2 + 4y^2 = 5, x + y + z = 1 \) which is an ellipse in space.

For \( x(t) = t \cos(t), y(t) = t \sin(t), z(t) = t \), then \( x = t \cos(z), y = t \sin(z) \) and we can see that \( x^2 + y^2 = z^2 \). The curve is located on a cone. We also have \( y/x = \tan(z) \) so that we could see the curve as an intersection of two surfaces. Detecting relations between \( x, y, z \) can help to understand the curve.

Curves describe the paths of particles, celestial bodies, or quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time. Here are some more places, where curves appear:

- **Strings or knots** are closed curves in space.
- **Molecules** like RNA or proteins.
- **Graphics:** surfaces are represented by mesh of curves.
- **Typography:** fonts represented by Bézier curves.
- **Relativity:** curve in space-time describes the motion of an object.
- **Topology:** space filling curves, boundaries of surfaces or knots.
If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is called the **velocity** at time $t$. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}''(t)$ is called the **acceleration**. The third derivative $\vec{r}'''(t)$ is called the **jerk**.

Any vector parallel to the velocity $\vec{r}'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

Here are where velocities, acceleration and jerk are computed:

<table>
<thead>
<tr>
<th></th>
<th>Position</th>
<th>Velocity</th>
<th>Acceleration</th>
<th>Jerk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\vec{r}(t)$</td>
<td>$\vec{r}'(t)$</td>
<td>$\vec{r}''(t)$</td>
<td>$\vec{r}'''(t)$</td>
</tr>
<tr>
<td></td>
<td>$\langle \cos(3t), \sin(2t), 2\sin(t) \rangle$</td>
<td>$\langle -3\sin(3t), 2\cos(2t), 2\cos(t) \rangle$</td>
<td>$\langle -9\cos(3t), -4\sin(2t), -2\sin(t) \rangle$</td>
<td>$\langle 27\sin(3t), 8\cos(2t), -2\cos(t) \rangle$</td>
</tr>
</tbody>
</table>

Let’s look at some examples of velocities and accelerations:

<table>
<thead>
<tr>
<th>Signals in nerves:</th>
<th>40 m/s</th>
<th>Train:</th>
<th>0.1-0.3 m/s²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane:</td>
<td>70-900 m/s</td>
<td>Car:</td>
<td>3-8 m/s²</td>
</tr>
<tr>
<td>Sound in air:</td>
<td>Mach 1=340 m/s</td>
<td>Free fall:</td>
<td>1G = 9.81 m/s²</td>
</tr>
<tr>
<td>Speed of bullet:</td>
<td>1200-1500 m/s</td>
<td>Space shuttle:</td>
<td>3G = 30m/s²</td>
</tr>
<tr>
<td>Earth around the sun:</td>
<td>30’000 m/s</td>
<td>Combat plane F16:</td>
<td>9G m/s²</td>
</tr>
<tr>
<td>Sun around galaxy center:</td>
<td>200’000 m/s</td>
<td>Ejection from F16:</td>
<td>14G m/s²</td>
</tr>
<tr>
<td>Light in vacuum:</td>
<td>300’000’000 m/s</td>
<td>Electron in vacuum tube:</td>
<td>$10^{15}$ m/s²</td>
</tr>
</tbody>
</table>

The **addition rule** in one dimension $(f + g)' = f' + g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ and the **chain rule** $(f(g))' = f'(g)g'$ generalize to vector-valued functions because in each component, we have the single variable rule.

$$(\vec{v} + \vec{w})' = \vec{v}' + \vec{w}', \quad (c\vec{v})' = c\vec{v}', \quad (\vec{v} \cdot \vec{w})' = \vec{v}' \cdot \vec{w} + \vec{v} \cdot \vec{w}' \quad (\vec{v} \times \vec{w})' = \vec{v}' \times \vec{w} + \vec{v} \times \vec{w}' \quad (\vec{v}(f(t)))' = \vec{v}'(f(t))f'(t).$$

The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by integration $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s) \, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ at all times as well as initial velocity and position $\vec{r}(0)$ and $\vec{v}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{v}(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s) \, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s) \, ds$.

The **free fall** is the case when acceleration is constant. The direction of the constant force defines what is “down”. If $\vec{r}''(t) = \langle 0, 0, -10 \rangle$, $\vec{r}'(0) = \langle 0, 1000, 2 \rangle$, $\vec{r}(0) = \langle 0, 0, h \rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2 \rangle$. 

![Free fall](image-url)
If \( r''(t) = \vec{F} \) is constant, then \( \vec{r}(t) = \vec{r}(0) + t \vec{r}'(0) - \vec{F} t^2/2 \).

**Homework**

1. Sketch the plane curve \( \vec{r}(t) = \langle x(t), y(t) \rangle = \langle 2 + t^3, t^2 - 1 \rangle \) for \( t \in [-1, 1] \) by plotting the points for different values of \( t \). Calculate its velocity \( \vec{r}'(t) \) as well as the acceleration \( \vec{r}''(t) \) at \( t = 2 \).

2. A device in a car measures the acceleration \( \vec{r}''(t) = \langle \cos(t), -\cos(5t) \rangle \) at time \( t \). Assume the car is at \( (0, 0) \) at time \( t = 0 \) and has velocity \( (1, 0) \) at \( t = 0 \), what is its position \( \vec{r}(t) \) at time \( t \)?

3. Verify that the curve \( \vec{r}(t) = \langle t \cos(t), 3t \sin(t), t^2 \rangle \) is located on the **elliptical paraboloid**

\[
 z = x^2 + \frac{y^2}{9} .
\]

Use this fact to sketch the curve.

4. Find the parameterization \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \) of the curve obtained by intersecting the elliptical cylinder \( x^2/9 + y^2/25 = 1 \) with the surface \( z = xy \). Find the velocity vector \( \vec{r}'(t) \) at the time \( t = \pi/2 \).

5. Consider the curve

\[
 \vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle t^2, 1 + t, 1 + t^3 \rangle .
\]

Check that it passes through the point \( (1, 0, 0) \) and find the velocity vector \( \vec{r}'(t) \), the acceleration vector \( \vec{r}''(t) \) as well as the jerk vector \( \vec{r}'''(t) \) at this point.
Lecture 8: Arc length and curvature

If \( t \in [a, b] \mapsto \vec{r}(t) \) is a curve with velocity \( \vec{r}'(t) \) and speed \( |\vec{r}'(t)| \), then \( L = \int_a^b |\vec{r}'(t)| \, dt \) is called the arc length of the curve.

We will justify in class why this formula is reasonable. In space, we have \( L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt \).

1. The arc length of the circle of radius \( R \) given by \( \vec{r}(t) = (R \cos(t), R \sin(t)) \) parameterized by \( 0 \leq t \leq 2\pi \) is \( 2\pi R \). The answer is \( 2\pi R \).

2. The helix \( \vec{r}(t) = (\cos(t), \sin(t), t) \) has velocity \( \vec{r}'(t) = (-\sin(t), \cos(t), 1) \) and constant speed \( |\vec{r}'(t)| = |(-\sin(t), \cos(t), 1)| = \sqrt{2} \).

3. What is the arc length of the curve \( \vec{r}(t) = (t, \log(t), t^2/2) \)

   \[
   \text{for } 1 \leq t \leq 2? \quad \text{Answer: Because } \vec{r}'(t) = (1, 1/t, t), \text{ we have } \vec{r}'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t| \text{ and } L = \int_1^2 \left(\frac{1}{t} + t \right) \, dt = \log(t) + \frac{t^2}{2} | \bigg|_1^2 = \log(2) + 2 - 1/2. \]

4. Find the arc length of the curve \( \vec{r}(t) = (3t^2, 6t, t^3) \) from \( t = 1 \) to \( t = 3 \).

5. What is the arc length of the curve \( \vec{r}(t) = (\cos^3(t), \sin^3(t)) \)?

   \[
   \text{Answer: We have } |\vec{r}'(t)| = 3\sqrt{\sin^2(t) \cos^4(t) + \cos^2(t) \sin^4(t)} = (3/2) |\sin^2(2t)|. \]

   Therefore, \( \int_0^{2\pi} (3/2) \sin(2t) \, dt = 6 \).

6. Find the arc length of \( \vec{r}(t) = (t^2/2, t^3/3) \) for \(-1 \leq t \leq 1 \). This cubic curve satisfies \( y^2 = x^3/8/9 \) and is an example of an elliptic curve. Because \( \int x\sqrt{1 + x^2} \, dx = (1 + x^2)^{3/2}/3 \), the integral can be evaluated as \( \int_{-1}^1 \frac{x}{\sqrt{1 + x^2}} \, dx = 2 \int_0^{\pi} x \sqrt{1 + x^2} \, dx = 2(1 + x^2)^{3/2}/3 | \bigg|_0^1 = 2(2\sqrt{2} - 1)/3 \).

7. The arc length of an epicycle \( \vec{r}(t) = (t + \sin(t), \cos(t)) \) parameterized by \( 0 \leq t \leq 2\pi \). We have \( |\vec{r}'(t)| = \sqrt{2 + 2 \cos(t)} \), so that \( L = \int_0^{2\pi} \sqrt{2 + 2 \cos(t)} \, dt \). A substitution \( t = 2u \) gives \( L = \int_0^{\pi} \sqrt{2 + 2 \cos(2u)} \, du = \frac{\pi}{2} \sqrt{2 + 2 \cos(2u)} - 2 \sin^2(u) 2du = \int_0^{\pi} \sqrt{4 \cos^2(u) \, 2du = 4 \int_0^{\pi} \sqrt{\cos^2(u) \, du = 8.} \)

8. Find the arc length of the catenary \( \vec{r}(t) = (t, \cosh(t)), \text{ where } \cosh(t) = (e^t + e^{-t})/2 \) is the hyperbolic cosine and \( t \in [-1, 1] \). We have

   \[
   \cosh^2(t)^2 - \sinh^2(t) = 1, \]

where \( \sinh(t) = (e^t - e^{-t})/2 \) is the hyperbolic sine. Solution: \( 2 \sinh(1) \).
Because a parameter change \( t = t(s) \) corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by \( \vec{r}(t) = (\cos(t^2), \sin(t^2)) \) on \( t = [0, \sqrt{2\pi}] \) has the velocity \( \vec{r}'(t) = 2t(-\sin(t), \cos(t)) \) and speed \( 2t \). The arc length is still \( \int_0^{2\pi} 2t \, dt = t^2 \big|_0^{2\pi} = 2\pi \).

Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** \( \vec{r}(t) = (\cos(3t), \sin(5t)) \) leads to the arc length integral \( \int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)} \, dt \) which can only be evaluated numerically.

Define the **unit tangent vector** \( \vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)| \) **unit tangent vector**.

The curvature of a curve at the point \( \vec{r}(t) \) is defined as \( \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \).

The curvature is the length of the acceleration vector if \( \vec{r}(t) \) traces the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You ”see” the curvature, while you ”feel” the acceleration.

The curvature does not depend on the parametrization.

9 Proof. Let \( s(t) \) be an other parametrization, then by the chain rule \( \frac{d}{dt}T'(s(t)) = T'(s(t))s'(t) \) and \( \frac{d}{dt}r(s(t)) = r'(s(t))s'(t) \). We see that the \( s' \) cancels in \( T'/r' \).

Especially, if the curve is parametrized by arc length, meaning that the velocity vector \( r'(t) \) has length 1, then \( \kappa(t) = |T'(t)| \). It measures the rate of change of the unit tangent vector.

11 The curve \( \vec{r}(t) = (t, f(t)) \), which is the graph of a function \( f \) has the velocity \( \vec{r}'(t) = (1, f'(t)) \) and the unit tangent vector \( \vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2} \). After some simplification we get

\[ \kappa(t) = |\vec{T}'(t)|/|\vec{r}'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3 \]

For example, for \( f(t) = \sin(t) \), then \( \kappa(t) = |\sin(t)|/\sqrt{1 + \cos^2(t)}^3 \).

If \( \vec{r}(t) \) is a curve which has nonzero speed at \( t \), then we can define \( \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \), the **unit tangent vector**, \( \vec{N}(t) = \frac{\vec{T}(t)}{|\vec{T}(t)|} \), the normal vector and \( \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \) the bi-normal vector. The plane spanned by \( N \) and \( B \)is called the normal plane. It is perpendicular to the curve. The plane spanned by \( T \) and \( N \) is called the osculating plane.

If we differentiate \( \vec{T}(t) \cdot \vec{T}(t) = 1 \), we get \( \vec{T}'(t) \cdot \vec{T}(t) = 0 \) and see that \( \vec{N}(t) \) is perpendicular to \( \vec{T}(t) \). Because \( B \) is automatically normal to \( T \) and \( N \), we have shown:
The three vectors \((\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))\) are unit vectors orthogonal to each other.

Here is an application of curvature: If a curve \(\mathbf{r}(t)\) represents a wave front and \(\mathbf{n}(t)\) is a unit vector normal to the curve at \(\mathbf{r}(t)\), then \(\mathbf{s}(t) = \mathbf{r}(t) + \mathbf{n}(t)/\kappa(t)\) defines a new curve called the caustic of the curve. Geometers call that curve the evolute of the original curve.

A useful formula for curvature is

\[
\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}
\]

We prove this in class. Finally, lets mention that curvature is important also in computer vision. If the gray level value of a picture is modeled as a function \(f(x, y)\) of two variables, places where the level curves of \(f\) have maximal curvature corresponds to corners in the picture. This is useful when tracking or identifying objects.

In this picture of John Harvard, software was looking for level curves for each color and started to draw at points where the curvature of the curves is large then follow the level curve.
Homework

1. Find the arc length of the curve 
   \[ \mathbf{r}'(t) = \langle 2t^2, 2\sin(t) - 2t\cos(t), 2\cos(t) + 2t\sin(t) \rangle, \]
   where the time parameter satisfies \(0 \leq t \leq \pi\).

2. Find the curvature of \( \mathbf{r}'(t) = \langle e^t\cos(t), e^t\sin(t), t \rangle \) at the point \((1, 0, 0)\).

3. Find the vectors \( \mathbf{T}(t), \mathbf{N}(t) \) and \( \mathbf{B}(t) \) for the curve \( \mathbf{r}(t) = \langle t^2, t^3, 0 \rangle \) for \( t = 2 \). Explore whether the vectors depend continuously on \( t \) for all \( t \).

4. Let \( \mathbf{r}(t) = \langle t, t^2 \rangle \). Find the equation for the caustic 
   \[ \mathbf{s}(t) = \mathbf{r}(t) + \frac{\mathbf{N}(t)}{\kappa(t)}. \]
   It is known also as the evolute of the curve.

5. If \( \mathbf{r}(t) = \langle -\sin(t), \cos(t) \rangle \) is the boundary of a coffee cup and light enters in the direction \( \langle -1, 0 \rangle \), then light focuses inside the cup on a curve which is called the coffee cup caustic. The light ray travels after the reflection for length \( \sin(\theta)/(2\kappa) \) until it reaches the caustic. Find a parameterization of the caustic.
Lecture 9: Partial derivatives

If \( f(x,y) \) is a function of two variables, then \( \frac{\partial}{\partial x} f(x,y) \) is defined as the derivative of the function \( g(x) = f(x,y) \), where \( y \) is considered a constant. It is called the \textbf{partial derivative} of \( f \) with respect to \( x \). The partial derivative with respect to \( y \) is defined similarly.

We use the short hand notation \( f_x(x,y) = \frac{\partial}{\partial x} f(x,y) \). For iterated derivatives, the notation is similar: for example \( f_{xy} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f \). The meaning of \( f_x(x_0, y_0) \) is the slope of the graph sliced at \((x_0, y_0)\) in the \( x \) direction. The second derivative \( f_{xx} \) is a measure of concavity in that direction. The meaning of \( f_{xy} \) is the rate of change of the slope if you change the slicing.

The notation for partial derivatives \( \partial_x f, \partial_y f \) was introduced by Carl Gustav Jacobi. Josef Lagrange had used the term ”partial differences”. Partial derivatives \( f_x \) and \( f_y \) measure the rate of change of the function in the \( x \) or \( y \) directions. For functions of more variables, the partial derivatives are defined in a similar way.

1. For \( f(x,y) = x^4 - 6x^2y^2 + y^4 \), we have \( f_x(x,y) = 4x^3 - 12xy^2 \), \( f_{xx} = 12x^2 - 12y^2 \), \( f_y(x,y) = -12x^2y + 4y^3 \), \( f_{yy} = -12x^2 + 12y^2 \) and see that \( f_{xx} + f_{yy} = 0 \). A function which satisfies this equation is also called \textbf{harmonic}. The equation \( f_{xx} + f_{yy} = 0 \) is an example of a \textbf{partial differential equation}: it is an equation for an unknown function \( f(x,y) \) which involves partial derivatives with respect to more than one variables.

\textbf{Clairaut’s theorem} If \( f_{xy} \) and \( f_{yx} \) are both continuous, then \( f_{xy} = f_{yx} \).

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant \( h \) is positive, then \( f_x(x,y) = [f(x+h,y) - f(x,y)]/h \). For \( h = 0 \) we define \( f_x \) as before. Compare the two sides for fixed \( h > 0 \):

\[
\begin{align*}
\frac{h}{x} f_x(x,y) &= f(x+h,y) - f(x,y) \\
\frac{h^2}{x^2} f_{xx}(x,y) &= f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x, y)
\end{align*}
\]

We have not taken any limits in this proof. We have established an identity which holds for all \( h > 0 \): the discrete derivatives \( f_x, f_y \) satisfy the relation \( f_{xy} = f_{yx} \). We could fancy the identity obtained in the proof as a ”quantum Clairaut” theorem. If the classical derivatives \( f_{xy}, f_{yx} \) are both continuous, we can take the limit \( h \to 0 \) to get the classical Clairaut’s theorem as a ”classical limit”. Note that the quantum Clairaut theorem shown first in this proof holds for all functions \( f(x,y) \) of two variables. We do not even need continuity.

2. Find \( f_{xxxxxyyyyy} \) for \( f(x) = \sin(x) + x^6y^{10} \cos(y) \). \textbf{Answer}: Do not compute, but think.

3. The continuity assumption for \( f_{xy} \) is necessary. The example

\[
f(x,y) = \frac{x^3y - xy^3}{x^2 + y^2}
\]

contradicts Clairaut’s theorem.
\[ f_x(x, y) = (3x^2y - y^3)/(x^2 + y^2) - 2x(x^3y - xy^3)/(x^2 + y^2)^2, \quad f_x(0, y) = -y, \quad f_{xy}(0, 0) = -1, \]
\[ f_y(x, y) = (x^3 - 3xy^2)/(x^2 + y^2) - 2y(x^3y - xy^3)/(x^2 + y^2)^2, \quad f_y(x, 0) = x, \quad f_{y,x}(0, 0) = 1. \]

An equation for an unknown function \( f(x, y) \) which involves partial derivatives with respect to at least two different variables is called a **partial differential equation** (PDE) If only the derivative with respect to one variable appears, it is an **ordinary differential equation** (ODE).

Here are examples of partial differential equations. You have to know the first four in the same way than a chemist has to know what \( H_2O, CO_2, CH_4, NaCl \) is. Of course, as more to know, as better: rubber \( C_5H_8 \), Asprin \( C_9H_8C_4 \), Ethanol \( C_2H_6 \), Ammonia \( NH_3 \) etc.

4 The **wave equation** \( f_{tt}(t, x) = f_{xx}(t, x) \) governs the motion of light or sound. The function \( f(t, x) = \sin(x - t) + \sin(x + t) \) satisfies the wave equation.

5 The **heat equation** \( f_t(t, x) = f_{xx}(t, x) \) describes diffusion of heat or spread of an epidemic. The function \( f(t, x) = \frac{1}{\sqrt{\pi}} e^{-x^2/(4t)} \) satisfies the heat equation.

6 The **Laplace equation** \( f_{xx} + f_{yy} = 0 \) determines the shape of a membrane. The function \( f(x, y) = x^3 - 3xy^2 \) is an example satisfying the Laplace equation.

7 The **advection equation** \( f_t = f_x \) is used to model transport in a wire. The function \( f(t, x) = e^{-(x+t)^2} \) satisfies the advection equation.

8 The **eiconal equation** \( f_x^2 + f_y^2 = 1 \) is used to see the evolution of wave fronts in optics. The function \( f(x, y) = \cos(x) + \sin(y) \) satisfies the eiconal equation.

9 The **Burgers equation** \( f_t + ff_x = f_{xx} \) describes waves at the beach which break. The function \( f(t, x) = \frac{x}{t} \sqrt{\frac{1}{2} e^{-x^2/(4t)}} \) satisfies the Burgers equation.

10 The **KdV equation** \( f_t + 6ff_x + f_{xxx} = 0 \) models water waves in a narrow channel. The function \( f(t, x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2t)) \) satisfies the KdV equation.

11 The **Schrödinger equation** \( f_t = i\hbar m f_{xx} \) is used to describe a quantum particle of mass \( m \). The function \( f(t, x) = e^{i(kx - \frac{\hbar k^2}{2m}t)} \) solves the Schrödinger equation. [Here \( i^2 = -1 \) is the imaginary \( i \) and \( \hbar \) is the Planck constant \( \hbar \sim 10^{-34} Js \).]

Here are the graphs of the solutions of the equations. Can you match them with the PDE’s?
Notice that in all these examples, we have just given one possible solution to the partial differential equation. There are in general many solutions and only additional conditions like initial or boundary conditions determine the solution uniquely. If we know \( f(0, x) \) for the Burgers equation, then the solution \( f(t, x) \) is determined. A course on partial differential equations would show you how to get the solution.

Paul Dirac once said: ”A great deal of my work is just playing with equations and seeing what they give. I don’t suppose that applies so much to other physicists; I think it’s a peculiarity of myself that I like to play about with equations, just looking for beautiful mathematical relations which maybe don’t have any physical meaning at all. Sometimes they do.” Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac’s equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an antiparticle: the existence of antiparticles was later confirmed. We will not learn here to find solutions to partial differential equations. But you should be able to verify that a given function is a solution of the equation.

**Homework**

1. Verify that \( f(t, x) = \tan(\sin(t + x)) \) is a solution of the transport equation \( f_t(t, x) = f_x(t, x) \).

2. a) Verify that \( f(x, y) = \cos(x)(\cos(2y) + \sin(2y)) \) satisfies the Klein Gordon equation \( u_{xx} - u_{yy} = 3u \). This PDE is useful in quantum mechanics.

   b) Verify that more generally, \( \cos(bx)(\cos(ay) + \sin(ay)) \) satisfies the Klein gordon equation \( u_{xx} - u_{yy} = (a^2 - b^2)u \).
3 Verify that \( f(x, t) = e^{-rt} \sin(x + ct) \) satisfies the driven transport equation \( f_t(x, t) = cf_x(x, t) - rf(x, t) \) It is sometimes also called the advection equation.

4 The partial differential equation \( f_{xx} + f_{yy} = f_{tt} \) is called the wave equation in two dimensions. It describes waves in a pool for example.

a) Show that if \( f(x, y, t) = \sin(nx + my) \sin(\sqrt{n^2 + m^2} t) \) satisfies the wave equation. It describes waves in a square where \( x \in [0, \pi] \) and \( y \in [0, \pi] \). The waves are zero at the boundary of the pool.

b) For which \( k \) is \( f(x, y, t) = \sin(nx) \cos(nt) + \sin(mx) \cos(mt) + \sin(nx+my) \cos(kt) \) do we get solution of the wave equation which is periodic in time? You might want to know that integers \( m, n, k \) which satisfy \( m^2 + n^2 = k^2 \) are called Pythagorean triples.
The partial differential equation $f_t + f f_x = f_{xx}$ is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the Navier-Stokes equation which are used to describe the weather. Verify that the function

$$f(t, x) = \left( \frac{1}{t} \right)^{3/2} xe^{-\frac{x^2}{4t}}$$

is a solution of the Burgers equation.

**Remark.** This calculation needs perseverance, when done by hand. You are welcome to use technology if you should get stuck. Here is an example on how to check that a function is a solution of the heat equation in Mathematica:

```mathematica
f[t_, x_] := (1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[D[f[t, x], t] == D[f[t, x], {x, 2}]]
```

And here is the function

$$f(t, x) := \frac{(1/t)^{(3/2)}*x*Exp[-(x^2)/(4t)]/((1/t)^{(1/2)}*Exp[-x^2/(4t)])}{(1/t)^{(1/2)}*Exp[-x^2/(4t)]}$$
Lecture 10: Linearization

In single variable calculus, you have seen the following definition:

The linear approximation of \( f(x) \) at a point \( a \) is the linear function

\[
L(x) = f(a) + f'(a)(x - a) .
\]

It's important to think about this in terms of functions and not graphs because for functions of three variables and more we can not draw graphs anymore.

The graph of the function \( L \) is close to the graph of \( f \) at \( a \). We generalize this to higher dimensions:

The linear approximation of \( f(x, y) \) at \( (a, b) \) is the linear function

\[
L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) .
\]

The linear approximation of a function \( f(x, y, z) \) at \( (a, b, c) \) is

\[
L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) .
\]

Using the gradient \( \nabla f(x, y) = \langle f_x, f_y \rangle \), \( \nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle \),

the linearization can be written more compactly as

\[
L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .
\]

How do we justify the linearization? If the second variable \( y = b \) is fixed, we have a one-dimensional situation, where the only variable is \( x \). Now \( f(x, b) = f(a, b) + f_x(a, b)(x - a) \) is the linear approximation. Similarly, if \( x = x_0 \) is fixed \( y \) is the single variable, then \( f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0) \). Knowing the linear approximations in both the \( x \) and \( y \) variables, we can get the general linear approximation by \( f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \).
1. What is the linear approximation of the function \( f(x, y) = \sin(\pi xy^2) \) at the point \((1, 1)\)? We have \((f_x(x, y), f_y(x, y)) = (\pi y^2 \cos(\pi xy^2), 2y \pi \cos(\pi xy^2))\) which is at the point \((1, 1)\) equal to \(\nabla f(1, 1) = (\pi \cos(\pi), 2\pi \cos(\pi)) = (-\pi, 2\pi)\).

2. Linearization can be used to estimate functions near a point. In the previous example, 

\[ -0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942 \, . \]

3. Here is an example in three dimensions: find the linear approximation to \( f(x, y, z) = xy + yz + zx \) at the point \((1, 1, 1)\). Since \( f(1, 1, 1) = 3 \), and \( \nabla f(x, y, z) = (y + z, x + z, y + x) \), \( \nabla f(1, 1, 1) = (2, 2, 2) \). we have \( L(x, y, z) = f(1, 1, 1) + (2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3 \).

4. Estimate \( f(0.01, 24.8, 1.02) \) for \( f(x, y, z) = e^{x \sqrt{yz}} \).

**Solution:** take \((x_0, y_0, z_0) = (0, 25, 1)\), where \( f(x_0, y_0, z_0) = 5 \). The gradient is \( \nabla f(x, y, z) = (e^{x \sqrt{yz}}, e^{x \sqrt{z}}, e^{x \sqrt{y}}) \). At the point \((x_0, y_0, z_0) = (0, 25, 1)\) the gradient is the vector \((5, 1/10, 5)\). The linear approximation is \( L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5 \). We can approximate \( f(0.01, 24.8, 1.02) \) by \( 5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13 \). The actual value is \( f(0.01, 24.8, 1.02) = 5.1306 \), very close to the estimate.

5. Find the tangent line to the graph of the function \( g(x) = x^2 \) at the point \((2, 4)\).

**Solution:** the level curve \( f(x, y) = y - x^2 = 0 \) is the graph of a function \( g(x) = x^2 \) and the tangent at a point \((2, g(2)) = (2, 4)\) is obtained by computing the gradient \( \langle a, b \rangle = \nabla f(2, 4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle \) and forming \(-4x + y = d\), where \( d = -4 \cdot 2 + 1 \cdot 4 = -4 \). The answer is \( -4x + y = -4 \) which is the line \( y = 4x - 4 \) of slope 4.

6. The **Barth surface** is defined as the level surface \( f = 0 \) of

\[
\begin{align*}
    f(x, y, z) &= (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 \\
    &+ 8(x^2 - t^4y^2)(-(t^4x^2 + z^2)(y^2 - t^4y^2)(y^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4),
\end{align*}
\]

where \( t = (\sqrt{5} + 1)/2 \) is a constant called the golden ratio. If we replace \( t \) with \( 1/t = (\sqrt{5} - 1)/2 \) we see the surface to the middle. For \( t = 1 \), we see to the right the surface \( f(x, y, z) = 8 \). Find the tangent plane of the later surface at the point \((1, 1, 0)\). **Answer:** We have \( \nabla f(1, 1, 0) = (64, 64, 0) \). The surface is \( x + y = d \) for some constant \( d \). By plugging in \((1, 1, 0)\) we see that \( x + y = 2 \).
The quartic surface

\[ f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0 \]

is called the **piriform**. What is the equation for the tangent plane at the point \( P = (2, 2, 2) \) of this pair shaped surface? We get \( \langle a, b, c \rangle = \langle 20, 4, 4 \rangle \) and so the equation of the plane \( 20x + 4y + 4z = 56 \), where we have obtained the constant to the right by plugging in the point \( (x, y, z) = (2, 2, 2) \).

**Remark:** some traditional text books like Stewart use **differentials** to describe linearizations. We recommend to avoid this 19th century notation and terminology. Newton has used used terms like “fluxions”, Leibniz ”differentials”, its time to move on. For us, the linearization of a function \( f \) at a point is a **linear function** \( L \) in the same number of variables defined above. 20th century mathematics has invented the notion of **differential forms** which is an extremely valuable mathematical notion, but it is a concept which needs some multi-linear algebra and becomes only useful in more advanced topics like differential geometry or topology. Similarly, the notion of **infinitesimal small quantities** has been made precise in a language called nonstandard analysis but it needs a considerable amount of background knowledge in logic to be appreciated and understood. The notion of ”differentials” comes from a time when calculus was still foggy in some areas. Unfortunately, the term has survived and appears even still in some calculus books. If you are not convinced by what was just said, try to find out (by looking on the web) what people mean with ”differential”: you find notions like ”change in the linearization of a function” or ”infinitesimals” which are both good examples of what is ”foggy terminology”. To add to the confusion, sometimes also terms like \( dx \) are called a “differentials”. This appears in integrals \( \int \sin(x) \, dx \) but note that in that case, it is just used as part of the **notation** to indicate with respect to which variable we integrate. Mathematica for example writes this as \( \text{Integrate}[\sin[x], x] \) and there is no mystery. It just means to find the anti-derivative of \( \sin(x) \) with respect to \( x \).
Homework

1. Given \( f(x, y) = \sin(x) - yx/\pi \). Estimate \( f(\pi + 0.01, 0.97) \) using linearization.

2. Estimate \( 10'000'000^{1/10} \) using linear approximation.

3. Estimate \( f(0.01, 0.9999) \) for \( f(x, y) = \cos(\pi xy)y + \sin(x + \pi y) \) using linearization.

4. Find the linear approximation \( L(x, y) \) of the function
   \[
   f(x, y) = \sqrt{10 - x^2 - 5y^2}
   \]
   at \((2, 1)\) and use it to estimate \( f(1.95, 1.04) \).

5. Sketch a contour map of the function
   \[
   f(x, y) = x^2 + 9y^2
   \]
   find the gradient vector \( \nabla f = \langle f_x, f_y \rangle \) of \( f \) at the point \((1, 1)\). Draw it together with the tangent line \( ax + by = d \) to the curve at \((1, 1)\).
Lecture 11: Chain rule

If \( f \) and \( g \) are functions of a single variable \( t \), the **single variable chain rule** tells us that \( \frac{d}{dt} f(g(t)) = f'(g(t))g'(t) \). For example, \( \frac{d}{dt} \sin(\log(t)) = \cos(\log(t))/t \).

It can be proven by linearizing the functions \( f \) and \( g \) and verifying the chain rule in the linear case. The **chain rule** is also useful:

For example, to find \( \arccos'(x) \), we differentiate \( x = \cos(\arccos(x)) \) to get \( 1 = \frac{d}{dx} \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))} \arccos'(x) = -\sqrt{1-x^2} \arccos'(x) \) so that \( \arccos'(x) = -1/\sqrt{1-x^2} \).

Define the **gradient** \( \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \) or \( \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \).

If \( \vec{r}(t) \) is curve and \( f \) is a function of several variables we can build a function \( t \mapsto f(\vec{r}(t)) \) of one variable. Similarly, If \( \vec{r}(t) \) is a parametrization of a curve in the plane and \( f \) is a function of two variables, then \( t \mapsto f(\vec{r}(t)) \) is a function of one variable.

The **multivariable chain rule** is \( \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \).

Proof. When written out in two dimensions, it is

\[
\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).
\]

Now, the identity

\[
\frac{f(x(t)+h), y(t+h)) - f(x(t), y(t))}{h} = \frac{f(x(t)+h), y(t+h)) - f(x(t), y(t+h))}{h} + \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{h}
\]

holds for every \( h > 0 \). The left hand side converges to \( \frac{df}{dt}(x(t), y(t)) \) in the limit \( h \to 0 \) and the right hand side to \( f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \) using the single variable chain rule twice. Here is the proof of the later, when we differentiate \( f \) with respect to \( t \) and \( y \) is treated as a constant:

\[
\frac{f(x(t+h)) - f(x(t))}{h} = \frac{\left[ f(x(t) + (x(t+h)-x(t))) - f(x(t)) \right]}{h} \cdot \frac{[x(t+h)-x(t)]}{h}.
\]

Write \( H(t) = x(t+h)-x(t) \) in the first part on the right hand side.

\[
\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t) + H) - f(x(t))]}{H} \cdot \frac{x(t+h) - x(t)}{h}.
\]

As \( h \to 0 \), we also have \( H \to 0 \) and the first part goes to \( f'(x(t)) \) and the second factor to \( x'(t) \).
1 We move on a circle \( r(t) = (\cos(t), \sin(t)) \) on a table with temperature distribution \( f(x, y) = x^2 - y^3 \). Find the rate of change of the temperature \( \nabla f(x, y) = (2x, -3y^2) \), \( r'(t) = (-\sin(t), \cos(t)) \) \( d/dt f(r(t)) = \nabla T(r(t)) \cdot r'(t) = (2 \cos(t), -3 \sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2 \cos(t) \sin(t) - 3 \sin^2(t) \cos(t) \).

From \( f(x, y) = 0 \) one can express \( y \) as a function of \( x \). From \( d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = \), we obtain \( y' = -f_x/f_y \). Even so, we do not know \( y(x) \), we can compute its derivative! Implicit differentiation works also in three variables. The equation \( f(x, y, z) = c \) defines a surface. Near a point where \( f_z \) is not zero, the surface can be described as a graph \( z = z(x, y) \). We can compute the derivative \( z_x \) without actually knowing the function \( z(x, y) \). To do so, we consider \( y \) a fixed parameter and compute using the chain rule

\[
f_x(x, y, z(x, y)) + f_z(x, y)z_x(x, y) = 0
\]

so that \( z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z) \).

2 The surface \( f(x, y, z) = x^2 + y^2/4 + z^2/9 = 6 \) is an ellipsoid. Compute \( z_x(x, y) \) at the point \( (x, y, z) = (2, 1, 1) \).

**Solution:** \( z_x(x, y) = -f_x(2, 1, 1)/f_z(2, 1, 1) = -4/(2/9) = -18 \).

The chain rule is powerful because it implies other differentiation rules like the addition, product and quotient rule in one dimensions: \( f(x, y) = x+y, x = u(t), y = v(t), d/dt(x+y) = f_xu' + f_yv' = u' + v' \).

\[
f(x, y) = xy, x = u(t), y = v(t), d/dt(xy) = f_xu' + f_yv' = uv' + vu'.
\]

\[
f(x, y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_xu' + f_yv' = u'/y - v'/y^2.
\]

As in one dimensions, the chain rule follows from linearization. If \( f \) is a linear function \( f(x, y) = ax + by - c \) and if the curve \( r(t) = (x_0 + tu, y_0 + tv) \) parametrizes a line. Then \( \frac{d}{dt} f(r(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv \) and this is the dot product of \( \nabla f = (a, b) \) with \( r'(t) = (u, v) \). Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.
Homework

1 You know that \( d/dt f(\vec{r}(t)) = 3 \) at \( t = 0 \) if \( \vec{r}(t) = \langle t, t \rangle \) and \( d/dt f(\vec{r}(t)) = 5 \) at \( t = 0 \). \( \vec{r}(t) = \langle t, -t \rangle \). Find the gradient of \( f \) at \((0,0)\).

2 The pressure in the space at the position \((x, y, z)\) is \( p(x, y, z) = x^2 + y^2 - z^3 \) and the trajectory of an observer is the curve \( \vec{r}(t) = \langle t, t, 1/t \rangle \). Using the chain rule, compute the rate of change of the pressure the observer measures at time \( t = 2 \).

3 The chain rule is closely related to linearization as it could be proven by linearization. Let’s get back to linearization a bit: A farm costs \( f(x, y) \), where \( x \) is the number of cows and \( y \) is the number of ducks. There are 10 cows and 20 ducks and \( f(10, 20) = 1000000 \). We know that \( f_x(x, y) = 2x \) and \( f_y(x, y) = y^2 \) for all \( x, y \). Estimate \( f(12, 19) \).

P.S. In the fall of 2013, Oliver made a song out of this:

”Old MacDonald had a million dollar farm, E-I-E-I-O, and on that farm he had \( x = 10 \) cows, E-I-E-I-O, and on that farm he had \( y = 20 \) ducks, E-I-E-I-O, with \( f_x = 2x \) here and \( f_y = y^2 \) there, and here two cows more, and there a duck less, how much does the farm cost now, E-I-E-I-O?”

4 Derive using implicit differentiation the derivative \( d/dx \arctanh(x) \), where

\[
\tanh(x) = \frac{\sinh(x)}{\cosh(x)}.
\]
The hyperbolic sine and hyperbolic cosine are defined as
are \( \sinh(x) = (e^x - e^{-x})/2 \) and \( \cosh(x) = (e^x + e^{-x})/2 \). We have \( \sinh' = \cosh \) and \( \cosh' = \sinh \) and \( \cosh^2(x) - \sinh^2(x) = 1 \).

The equation \( f(x, y, z) = e^{xyz} + z = 1 + e \) implicitly defines \( z \) as a function \( z = g(x, y) \) of \( x \) and \( y \). Find formulas (in terms of \( x, y \) and \( z \)) for \( g_x(x, y) \) and \( g_y(x, y) \). Estimate \( g(1.01, 0.99) \) using linear approximation.
Lecture 12: Gradient

The gradient of a function \( f(x, y) \) is defined as

\[
\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.
\]

For functions of three dimensions, define

\[
\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle.
\]

The symbol \( \nabla \) is spelled "Nabla" and named after an Egyptian harp. Here is a very important fact:

Gradients are orthogonal to level curves and level surfaces.

Proof. Every curve \( \vec{r}(t) \) on the level curve or level surface satisfies \( \frac{d}{dt} f(\vec{r}(t)) = 0 \). By the chain rule, \( \nabla f(\vec{r}(t)) \) is perpendicular to the tangent vector \( \vec{r}'(t) \).

Because \( \vec{n} = \nabla f(p, q) = \langle a, b \rangle \) is perpendicular to the level curve \( f(x, y) = c \) through \((p, q)\), the equation for the tangent line is \( ax + by = d \), \( a = f_x(p, q) \), \( b = f_y(p, q) \), \( d = ap + bq \). Compactley written, this is

\[
\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0
\]

and means that the gradient of \( f \) is perpendicular to any vector \((\vec{x} - \vec{x}_0)\) in the plane. It is one of the most important statements in multivariable calculus since it provides a crucial link between calculus and geometry. The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines:
1 Compute the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1, 1, 1)$. **Solution:**
\[ \nabla f(x, y, z) = (6xy, 3x^2, 2z). \] And \( \nabla f(1, 1, 1) = (6, 3, 2) \). The plane is \( 6x + 3y + 2z = d \) where \( d \) is a constant. We can find the constant \( d \) by plugging in a point and get \( 6x + 3y + 2z = 11 \).

2 **Problem:** reflect the ray \( \vec{r}(t) = (1 - t, -t, 1) \) at the surface
\[ x^4 + y^2 + z^6 = 6. \]
**Solution:** \( \vec{r}(t) \) hits the surface at the time \( t = 2 \) in the point \((-1, -2, 1)\). The velocity vector in that ray is \( \vec{v} = (-1, -1, 0) \) The normal vector at this point is \( \nabla f(-1, -2, 1) = (-4, -4, 6) = \vec{n} \). The reflected vector is
\[ R(\vec{v}) = 2\text{Proj}_n(\vec{v}) - \vec{v}. \]
We have \( \text{Proj}_n(\vec{v}) = 8/68(-4, -4, 6) \). Therefore, the reflected ray is \( \vec{w} = (4/17)(-4, -4, 6) - (-1, -1, 0) \).
If \( f \) is a function of several variables and \( \vec{v} \) is a unit vector then \( D_{\vec{v}} f = \nabla f \cdot \vec{v} \) is called the \textbf{directional derivative} of \( f \) in the direction \( \vec{v} \).

The name directional derivative is related to the fact that every unit vector gives a direction. If \( \vec{v} \) is a unit vector, then the chain rule tells us \( \frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v}) \).

The directional derivative tells us how the function changes when we move in a given direction. Assume for example that \( T(x, y, z) \) is the temperature at position \((x, y, z)\). If we move with velocity \( \vec{v} \) through space, then \( D_{\vec{v}} T \) tells us at which rate the temperature changes for us. If we move with velocity \( \vec{v} \) on a hilly surface of height \( h(x, y) \), then \( D_{\vec{v}} h(x, y) \) gives us the slope we drive on.

If \( \vec{r}(t) \) is a curve with velocity \( \vec{r}'(t) \) and the speed is 1, then \( D_{\vec{r}(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \) is the temperature change, one measures at \( \vec{r}(t) \). The chain rule told us that this is \( d/dt f(\vec{r}(t)) \).

For \( \vec{v} = (1, 0, 0) \), then \( D_{\vec{v}} f = \nabla f \cdot \vec{v} = f_x \), the directional derivative is a generalization of the partial derivatives. It measures the rate of change of \( f \), if we walk with unit speed into that direction. But as with partial derivatives, it is a \textbf{scalar}.

The directional derivative satisfies \( |D_{\vec{v}} f| \leq |\nabla f||\vec{v}| \) because \( \nabla f \cdot \vec{v} = |\nabla f||\vec{v}||\cos(\phi)| \leq |\nabla f||\vec{v}| \).

The direction \( \vec{v} = \nabla f/|\nabla f| \) is the direction, where \( f \) \textbf{increases} most. It is the \textbf{direction of steepest ascent}.

If \( \vec{v} = \nabla f/|\nabla f| \), then the directional derivative is \( \nabla f \cdot \nabla f/|\nabla f| = |\nabla f| \). This means \( f \) \textbf{increases}, if we move into the direction of the gradient. The slope in that direction is \(|\nabla f|\).

You are on a trip in a air-ship over Cambridge at \((1, 2)\) and you want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function \( p(x, y) = x^2 + 2y^2 \). In which direction do you have to fly so that the pressure change is largest?

\textbf{Solution:} The gradient \( \nabla p(x, y) = \langle 2x, 4y \rangle \) at the point \((1, 2)\) is \( \langle 2, 8 \rangle \). Normalize to get the direction \( \langle 1, 4 \rangle/\sqrt{17} \).

The directional derivative has the same properties than any derivative: \( D_{\vec{v}}(\lambda f) = \lambda D_{\vec{v}} f \), \( D_{\vec{v}}(f + g) = D_{\vec{v}} f + D_{\vec{v}} g \) and \( D_{\vec{v}}(fg) = D_{\vec{v}} f g + f D_{\vec{v}} g \).

We will see later that points with \( \nabla f = 0 \) are candidates for \textbf{local maxima} or \textbf{minima} of \( f \). Points \((x, y)\), where \( \nabla f(x, y) = (0, 0) \) are called \textbf{critical points} and help to understand the function \( f \).

The Matterhorn is a 4’478 meter high mountain in Switzerland. It is quite easy to climb with a guide because there are ropes and ladders at difficult places. Evenso there are quite many climbing accidents at the Matterhorn, this does not stop you from trying an
ascent. In suitable units on the ground, the height \( f(x, y) \) of the Matterhorn is approximated by the function \( f(x, y) = 4000 - x^2 - y^2 \). At height \( f(-10, 10) = 3800 \), at the point \((-10, 10, 3800)\), you rest. The climbing route continues into the south-east direction \( v = \langle 1, -1 \rangle / \sqrt{2} \). Calculate the rate of change in that direction. We have \( \nabla f(x, y) = \langle -2x, -2y \rangle \), so that \( 20 \cdot \langle 1, -1 \rangle / \sqrt{2} = 40 / \sqrt{2} \). This is a place, with a ladder, where you climb 40 / \sqrt{2} meters up when advancing 1m forward.

The rate of change in all directions is zero if and only if \( \nabla f(x, y) = 0 \): if \( \nabla f \neq 0 \), we can choose \( \vec{v} = \nabla f / |\nabla f| \) and get \( D_{\nabla f} f = |\nabla f| \).

Assume we know \( D_v f(1, 1) = 3 / \sqrt{5} \) and \( D_w f(1, 1) = 5 / \sqrt{5} \), where \( v = \langle 1, 2 \rangle / \sqrt{5} \) and \( w = \langle 2, 1 \rangle / \sqrt{5} \). Find the gradient of \( f \). Note that we do not know anything else about the function \( f \).

**Solution:** Let \( \nabla f(1, 1) = \langle a, b \rangle \). We know \( a + 2b = 3 \) and \( 2a + b = 5 \). This allows us to get \( a = 7/3, b = 1/3 \).

**Homework**

1. Find the directional derivative \( D_{\vec{v}} f(2, 1) = \nabla f(2, 1) \cdot \vec{v} \) into the direction \( \vec{v} = \langle -3, 4 \rangle / 5 \) for the function \( f(x, y) = x^5 y + y^3 + x + y \).

2. A surface \( x^2 + y^2 - z = 1 \) radiates light away. It can be parametrized as \( \vec{r}(x, y) = \langle x, y, x^2 + y^2 - 1 \rangle \). Find the parametrization of the wave front which is distance 1 from the surface.

3. Assume \( f(x, y) = 1 - x^2 + y^2 \). Compute the directional derivative \( D_{\vec{v}} f(x, y) \) at \((0, 0)\), where \( \vec{v} = \langle \cos(t), \sin(t) \rangle \) is a unit vector. Now compute \( D_v D_v f(x, y) \) at \((0, 0)\), for any unit vector. For which directions is this second directional derivative positive?

4. The **Kitchen-Rosenberg formula** gives the curvature of a level curve \( f(x, y) = c \) as

\[
\kappa = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x^2 + f_y^2)^{3/2}}
\]
Use this formula to find the curvature of the ellipsoid \( f(x, y) = x^2 + 2y^2 = 1 \) at the point \((1, 0)\).

This formula is useful in computer vision. If you want to derive the formula, you can check that the angle

\[
g(x, y) = \arctan\left(\frac{f_y}{f_x}\right)
\]

of the gradient vector has \( \kappa \) as the directional derivative in the direction \( \vec{v} = \langle -f_y, f_x \rangle / \sqrt{f_x^2 + f_y^2} \) tangent to the curve.

5 One numerical method to find the maximum of a function of two variables is to move in the direction of the gradient. This is called the **steepest ascent method**. You start at a point \((x_0, y_0)\) then move in the direction of the gradient for some time \(c\) to be at \((x_1, y_1) = (x_0, y_0) + c \nabla f(x_0, y_0)\). Now you continue to get to \((x_2, y_2) = (x_1, y_1) + c \nabla f(x_1, y_1)\). This works well in many cases like the function \( f(x, y) = 1 - x^2 - y^2 \). It can have problems if the function has a flat ridge like in the **Rosenbrock function**

\[
f(x, y) = 1 - (1 - x)^2 - 100(y - x^2)^2.
\]

Plot using a computer the Contour map of this function on \(-0.6 \leq x \leq 1, -0.1 \leq y \leq 1.1\) and find the directional derivative at \((1/5, 0)\) in the direction \((1, 1)/\sqrt{2}\). Why is it also called the **banana function**?
Lecture 13: Extrema

An important problem in calculus is to extremize a function \( f \). We do this now in higher dimensions. As in one dimensions, in order to look for maxima or minima, we consider points, where the "derivative" is zero.

A point \((a, b)\) in the plane is called a **critical point** of a function \( f(x, y) \) if \( \nabla f(a, b) = (0, 0) \).

Critical points are candidates for extrema because at critical points, all directional derivatives \( D_{\vec{v}} f = \nabla f \cdot \vec{v} \) are zero. We can not increase the value of \( f \) by moving into any direction.

This definition does not include points, where \( f \) or its derivative is not defined. We usually assume that a function can be differentiated arbitrarily often. Points where the function has no derivatives are not considered part of the domain and need to be studied separately. For the continuous function \( f(x, y) = |xy| \) for example, we would have to look at the points on the coordinate axes separately because these points are not in the domain of \( \nabla f \).

1. Find the critical points of \( f(x, y) = x^4 + y^4 - 4xy + 2 \). The gradient is \( \nabla f(x, y) = \langle 4(x^3 - y), 4(y^3 - x) \rangle \) with critical points \((0, 0), (1, 1), (-1, -1)\).

2. \( f(x, y) = \sin(x^2 + y) + y \). The gradient is \( \nabla f(x, y) = \langle 2x \cos(x^2 + y), \cos(x^2 + y) + 1 \rangle \). For a critical points, we must have \( x = 0 \) and \( \cos(y) + 1 = 0 \) which means \( \pi + k2\pi \). The critical points are at \( \ldots (0, -\pi), (0, \pi), (0, 3\pi), \ldots \).

3. The graph of \( f(x, y) = (x^2 + y^2)e^{-x^2-y^2} \) looks like a volcano. The gradient \( \nabla f = \langle 2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2) \rangle e^{-x^2-y^2} \) vanishes at \((0, 0)\) and on the circle \( x^2 + y^2 = 1 \). This function has infinitely many critical points.

4. The function \( f(x, y) = y^2/2 - g \cos(x) \) is the energy of the pendulum. The variable \( g \) is a constant. We have \( \nabla f = (y, -g \sin(x)) = \langle 0, 0 \rangle \) for \((x, y) = \ldots, (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), \ldots \). These points are equilibrium points, angles for which the pendulum is at rest.

5. The function \( f(x, y) = a \log(y) - by + c \log(x) - dx \) is left invariant by the flow of the **Volterra-Lodka** differential equation \( \dot{x} = ax - bxy, \dot{y} = -cy + dxy \). The point \((c/d, a/b)\) is a critical point. It is a place, where the differential equation has stationary points.

6. The function \( f(x, y) = |x| + |y| \) is smooth on the first quadrant. It does not have critical points there. The function has a minimum at \((0, 0)\) but it is not in the domain, where \( f \) and \( \nabla f \) are defined.

In one dimension, we needed \( f'(x) = 0, f''(x) > 0 \) to have a local minimum, \( f'(x) = 0, f''(x) < 0 \) for a local maximum. If \( f'(x) = 0, f''(x) = 0 \), then the critical point was undetermined and could be a maximum like for \( f(x) = -x^4 \), or a minimum like for \( f(x) = x^4 \) or a flat inflection point like for \( f(x) = x^3 \).
Let now \( f(x, y) \) be a function of two variables with a critical point \((a, b)\). Define \( D = f_{xx}f_{yy} - f_{xy}^2 \). It is called the **discriminant** of the critical point.

Remark: The discriminant can be remembered better if it is seen as the determinant of the **Hessian matrix** \( H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \).

**Second derivative test.** Assume \((a, b)\) is a critical point for \( f(x, y) \).
- If \( D > 0 \) and \( f_{xx}(a, b) > 0 \) then \((a, b)\) is a local minimum.
- If \( D > 0 \) and \( f_{xx}(a, b) < 0 \) then \((a, b)\) is a local maximum.
- If \( D < 0 \) then \((a, b)\) is a saddle point.

In the case \( D = 0 \), we need higher derivatives to determine the nature of the critical point.

7 The function \( f(x, y) = x^3/3 - x - (y^3/3 - y) \) has a graph which looks like a "napkin". It has the gradient \( \nabla f(x, y) = (x^2 - 1, -y^2+1) \). There are 4 critical points \((1, 1), (-1, 1), (1, -1)\) and \((-1, -1)\). The Hessian matrix which includes all partial derivatives is \( H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix} \).

For \((1, 1)\) we have \( D = -4 \) and so a saddle point,
For \((-1, 1)\) we have \( D = 4, f_{xx} = -2 \) and so a local maximum,
For \((1, -1)\) we have \( D = 4, f_{xx} = 2 \) and so a local minimum.
For \((-1, -1)\) we have \( D = -4 \) and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.

To determine the maximum or minimum of \( f(x, y) \) on a domain, determine all critical points in the **interior the domain**, and compare their values with maxima or minima at the boundary. We will see next time how to get extrema on the boundary.

8 Find the maximum of \( f(x, y) = 2x^2 - x^3 - y^2 \) on \( y \geq -1 \). With \( \nabla f(x, y) = 4x - 3x^2, -2y \), the critical points are \((4/3, 0)\) and \((0, 0)\). The Hessian is \( H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix} \). At \((0, 0)\), the discriminant is \(-8\) so that this is a saddle point. At \((4/3, 0)\), the discriminant is \(8\) and \(H_{11} = 4/3\), so that \((4/3, 0)\) is a local maximum. We have now also to look at the boundary \( y = -1 \) where the function is \( g(x) = f(x, -1) = 2x^2 - x^3 - 1 \). Since \( g'(x) = 0 \) at \( x = 0, 4/3 \), where 0 is a local minimum, and \(4/3\) is a local maximum on the line \( y = -1 \). Comparing \( f(4/3, 0), f(4/3, -1) \) shows that \((4/3, 0)\) is the global maximum.
As in one dimensions, knowing the critical points helps to understand the function. Critical points are also physically relevant. Examples are configurations with lowest energy. Many physical laws are based on the principle that the equations are critical points. Newton equations in Classical mechanics are an example: a particle of mass \( m \) moving in a field \( V \) along a path \( \gamma : t \mapsto \mathbf{r}(t) \) extremizes the integral \( S(\gamma) = \int_a^b m \mathbf{r}'(t)^2/2 - V(\mathbf{r}(t)) \, dt \) among all possible paths. Critical points \( \gamma \) satisfy the Newton equations \( m \mathbf{r}''(t)/2 - \nabla V(\mathbf{r}(t)) = 0 \).

Why is the second derivative test true? Assume \( f(x, y) \) has the critical point \((0, 0)\) and is a quadratic function satisfying \( f(0, 0) = 0 \). Then
\[
ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)
\]
with \( A = (x + \frac{b}{a}y), B = \frac{b^2}{a^2} \) and discriminant \( D \). You see that if \( a = f_{xx} > 0 \) and \( D > 0 \) then \( c - \frac{b^2}{a} > 0 \) and the function has positive values for all \((x, y) \neq (0, 0)\). The point \((0, 0)\) is a minimum. If \( a = f_{xx} < 0 \) and \( D > 0 \), then \( c - \frac{b^2}{a} < 0 \) and the function has negative values for all \((x, y) \neq (0, 0)\) and the point \((x, y)\) is a local maximum. If \( D < 0 \), then the function can take both negative and positive values. A general smooth function can be approximated by a quadratic function near \((0, 0)\).

Sometimes, we want to find the overall maximum and not only the local ones.

A point \((a, b)\) in the plane is called a **global maximum** of \( f(x, y) \) if \( f(x, y) \leq f(a, b) \) for all \((x, y)\). For example, the point \((0, 0)\) is a global maximum of the function \( f(x, y) = 1 - x^2 - y^2 \). Similarly, we call \((a, b)\) a **global minimum**, if \( f(x, y) \geq f(a, b) \) for all \((x, y)\).

Does the function \( f(x, y) = x^4 + y^4 - 2x^2 - 2y^2 \) have a global maximum or a global minimum? If yes, find them. **Solution:** the function has no global maximum. This can be seen by restricting the function to the \( x \)-axis, where \( f(x, 0) = x^4 - 2x^2 \) is a function without maximum. The function has four global minima however. They are located on the 4 points \((\pm 1, \pm 1)\). The best way to see this is to note that \( f(x, y) = (x^2 - 1)^2 + (y - 1)^2 - 2 \) which is minimal when \( x^2 = 1, y^2 = 1 \).

Let \( f(x, y) \) be the height function of an island for which only finitely many critical points exist. Assume all of them have nonzero discriminant \( D \). Label each critical point with a +1 if it is a maximum or minimum, and with −1 if it is a saddle point. If you sum up all these integers you will get 1, independent of the function. This theorem of Poincare-Hopf is an example of an "index theorem", a prototype for important theorems in physics and mathematics. The indices \( \pm 1 \) add up to one, whatever island you consider, whether it has only one mountain peak, or two mountain peaks and a mountain pass. Place your hand into water so that it forms an island, then and count peaks = knuckles) and mountain passes.

The following remarks can be skipped as they are more theoretical and point out connections to other fields:
1) if you have seen some linear algebra, you see that the discriminant \( D \) is a determinant \( \det(H) \) of the matrix \( H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \). Besides the determinant, also the trace \( f_{xx} + f_{yy} \) is independent of the coordinate system. The determinant is the product \( \lambda_1 \lambda_2 \) of the eigenvalues of \( H \) and the trace is the sum of the eigenvalues. If the determinant \( D \) is positive, then \( \lambda_1, \lambda_2 \) have the same sign and this is also the sign of the trace. If the trace is positive then both eigenvalues are positive. This means that in in the eigendirections, the graph is concave up. We have a minimum. On the other hand, if the determinant \( D \) is negative, then \( \lambda_1, \lambda_2 \) have different signs and the function is concave up in one eigendirection and concave down in another. In any case, if \( D \) is not zero, we have an orthonormal eigenbasis of the symmetric matrix \( A \). In that basis, the matrix \( H \) is diagonal.

2) The discriminant \( D \) can be considered also at points where we have no critical point. The number \( K = D/(1 + |\nabla f|^2)^2 \) is called the Gaussian curvature of the surface. It is a remarkable quantity since it only depends on the intrinsic geometry of the surface and not on the way how the surface is embedded in space. This is the famous Theorema Egregia (=great theorem) of Gauss. Note that at a critical point \( \nabla f(x) = 0 \), the discriminant agrees with the curvature \( D = K \) at that point. Since we mentioned the ”island” already: here is another one which follows from the Gauss-Bonnet theorem: assume you measure the curvature \( K \) at each point of an island and assume there is a nice beach all around so that the land disappears flat into the water. In that case the average curvature over the entire island is zero.

3) You might wonder what happens in higher dimensions. The second derivative test needs then more linear algebra. In three dimensions for example, one can form the second derivative matrix \( H \) and look at all the eigenvalues of \( H \). If all eigenvalues are negative, we have a local maximum, if all eigenvalues are positive, we have a local minimum. If the eigenvalues have different signs, we have a saddle point situation where in some directions the function increases and other directions the function decreases.

4) We have mentioned in the intro lecture how many ideas of calculus also go over to the discrete. Indeed, one can also look at extrema problems on graphs. The analogue of a two dimensional space is a graph for which every unit circle is a circular graph of length larger than 3. One can now look at a function \( f \) on the vertices of a graph. One knows looks at the set \( S_{\bar{f}}(x) \) on the unit circle, where \( f(y) < f(x) \). If this set \( S_{\bar{f}}(x) \) is the entire circle, we have a maximum. If it is empty, then we have a minimum. And if it is a disconnected set, then we have a saddle point. If \( \chi(S_{\bar{f}}(x)) \) is the number of vertices in \( S_{\bar{f}}(x) \) minus the number of edges in \( S_{\bar{f}}(x) \), then the discriminant \( D \) is defined as \( 1 - \chi(S_{\bar{f}}(x)) \). We see that it is equal to 1 at maxima and minima and -1 at saddle points. At a ”monkey saddle” a point where \( S_{\bar{f}}(x) \) has three components, then \( D = -2 \).
Homework

1. Find all the extrema of the function $f(x, y) = 1 + 8y^4 - 8x^3 - 16y^2 + 24x$ and determine whether they are maxima, minima or saddle points.

2. Where on the parametrized surface $\vec{r}(u, v) = \langle u^2, v^3, uv \rangle$ is the temperature $T(x, y, z) = 12x + y - 12z$ minimal? To find the minimum, look where the function $f(u, v) = T(\vec{r}(u, v))$ has an extremum. Find all local maxima, local minima or saddle points of $f$.

Remark. After you have found the function $f(u, v)$, you could replace the variables $u, v$ again with $x, y$ if you like and look at a function $f(x, y)$.

3. Find and classify all the extrema of the function $f(x, y) = e^{-x^2 - y^2}(x^2 + 2y^2)$.

4. Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

A minigolf on the cape has a hole at a local minimum of the function

$$f(x, y) = 3x^2 + 2x^3 + 2y^5 - 5y^2 \, .$$

5. Find all the critical points and classify them.

About minigolf: the first standardized minigolf course appeared in 1916 in North Carolina. The world record on a round of minigolf is 18 strokes on 18 holes on eternite. No perfect round on concrete has been scored. The highest prizes reach 5000 dollars only so that nobody is known to make a living by competing in minigolf.
Lecture 14: Lagrange

We look now for maxima and minima of a function \( f(x, y) \) in the presence of a constraint \( g(x, y) = 0 \). A necessary condition for a critical point is that the gradients of \( f \) and \( g \) are parallel because otherwise, we can move along the curve \( g \) and increase the value of \( f \). The directional derivative of \( f \) in the direction tangent to the level curve is zero if and only if the tangent vector to \( g \) is perpendicular to the gradient of \( f \) or if there is no tangent vector.

The system of equations \( \nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0 \) for the three unknowns \( x, y, \lambda \) are called Lagrange equations. The variable \( \lambda \) is a Lagrange multiplier.

Lagrange theorem: A maximum or minimum of \( f(x, y) \) on the curve \( g(x, y) = c \) is either a solution of the Lagrange equations or is a critical point of \( g \).

Proof. The condition that \( \nabla f \) is parallel to \( \nabla g \) either means \( \nabla f = \lambda \nabla g \) or \( \nabla f = 0 \) or \( \nabla g = 0 \). The case \( \nabla f = 0 \) can be included in the Lagrange equation case with \( \lambda = 0 \).

1. Minimize \( f(x, y) = x^2 + 2y^2 \) under the constraint \( g(x, y) = x + y^2 = 1 \). Solution: The Lagrange equations are \( 2x = \lambda, 4y = 2\lambda y \). If \( y = 0 \) then \( x = 1 \). If \( y \neq 0 \) we can divide the second equation by \( y \) and get \( 2x = \lambda, 4 = \lambda 2 \) again showing \( x = 1 \). The point \( x = 1, y = 0 \) is the only solution.

2. Find the shortest distance from the origin to the curve \( x^6 + 3y^2 = 1 \). Solution: Minimize the function \( f(x, y) = x^2 + y^2 \) under the constraint \( g(x, y) = x^6 + 3y^2 = 1 \). The gradients are \( \nabla f = (2x, 2y), \nabla g = (6x^5, 6y) \). The Lagrange equations \( \nabla f = \lambda \nabla g \) lead to the system \( 2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0 \). We get \( \lambda = 1/3, x = x^5 \), so that either \( x = 0 \) or 1 or \(-1\). From the constraint equation \( g = 1 \), we obtain \( y = \sqrt{(1 - x^6)/3} \). So, we have the solutions \((0, \pm \sqrt{1/3})\) and \((1, 0), (-1, 0)\). To see which is the minimum, just evaluate \( f \) on each of the points. We see that \((0, \pm \sqrt{1/3})\) are the minima.
3 Which cylindrical soda cans of height $h$ and radius $r$ has minimal surface for fixed volume? **Solution:** The volume is $V(r, h) = h\pi r^2 = 1$. The surface area is $A(r, h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x, y) = 2xy + 2\pi y^2$ under the constrained $g(x, y) = xy^2 = 1$. Calculate $\nabla f(x, y) = (2y, 2x + 4\pi y), \nabla g(x, y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^2 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54...$, $r = 2h = 1.08$. Remark: Other factors can influence the shape. For example, the can has to withstand a pressure up to 100 psi. A typical can of "Coca-Cola classic" with 3.7 volumes of $CO_2$ dissolve has at 75F an internal pressure of 55 psi, where PSI stands for pounds per square inch.

4 On the curve $g(x, y) = x^2 - y^2$ the function $f(x, y) = x$ obviously has a minimum $(0, 0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x, y) = 0$.

**Remarks.**

1) Either of the two properties equated in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.

2) With $g(x, y) = 0$, the Lagrange equations can also be written as $\nabla F(x, y, \lambda) = 0$ where $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$.

3) Either of the two properties equated in the Lagrange theorem are equivalent to "$\nabla g = \lambda \nabla f$ or $f$ has a critical point".

4) Constrained optimization problems work also in higher dimensions. The proof is the same:

**Extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$.**

5 Find the extrema of $f(x, y, z) = z$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x, y, z) = (0, 0, 1), \nabla g(x, y, z) = (2x, 2y, 2z)$ and solve $(0, 0, 1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0, 0, -1)$ the maximum the north pole $(0, 0, 1)$.

6 A dice shows $k$ eyes with probability $p_k$ with $k$ in $\Omega = \{1, 2, 3, 4, 5, 6\}$. A probability distribution is a nonnegative function $p$ on $\Omega$ which sums up to 1. It can be written as a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The entropy of the probability vector $\vec{p}$ is defined as $f(\vec{p}) = -\sum_{i=1}^{6} p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - ... - p_6 \log(p_6)$. Find the distribution $p$ which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = (-1 - \log(p_1), ..., -1 - \log(p_6)), \nabla g = (1, ..., 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + ... + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-\lambda) = 6 \exp(-\lambda) = \exp(-\lambda + 1)$ fixes $\lambda = -\log(1/6) = -1$ so that $p_i = 1/6$. The distribution, where each event has the same probability is the distribution of maximal entropy. Maximal entropy means least information content. An unfair dice allows a cheating gambler or casino to gain profit. Cheating through asymmetric weight distributions can be avoided by making the dices transparent.

7 Assume that the probability that a physical or chemical system is in a state $k$ is $p_k$ and that the energy of the state $k$ is $E_k$. Nature tries to minimize the free energy $f(p_1, \ldots, p_n) =$
Find the cylindrical basket which is open on the top has has the largest volume for fixed area $\pi$. If $x$ is the radius and $y$ is the height, we have to extremize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

2 Find the extrema of the same function

$$f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$$
as in problem 4.1.3 but now on the entire disc \( \{ x^2 + y^2 \leq 4 \} \) of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

3 After having watched the Disney movie “Tangled”, we want to build a hot air balloon with a cuboid mesh of dimension \( x, y, z \) which together with the top and bottom fortifications uses wires of total length \( g(x, y, z) = 6x + 6y + 4z = 32 \). Find the balloon with maximal volume \( f(x, y, z) = xyz \).

4 A solid bullet made of a half sphere and a cylinder has the volume \( V = \frac{2\pi r^3}{3} + \pi r^2 h \) and surface area \( A = 2\pi r^2 + 2\pi rh + \pi r^2 \). Doctor Manhattan designs a bullet with fixed volume and minimal area. With \( g = 3V/\pi = 1 \) and \( f = A/\pi \) he therefore minimizes \( f(h, r) = 3r^2 + 2rh \) under the constraint \( g(h, r) = 2r^3 + 3r^2 h = 1 \). Use the Lagrange method to find a local minimum of \( f \) under the constraint \( g = 1 \).

5 Minimize the material cost of an office tray

\[
f(x, y) = xy + x + 2y
\]

of length \( x \), width \( y \) and height 1 under the constraint that the volume \( g(x, y) = xy \) is constant and equal to 4.
Lecture 15: Double integrals

We start with a review of single variable calculus: if $f(x)$ is a differentiable function, then the Riemann integral $\int_a^b f(x) \, dx$ is defined as the limit of the Riemann sums $S_n f(x) = \frac{1}{n} \sum_{k/n \in [a,b]} f(k/n)$ for $n \to \infty$. The derivative of $f$ is the limit of difference quotients $D_n f(x) = n [f(x + 1/n) - f(x)]$ as $n \to \infty$. The integral $\int_a^b f(x) \, dx$ is the signed area under the graph of $f$ and above the $x$-axes, where ”signed” indicates that parts below have a negative sign. The function $F(x) = \int_0^x f(y) \, dy$ is called an anti-derivative of $f$. It is determined up a constant. The fundamental theorem of calculus states

$$F'(x) = f(x), \quad \int_0^x f(x) = F(x) - F(0),$$

and allows to compute integrals by inverting differentiation. Differentiation rules become integration rules: the product rule leads to integration by parts, the chain rule becomes partial integration. For a 20 × 20 second version, see www.math.harvard.edu/~knill/pedagogy/pechakucha, For a 140 character version, see https://twitter.com/oliverknill/status/320289197653106688.

If $f(x, y)$ is differentiable on a region $R$, the integral $\int_R f(x, y) \, dxdy$ is defined as the limit of the Riemann sum

$$\frac{1}{n^2} \sum_{(i/n, j/n) \in R} f(i/n, j/n)$$

when $n \to \infty$. We write also $\int_R f(x, y) \, dA$ and think of $dA$ as an area element.

1. If we integrate $f(x, y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over $y$ to get $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2. If $f(x, y) = 1$, then the integral is the area of the region $R$. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside $R$. 
3 The integral $\iint_R f(x, y) \, dA$ divided by the area of $R$ is the \textbf{average} value of $f$ on $R$.

4 One can interpret $\iint_R f(x, y) \, dy \, dx$ as the \textbf{signed volume} of the solid below the graph of $f$ and above $R$ in the $x-y$ plane. As in 1D integration, the volume of the solid below the xy-plane is counted negatively.

\textbf{Fubini’s theorem} allows to switch the order of integration over a rectangle if the function $f$ is continuous: $\int_a^b \int_c^d f(x, y) \, dx \, dy = \int_c^d \int_a^b f(x, y) \, dy \, dx$.

Proof. We have for every $n$ the ”quantum Fubini identity”

\[ \sum_{\frac{i}{n} \in [a,b]} \sum_{\frac{j}{n} \in [c,d]} f\left(\frac{i}{n}, \frac{j}{n}\right) = \sum_{\frac{i}{n} \in [a,b]} \sum_{\frac{j}{n} \in [c,d]} f\left(\frac{i}{n}, \frac{j}{n}\right) \]

which holds for all functions. Now divide both sides by $n^2$ and take the limit $n \to \infty$.

Fubini’s theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

A \textbf{type I region} is of the form

$$ R = \{(x, y) \mid a \leq x \leq b, \ c(x) \leq y \leq d(x) \}.$$  

An integral over such a region is called a \textbf{type I integral}

$$ \iint_R f \, dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) \, dy \, dx.$$

\[ \]
A **type II region** is of the form

\[ R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y) \} . \]

An integral over such a region is called a **type II integral**

\[ \iint_{R} f \, dA = \int_{c}^{d} \int_{a(y)}^{b(y)} f(x, y) \, dx \, dy . \]

Integrate \( f(x, y) = x^2 \) over the region bounded above by \( \sin(x^3) \) and bounded below by the graph of \( -\sin(x^3) \) for \( 0 \leq x \leq \pi \). The value of this integral has a physical meaning. It is called **moment of inertia**.

\[
\int_{0}^{1/\sqrt[3]{\pi}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2 \, dy \, dx = 2 \int_{0}^{\pi^{1/3}} \sin(x^3) x^2 \, dx
\]

We have now an integral, which we can solve by substitution

\[ = -\frac{2}{3} \cos(x^3)|_{0}^{\pi^{1/3}} = \frac{4}{3} . \]

Integrate \( f(x, y) = y^2 \) over the region bound by the \( x \)-axes, the lines \( y = x + 1 \) and \( y = 1 - x \). The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of \( y \): they are \( x = y - 1 \) and \( x = 1 - y \)

\[
\int_{0}^{1} \int_{y-1}^{1-y} y^3 \, dx \, dy = 2 \int_{0}^{1} y^3 (1 - y) \, dy = 2 \left( \frac{1}{4} - \frac{1}{3} \right) = \frac{1}{10} .
\]

Let \( R \) be the triangle \( 1 \geq x \geq 0, 0 \leq y \leq x \). What is

\[ \iint_{R} e^{-x^2} \, dxdy ? \]

The type II integral \( \int_{0}^{1} [\int_{y}^{1} e^{-x^2} \, dx] \, dy \) can not be solved because \( e^{-x^2} \) has no anti-derivative in terms of elementary functions.

The type I integral \( \int_{0}^{1} [\int_{0}^{x} e^{-x^2} \, dy] \, dx \) however can be solved:

\[
= \int_{0}^{1} xe^{-x^2} \, dx = -\frac{e^{-x^2}}{2} \Big|_{0}^{1} = \frac{(1 - e^{-1})}{2} = 0.316... .
\]
The area of a disc of radius $R$ is

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \, dy \, dx = \int_{-R}^{R} 2\sqrt{R^2-x^2} \, dx .$$

Substitute $x = R \sin(u), dx = R \cos(u)$, to get

$$\int_{-\pi/2}^{\pi/2} 2 \sqrt{R^2 - R^2 \sin^2(u)} R \cos(u) \, du = \int_{-\pi/2}^{\pi/2} 2R^2 \cos^2(u) \, du.$$ Using a double angle formula, this gives $R^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2u)}{2} \, du = R^2 \pi$.

**Remark:** The Riemann integral just defined works well for continuous functions. In other branches of mathematics like probability theory, a better integral is needed. The Lebesgue integral fits the bill. Its definition is close to the Riemann integral which we have given as the limit $\frac{1}{n^2} \sum_{(x_k, y_l) \in R} f(x_k, y_l)$ where $x_k = k/n, y_l = l/n$. The Lebesgue integral replaces the regularly spaced $(x_k, y_l)$ grid with random points $x_k, y_l$ and uses the same formula. The following Mathematica code computes the integral $\int_0^1 \int_0^1 x^2y$ using this Monte Carlo definition of the Lebesgue integral.

```
M=10000; R:=Random[]; f[x_, y_]:=x^2 y; Sum[f[R,R],{M}]/M
```

It is as elegant than the numerical Riemann sum computation

```
M=100; f[x_, y_]:=x^2 y; Sum[f[k/M,1/M],{k,M},{1,M}]/M^2
```

but the Lebesgue integral is usually closer to the actual answer $1/6$ than the Riemann integral.

Note that for all continuous functions, the Lebesgue integral gives the same results than the Riemann integral. It does not change calculus. But it is useful for example to compute nasty integrals like the area of the Mandelbrot set.

**Homework**

1. Find the double integral $\int_0^4 \int_0^2 (3x - \sqrt{y}) \, dx \, dy$.

2. Find the area of the region

$$R = \{(x, y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value $\int \int_R f(x, y) \, dx \, dy / \text{area}(R)$ of $f(x, y) = y$ over that region.

3. Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3] = \{(x, y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3 \}$. 
4 Calculate the iterated integral \( \int_0^1 \int_{2-x}^{x^2} (x^2 - y) \, dy \, dx \). Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

5 It turns out that there is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!

a) (6 points) Find the integral
\[
\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{x^7}{\sqrt{x - x^2}} \, dx \, dy.
\]

b) (4 points) Integrate
\[
\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{10} \, dx \, dy.
\]
You might want to time travel a lecture ahead here (and transcend from horror to sci-fi movies for a moment) to solve this problem.
Lecture 16: Surface area

A polar region is a region bound by a simple closed curve given in polar coordinates as the curve \((r(t), \theta(t))\).

In Cartesian coordinates the parametrization of the boundary of a polar region is \(\vec{r}(t) = (r(t) \cos(\theta(t)), r(t) \sin(\theta(t)))\).

1. The polar graph defined by \(r(\theta) = |\cos(3\theta)|\) belongs to the class of roses \(r(t) = |\cos(nt)|\). Regions enclosed by this graph are also called rhododenea.

2. The polar curve \(r(\theta) = 1 + \sin(\theta)\) is called a cardioid. It looks like a heart. It is a special case of limacon curves \(r(\theta) = 1 + b \sin(\theta)\).

3. The polar curve \(r(\theta) = |\sqrt{\cos(2\theta)}|\) is called a lemniscate. It looks like an infinity sign.

To integrate in polar coordinates, we evaluate the integral

\[
\iint_R f(x, y) \, dxdy = \iint_R f(r \cos(\theta), r \sin(\theta)) r \, drd\theta
\]

4. Integrate \(f(x, y) = x^2 + y^2 + xy\) over the unit disc. We have \(f(x, y) = f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \cos(\theta) \sin(\theta)\) so that \(\iint_R f(x, y) \, dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2 \cos(\theta) \sin(\theta)) r \, d\theta dr = 2\pi/4\).

5. We have earlier computed area of the disc \(\{x^2 + y^2 \leq 1\}\) using substitution. It is more elegant to do this integral in polar coordinates:

\[
\int_0^{2\pi} \int_0^1 r \, drd\theta = 2\pi r^2/2|_0^1 = \pi.
\]
Why do we have to include the factor \( r \), when we move to polar coordinates? The reason is that a small rectangle \( R \) with dimensions \( d\theta dr \) in the \((r, \theta)\) plane is mapped by \( T : (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta)) \) to a sector segment \( S \) in the \((x, y)\) plane. It has the area \( r \, d\theta dr \). If you have seen some linear algebra, note that the Jacobean matrix \( dT \) has the determinant \( r \).

We can now integrate over type I or type II regions in the \((\theta, r)\) plane. like flowers: \( \{(\theta, r) \mid 0 \leq r \leq f(\theta)\} \) where \( f(\theta) \) is a periodic function of \( \theta \).

6 Integrate the function \( f(x, y) = 1 \ \{(\theta, r(\theta)) \mid r(\theta) \leq |\cos(3\theta)| \} \).

\[
\int \int_R 1 \, dx \, dy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r \, dr \, d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} \, d\theta = \pi/2.
\]

7 Integrate \( f(x, y) = y \sqrt{x^2 + y^2} \) over the region \( R = \{(x, y) \mid 1 < x^2 + y^2 < 4, y > 0 \} \).

\[
\int_1^2 \int_0^\pi r \sin(\theta) r \, d\theta \, dr = \int_1^2 r^3 \int_0^\pi \sin(\theta) \, d\theta \, dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) \, d\theta = 15/2
\]

For integration problems, where the region is part of an annular region, or if you see function with terms \( x^2 + y^2 \) try to use polar coordinates \( x = r \cos(\theta), y = r \sin(\theta) \).
The Belgian Biologist Johan Gielis defined in 1997 with the family of curves given in polar coordinates as

\[ r(\phi) = \left( \frac{\cos\left(\frac{m\phi}{4}\right)}{a} + \frac{\sin\left(\frac{m\phi}{4}\right)}{b}\right)^{-1/n} \]

This super-curve can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the super-ellipse which had been discussed in 1818 by Lamé and helps to describe forms in biology. 

\[ A \text{ surface } \vec{r}(u, v) \text{ parametrized on a parameter domain } R \text{ has the surface area } \]

\[ \int \int_{R} |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, dudv. \]

Proof. The vector \( \vec{r}_u \) is tangent to the grid curve \( u \mapsto \vec{r}(u, v) \) and \( \vec{r}_v \) is tangent to \( v \mapsto \vec{r}(u, v) \), the two vectors span a parallelogram with area \( |\vec{r}_u \times \vec{r}_v| \). A small rectangle \([u, u + du] \times [v, v + dv]\) is mapped by \( \vec{r} \) to a parallelogram spanned by \([\vec{r}, \vec{r} + \vec{r}_u]\) and \([\vec{r}, \vec{r} + \vec{r}_v]\) which has the area \( |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, dudv. \)

The parametrized surface \( \vec{r}(u, v) = (2u, 3v, 0) \) is part of the xy-plane. The parameter region \( G \) just gets stretched by a factor 2 in the \( x \) coordinate and by a factor 3 in the \( y \) coordinate. \( \vec{r}_u \times \vec{r}_v = (0, 0, 6) \) and we see for example that the area of \( \vec{r}(G) \) is 6 times the area of \( G \).

\[ ^{19} \text{Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).} \]
The map \( \mathbf{r}(u, v) = \langle L \cos(u) \sin(v), L \sin(u) \sin(v), L \cos(v) \rangle \) maps the rectangle \( G = [0, 2\pi] \times [0, \pi] \) onto the sphere of radius \( L \). We compute \( \mathbf{r}_u \times \mathbf{r}_v = L \sin(v) \) \( \mathbf{r}(u, v) \). So, \( |\mathbf{r}_u \times \mathbf{r}_v| = L^2 |\sin(v)| \) and \( \int \int_R 1 \, dS = \int_0^{2\pi} \int_0^\pi L^2 \sin(v) \, dv \, du = 4\pi L^2 \).

For graphs \((u, v) \mapsto \langle u, v, f(u, v) \rangle\), we have \( \mathbf{r}_u = (1, 0, f_u(u, v)) \) and \( \mathbf{r}_v = (0, 1, f_v(u, v)) \). The cross product \( \mathbf{r}_u \times \mathbf{r}_v = (-f_u, -f_v, 1) \) has the length \( \sqrt{1 + f_u^2 + f_v^2} \). The area of the surface above a region \( G \) is \( \int \int_G \sqrt{1 + f_u^2 + f_v^2} \, dudv \).

Let's take a surface of revolution \( \mathbf{r}(u, v) = \langle v, f(v) \cos(u), f(v) \sin(u) \rangle \) on \( R = [0, 2\pi] \times [a, b] \). We have \( \mathbf{r}_u = (0, -f(v) \sin(u), f(v) \cos(u)), \mathbf{r}_v = (1, f'(v) \cos(u), f'(v) \sin(u)) \) and \( \mathbf{r}_u \times \mathbf{r}_v = (f(v) f'(v) \cos(u), f(v) \cos(u), f(v) \sin(u)) = f(v)(-f'(v), \cos(u), \sin(u)) \). The surface area is \( \int \int \mathbf{r}_u \times \mathbf{r}_v \, dudv = 2\pi \int_0^b |f(v)| \sqrt{1 + f'(v)^2} \, dv \).

**Homework**

1. Integrate \( f(x, y) = 4x^2 + 8y^2 \) over the unit disc \( \{x^2 + y^2 \leq 1 \} \) in two ways, first using Cartesian coordinates, then using polar coordinates.

2. Find \( \int \int_R (x^2 + y^2)^{40} \, dA \), where \( R \) is the part of the unit disc \( \{x^2 + y^2 \leq 1 \} \) for which \( y > x \).

3. What is the area of the region which is bounded by the following three curves, first by the polar curve \( r(\theta) = \theta \) with \( \theta \in [0, 2\pi] \), second by the polar curve \( r(\theta) = 2\theta \) with \( \theta \in [0, 2\pi] \) and third by the positive \( x \)-axis?

4. Find the average value of \( f(x, y) = 2(x^2 + y^2) \) on the annular region \( R : 1 \leq |(x, y)| \leq 2 \). The average is \( \frac{\int_R \int f \, dx \, dy}{\int_R 1 \, dx \, dy} \).

5. Find the surface area of the part of the paraboloid \( x = y^2 + z^2 \) which is inside the cylinder \( y^2 + z^2 \leq 9 \).
Lecture 17: Triple integrals

If \( f(x, y, z) \) is a function of three variables and \( E \) is a solid region in space, then
\[
\int \int \int_E f(x, y, z) \, dx \, dy \, dz
\]
is defined as the \( n \to \infty \) limit of the Riemann sum
\[
\frac{1}{n^3} \sum_{(i/n, j/n, k/n) \in E} f(i/n, j/n, k/n).
\]

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1. Assume \( E \) is the box \([0, 1] \times [0, 1] \times [0, 1]\) and \( f(x, y, z) = 24x^2y^3z \).

\[
\int_0^1 \int_0^1 \int_0^1 24x^2y^3z \, dz \, dy \, dx.
\]

To compute the integral we start from the core \( \int_0^1 24x^2y^3z \, dz = 12x^3y^3 \), then integrate the middle layer, \( \int_0^1 12x^3y^3 \, dy = 3x^2 \) and finally and finally handle the outer layer: \( \int_0^1 3x^2 \, dx = 1 \).

When we calculate the most inner integral, we fix \( x \) and \( y \). The integral is integrating up \( f(x, y, z) \) along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane \( z = \text{const} \) intersected with \( R \). The most outer integral sums up all these two dimensional sections.

In calculus, two important reductions are used to compute triple integrals. In single variable calculus, one reduces the problem directly to a one dimensional integral by slicing the body along an axes. Here in multi-variable calculus, we reduce the problem to a two-dimensional integration problem. If the single variable calculus was "burger" with cheese, beef and lattice slices, then we cut the potato into "fries".

The single variable method slices the solid along a line. If \( g(z) \) is the double integral along the two dimensional slice, then \( \int_a^b g(z) \, dz \). The multi-variable method sees the solid sandwiched between the graphs of two functions \( g(x, y) \) and \( h(x, y) \) over a common two dimensional region \( R \). The integral reduces to a double integral \( \int \int_R [f(x, y, z) \, dz] \, dA \).
An important special case of the sandwich method is the volume
\[ \int_R \int_0^{f(x,y)} 1 \, dz \, dx \, dy. \]
below the graph of a function \( f(x, y) \) and above a region \( R \), considered part of the \( xy \) plane. It is the integral \( \int_R f(x, y) \, dA \). We actually have computed is a triple integral. Think of volume as a triple integral from now on.

Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions. Let \( R \) be the unit disc in the \( xy \) plane. If we use the sandwich method, we get
\[ V = \int \int_R [\int_0^{\sqrt{1-x^2-y^2}} 1 \, dz] \, dA, \]
which gives a double integral \( \int \int_R 2\sqrt{1-x^2-y^2} \, dA \) which is of course best solved in polar coordinates. We have \( \int_0^{2\pi} \int_0^1 \sqrt{1-r^2} \, dr \, d\theta = 4\pi/3. \)
With the washer method which is in this case also called disc method, we slice along the \( z \) axes and get a disc of radius \( \sqrt{1-z^2} \) with area \( \pi(1-z^2) \). This is a method suitable for single variable calculus because we get directly \( \int_{-1}^{1} \pi(1-z^2) \, dz = 4\pi/3. \)

The mass of a body with density \( \rho(x, y, z) \) is defined as \( \int \int \int_R \rho(x, y, z) \, dV \). For bodies with constant density \( \rho \) the mass is \( \rho V \), where \( V \) is the volume. Compute the mass of a body which is bounded by the parabolic cylinder \( z = 4 - x^2 \), and the planes \( x = 0, y = 0, y = 6, z = 0 \) if the density of the body is 1. **Solution:**
\[
\int_0^2 \int_0^6 \int_0^{4-x^2} dz \, dy \, dx = \int_0^2 \int_0^6 (4-x^2) \, dy \, dx \\
= 6 \int_0^2 (4-x^2) \, dx = 6(4x-x^3/3)|_0^2 = 32
\]
The solid region bound by \( x^2 + y^2 = 1, x = z \) and \( z = 0 \) is called the hoofs of Archimedes. It is historically significant because it is one of the first examples, on which Archimedes probbed his Riemann sum integration technique. It appears in every calculus text book. Find the volume. **Solution.** Look from the situation from above and picture it in the \( x-y \) plane. You see a half disc \( R \). It is the floor of the solid. The roof is the function \( z = x \). We have to integrate \( \int_R x \, dxdy \). We got a double integral problems which is best done in polar coordinates; \( \int_{\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta) \, dr \, d\theta = 2/3. \)
Finding the volume of the solid region bound by the three cylinders \( x^2 + y^2 = 1, \ x^2 + z^2 = 1 \) and \( y^2 + z^2 = 1 \) is one of the most famous volume integration problems.

**Solution:** look at \( 1/16'\)th of the body given in cylindrical coordinates \( 0 \leq \theta \leq \pi/4, \ r \leq 1, \ z > 0 \). The roof is \( z = \sqrt{1-x^2} \) because above the ”one eighth disc” \( R \) only the cylinder \( x^2 + z^2 = 1 \) matters. The polar integration problem

\[
16 \int_{0}^{\pi/4} \int_{0}^{1} \sqrt{1-r^2 \cos^2(\theta)} \ r \ dr \ d\theta
\]

has an inner \( r \)-integral of \((16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)\). Integrating this over \( \theta \) can be done by integrating \( (1 + \sin(x)^3) \sec^2(x) \) by parts using \( \tan'(x) = \sec^2(x) \) leading to the anti derivative \( \cos(x) + \sec(x) + \tan(x) \). The result is \( 16 - 8\sqrt{2} \).

The problem of computing volumes has been tackled early in mathematics:

**Archimedes** (287-212 BC) designed an integration method which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of exhaustion is close to the numerical method of integration by Riemann sum. In our terminology, Archimedes used the washer method to reduce the problem to a single variable problem. The Archimedes principle states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. His displacement method later would morph into Cavalieri principle and modern rearrangement techniques in modern analysis. Heureka!

**Cavalieri** (1598-1647) would build on Archimedes ideas and determine area and volume using tricks like the Cavalieri principle. An example already due to Archimedes is the computation of the volume the half sphere of radius \( R \), cut away a cone of height and radius \( R \) from a cylinder of height \( R \) and radius \( R \). At height \( z \), this body has a cross section with area \( R^2 \pi - r^2 \pi \). If we cut the half sphere at height \( z \), we obtain a disc of area \( (R^2 - r^2) \pi \). Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: \( \pi R^3 - \pi R^3/3 + 2\pi R^3/3 = 2\pi R^3/3 \) and the volume of the sphere is \( 4\pi R^3/3 \).

**Newton** (1643-1727) and **Leibniz** (1646-1716): Newton and Leibniz, developed calculus independently. The new tool made it possible to compute integrals through ”anti-derivation”. Suddenly, it became possible to find integrals using analytic tools as we do here.
Remarks which can be skipped. 1) There is an other integral called Lebesgue integral which is more powerful than the Riemann integral: suppose we want to calculate the volume of some solid body \( R \) which we assumed to be contained inside the unit cube \([0, 1] \times [0, 1] \times [0, 1]\). The **Monte Carlo method** shoots randomly \( n \) times onto the unit cube and count the number \( k \) of times, we hit the solid. The result \( k/n \) approximates the volume. Here is a Mathematica example where an octant of the sphere is computed:

\[
R := \text{Random}[]; k = 0; \text{Do}[x = R; y = R; z = R; \text{If}[x^2 + y^2 + z^2 < 1, k++]], \{10000\}]; k/10000
\]

Assume, we hit 5277 of \( n=10000 \) times. The volume so measured is 0.5277. The actual volume of \( 1/8 \)'th of the sphere is \( \pi/6 = 0.524 \). For \( n \to \infty \) the Monte Carlo computation gives the actual volume. The Monte-Carlo integral is stronger than the Riemann integral. From probability theory, one can deduce that it is equivalent to the **Lebesgue integral** and allows to measure much more sets than solids with piecewise smooth boundaries.

2) Is there an "integral" which can measure every solid in space and which has the property that the volume of a rotated or translated body remains the same? No! Most sets turn out to be "crazy" in the sense that one can not measure their volume. An example is the **paradox of Banach and Tarski** which tells that one can slice up the unit ball \( x^2 + y^2 + z^2 \leq 1 \) into 5 pieces \( A, B, C, D, E \), rotate and translate them in space so that the pieces \( A, B, C \) fit together to be a unit ball and \( D, E \) again form an other unit ball. Since the volume has obviously doubled and volume should be additive, some of the sets \( A, B, C, D, E \) have no well defined volume.

**Homework**

1. Evaluate the triple integral
   \[
   \int_0^1 \int_0^z \int_0^{2y} z e^{-y^2} \, dx \, dy \, dz.
   \]

2. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 16 - (x^2 + y^2) \) and satisfying \( x \geq 0 \).

3. Find the moment of inertia \( \int \int \int_E (x^2 + y^2) \, dV \) of a cone
   \[
   E = \{ x^2 + y^2 \leq z^2 \, \, 0 \leq z \leq 2 \} ,
   \]
   which has the \( z \)-axis as its center of symmetry.

4. Integrate \( f(x, y, z) = x^2+y^2-z \) over the tetrahedron with vertices \((0, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 3)\).

5. Also solved by Archimedes: What is the volume of the body obtained by intersecting the solid cylinders \( x^2 + z^2 \leq 1 \) and \( y^2 + z^2 \leq 1 \)?
Cylindrical coordinates are coordinates in space in which polar coordinates are chosen in the xy-plane and where the z-coordinate is untouched. A surface of revolution can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$, produces the same integration factor $r$ as in polar coordinates.

$$\iint_{T(R)} f(x, y, z) \, dx \, dy \, dz = \iint_{R} g(r, \theta, z) \, r \, dr \, d\theta \, dz$$

Remember also that spherical coordinates use $\rho$, the distance to the origin as well as two angles: $\theta$ the polar angle and $\phi$, the angle between the vector and the z axis. The coordinate change is

$$T : (x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) .$$

The integration factor can be seen by measuring the volume of a spherical wedge which is $d\rho, \rho \sin(\phi) \, d\theta, \rho d\phi = \rho^2 \sin(\phi) d\theta d\phi d\rho$.

$$\iint_{T(R)} f(x, y, z) \, dx \, dy \, dz = \iint_{R} g(\rho, \theta, z) \, \rho^2 \sin(\phi) \, d\rho d\theta d\phi$$

1 A sphere of radius $R$ has the volume

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 \sin(\phi) \, d\phi d\theta d\rho .$$

The most inner integral $\int_0^\pi \rho^2 \sin(\phi) d\phi = -\rho^2 \cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because $\phi$ does not appear: $\int_0^{2\pi} 2\rho^2 \, d\phi = 4\pi \rho^2$. The final integral is $\int_0^R 4\pi \rho^2 \, d\rho = 4\pi R^3/3$. 
The moment of inertia of a body $G$ with respect to an axis $L$ is defined as the triple integral 
\[ \int \int \int_G r(x, y, z)^2 \, dz \, dy \, dx, \]
where $r(x, y, z) = R \sin(\phi)$ is the distance from the axis $L$.

2 For a sphere of radius $R$ we obtain with respect to the $z$-axis:

\[
I = \int_0^R \int_0^{2\pi} \int_0^{\pi/6} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho
\]
\[
= \left( \int_0^\pi \sin^3(\phi) \, d\phi \right) \left( \int_0^R \rho^4 \, d\rho \right) \left( \int_0^{2\pi} \, d\theta \right)
\]
\[
= \left( -\cos(\phi) + \frac{\cos(\phi)^3}{3} \right) \int_0^\pi \left( \frac{R^5}{5} \right) (2\pi) = \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15}.
\]

If the sphere rotates with angular velocity $\omega$, then $I\omega^2/2$ is the kinetic energy of that sphere.

Example: the moment of inertia of the earth is $8 \cdot 10^{37} \text{kgm}^2$. The angular velocity is $\omega = 2\pi/\text{day} = 2\pi/(86400 \text{s})$. The rotational energy is $8 \cdot 10^{37} \text{kgm}^2/(7464960000 \text{s}^2) \sim 10^{29} \text{J} \sim 2.510^{24} \text{kcal}$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.

Solution: we use spherical coordinates to find the center of mass

\[
\tau = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta) \, d\phi \, d\theta \, d\rho \frac{1}{V} = 0
\]
\[
\eta = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta) \, d\phi \, d\theta \, d\rho \frac{1}{V} = 0
\]
\[
\zeta = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi) \sin(\phi) \, d\phi \, d\theta \, d\rho \frac{1}{V} = \frac{2\pi}{32}
\]

4 Find $\int \int_R z^2 \, dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4 \}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.

Solution: since the result for the double cone is twice the result for the single cone, we
work with the diamond shaped region $R$ in $\{ z > 0 \}$ and multiply the result at the end with 2. In spherical coordinates, the solid $R$ is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho \cos(\phi)$, we have
\[
\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi) \, d\phi d\theta d\rho
= \left( \frac{2^5}{5} - \frac{1^5}{5} \right) 2\pi \left( -\cos^3(\phi) \right) \bigg|_0^{\pi/4} = 2\pi \frac{31}{5} \left( 1 - 2^{-3/2} \right).
\]
The result for the double cone is $4\pi (31/5) (1 - 1/\sqrt{2})$.

Remarks: There are other coordinate systems besides Euclidean, cylindrical and spherical. One of them are torus coordinates, where $T(r, \phi, \theta) = (1 + r \cos(\phi)) \cos(\theta), (1 + r \cos(\phi)) \sin(\theta), r \sin(\phi)$), a coordinate system which works inside the solid torus $r \leq 1$. Are there spherical coordinates in higher dimensions? Yes, there are. They are called hyperspherical coordinates. In four dimensions for example we would have a third angle $\psi$ and get
\[
(x, y, z, w) = (\rho \sin(\psi) \sin(\phi) \sin(\theta), \rho \sin(\psi) \sin(\phi) \cos(\theta), \rho \sin(\psi) \cos(\phi), \rho \cos(\psi)).
\]
The four dimensional case is especially interesting because one can write the sphere $S^3$ in four dimensions as the set of pairs of complex numbers $z, w$ with $|z|^2 + |w|^2 = 1$. The 3 sphere is special because it is equal to the group $SU(2)$ of all unitary $2 \times 2$ matrices of determinant 1. It is also the set of all quaternions of length 1. The quaternions are historically interesting for multivariable calculus because they predated vector calculus we teach today and incorporate both the dot and cross product.
Homework

1. The density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1$. is given by the forth power of the distance to the z-axes: $\sigma(x, y, z) = (x^2 + y^2)^2$. Find its mass

$$M = \int \int \int_E (x^2 + y^2)^2 \, dx \, dy \, dz.$$ 

2. Find the moment of inertia $\int \int \int_E (x^2 + y^2) \, dV$ of the body $E$ whose volume is given by the integral

$$\text{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi.$$ 

3. A solid is described in spherical coordinates by the inequality $\rho \leq \sin(\phi)$. Find its volume.

4. Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

5. Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 1$. 


A vector field in the plane is a map, which assigns to each point \((x, y)\) in the plane a vector \(\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle\). A vector field in space is a map, which assigns to each point \((x, y, z)\) in space a vector \(\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle\).

For example \(\vec{F}(x, y) = \langle \frac{x-1}{((x-1)^2+y^2)^{3/2}} - \frac{x+1}{((x+1)^2+y^2)^{3/2}} \rangle\) is the electric field of positive and negative point charge. It is called dipole field. It is shown in the picture below.

If \(f(x, y)\) is a function of two variables, then \(\vec{F}(x, y) = \nabla f(x, y)\) is called a gradient field. Gradient fields in space are of the form \(\vec{F}(x, y, z) = \nabla f(x, y, z)\).

When is a vector field a gradient field? \(\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \nabla f(x, y)\) implies \(Q_x(x, y) = P_y(x, y)\). If this does not hold at some point, \(F\) is no gradient field.

**Clairaut test:** If \(Q_x(x, y) - P_y(x, y)\) is not zero at some point, then \(\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle\) is not a gradient field.

We will see next week that the condition \(\text{curl}(F) = Q_x - P_y = 0\) is also necessary for \(F\) to be a gradient field. In class, we see more examples on how to construct the potential \(f\) from the gradient field \(F\).

1. Is the vector field \(\vec{F}(x, y) = \langle P, Q \rangle = \langle 3x^2y + y + 2, x^3 + x - 1 \rangle\) a gradient field? **Solution:**

   the Clairaut test shows \(Q_x - P_y = 0\). We integrate the equation \(f_x = P = 3x^2y + y + 2\) and get \(f(x, y) = 2x + xy + x^3y + c(y)\). Now take the derivative of this with respect to \(y\) to get \(x + x^3 + c'(y)\) and compare with \(x^3 + x - 1\). We see \(c'(y) = -1\) and so \(c(y) = -y + c\). We see the solution \(x^3y + xy - y + 2x\).
2 Is the vector field $\vec{F}(x, y) = (xy, 2xy^2)$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

Vector fields are important in differential equations. We look at some examples in population dynamics and mechanics. You can skip this part. This is more motivational.

3 Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive $a, b$ because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $-cy(t) + dxy$, where $c, d$ are positive. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

$$
\begin{align*}
\dot{x} &= 0.4x - 0.4xy \\
\dot{y} &= -0.1y + 0.2xy 
\end{align*}
$$

Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x, y) = (x(t), y(t))$, there is a curve $\vec{r}(t) = (x(t), y(t))$ through that point for which the tangent $\vec{r}'(t) = (x'(t), y'(t))$ is the vector $(0.4x - 0.4xy, -0.1y + 0.2xy)$.

4 A class vector fields important in mechanics are **Hamiltonian fields**: If $H(x, y)$ is a function of two variables, then $(H_y(x, y), -H_x(x, y))$ is called a Hamiltonian vector field. An example is the harmonic oscillator $H(x, y) = (x^2 + y^2)/2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of $H$ (as you have shown in the homework with the chain rule).

5 Newton’s law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force $F$ acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of $t$, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course $y$ is the velocity of the mass point, so a pair $(x, y)$, thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.
We don't yet know yet the curve $t \mapsto \vec{r}(t) = (x(t), y(t))$, but we know the tangents $\vec{r}'(t) = (x'(t), y'(t)) = (y(t), -x(t))$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE's). More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point $(x, y)$, the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r \sin(t), r \cos(t))$ are solutions. They form a family of curves.

If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $G(x) = -gm \sin(x)$, where $m$ is the mass of the pendulum and where $g$ is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero. The Newton equation "mass times acceleration = force" gives

$$\ddot{x}(t) = -g \sin(x(t)) .$$

The equation of motion for the pendulum $\ddot{x}(t) = -g \sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g \sin(x(t))) .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $\vec{r}(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like $\sin, \cos, \exp, \log$ etc. However, one still can understand the curves.

Curves on the top of the picture represent situations where the velocity $y$ is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point
(0, 0), where the pendulum is at a stable rest, describe small oscillations of the pendulum.

**Vector fields in weather forecast** On weather maps, one can see *isotems*, curves of constant temperature or *isobars*, curves \( p(x, y) = c \) of constant pressure. These are level curves. The wind velocity \( \vec{F}(x, y) \) is close but not always exactly perpendicular to the *isobars*, the lines of equal pressure \( p \). In reality, the scalar pressure field \( p \) and the velocity field \( \vec{F} \) also depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

\[
\frac{d}{dt} \vec{F} + \vec{F} \cdot \nabla \vec{F} = \nu \Delta \vec{F} - \nabla p + f, \quad \text{div} \vec{F} = 0
\]

(where \( \Delta \) and div are defined later. This is an other example of a **partial differential equation**. It is one of the millenium problems to prove that these equations have smooth solutions in space.

**Homework**

1. The vector field \( \vec{F}(x, y) = \langle x/r^3, y/r^3 \rangle \) appears in electrostatics, where \( r = \sqrt{x^2 + y^2} \) is the distance to the charge. Find a function \( f(x, y) \) such that \( \vec{F} = \nabla f \). Hint. Write first \( \vec{F} = \langle P, Q \rangle \), where \( P, Q \) are functions of \( x, y \). Then integrate \( P \) with respect to \( x \).

2. a) Draw the gradient vector field of the function \( f(x, y) = \exp(x^2 - y^2) \).
   
   b) Draw the gradient vector field of the function \( f(x, y) = (x - 1)^2 + (y - 2)^2 \).
   
   **Hint:** In both cases, draw first a contour map of \( f \) and use a property of gradients to draw the vector field \( \vec{F}(x, y) = \nabla f \).

3. a) Is the vector field \( \vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x^2 \rangle \) a gradient field?
   
   b) Is the vector field \( \vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle \sin(x) + y, \cos(y) + x \rangle \) a gradient field?
   
   In both cases, give the potential \( f(x, y) \) satisfying \( \nabla f(x, y) = \vec{F}(x, y) \) if it exists and if there is no gradient, give a reason, why it is not a gradient field.

4. Which of the following vector fields \( \vec{F} = \langle P, Q \rangle \) can be written as \( \vec{F} = \langle P, Q \rangle = \langle f_x, f_y \rangle \)? Make use of Clairaut’s identity which
implies that $Q_x = P_y$, if a function $f$ exists. If $f$ exists, find the potential $f$.

a) $\vec{F}(x, y) = \langle x^7, y^9 \rangle$.
b) $\vec{F}(x, y) = \langle y^9, x^7 \rangle$.
c) $\vec{F}(x, y) = \langle 2y, 2x \rangle$.
d) $\vec{F}(x, y) = \langle y^2 + x^2, y^2 + x^2 \rangle$.
e) $\vec{F}(x, y) = \langle 9 - y^2 + 4x^3y^3, -2xy + 3x^4y^2 \rangle$.

The vector field

$$\vec{F}(x, y, z) = \langle 5x^4y + z^4 + y \cos(xy), x^5 + x \cos(xy), 4xz^3 \rangle$$

is a gradient field. Find the potential function $f$. 
Lecture 20: Line integral Theorem

If $\vec{F}$ is a vector field in the plane or in three dimensional space and $C : t \mapsto \vec{r}(t)$ is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

is called the **line integral** of $\vec{F}$ along the curve $C$.

The short-hand notation $\int_C \vec{F} \cdot \vec{dr}$ is also used. In physics, if $\vec{F}(x,y,z)$ is a force field, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$ is called **work**. In electrodynamics, if $\vec{F}(x,y,z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$ gives the electric potential.

1. Let $C : t \mapsto \vec{r}(t) = (\cos(t), \sin(t))$ be a circle parametrized by $t \in [0, 2\pi]$ and let $\vec{F}(x,y) = (-y,x)$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.

   **Solution:** We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) \, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt = 2\pi$.

2. Let $\vec{r}(t)$ be a curve given in polar coordinates as $\vec{r}(t) = (\cos(t), \phi(t)) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x,y) = (-xy,0)$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

   **Solution:** In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t) \sin(t))$. The velocity vector is then $\vec{r}'(t) = (-2\sin(t) \cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

   $$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^\pi (\cos^3(t) \sin(t), 0) \cdot (-2\sin(t) \cos(t), -\sin^2(t) + \cos^2(t)) \, dt$$

   $$= -2 \int_0^\pi \sin^2(t) \cos^4(t) \, dt = -2(t/16 + \sin(2t)/64 - sin(4t)/64 - sin(6t)/192)|_0^\pi = -\pi/8.$$  

Here is the first generalization of the fundamental theorem of calculus to higher dimensions. It is called the **fundamental theorem of line integrals**.

**Fundamental theorem of line integrals:** If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

In other words, the line integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if $\vec{F}$ is a gradient field.
3 Let $f(x, y, z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time $t$. The change of temperature for the fly is $\frac{d}{dt}f(\vec{r}(t))$. The line-integral of the temperature gradient $\nabla f$ along the path of the fly coincides with the temperature difference between the end point and initial point.

If $\vec{r}(t)$ is parallel to the level curve of $f$, then $\frac{d}{dt}f(\vec{r}(t)) = 0$ and $\vec{r}'(t)$ orthogonal to $\nabla f(\vec{r}(t))$.

If $\vec{r}(t)$ is orthogonal to the level curve, then $|\frac{d}{dt}f(\vec{r}(t))| = |\nabla f||\vec{r}'(t)|$ and $\vec{r}'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_a^b \frac{d}{dt}f(\vec{r}(t)) \, dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

For a gradient field, the line-integral along any closed curve is zero.

When is a vector field a gradient field? $\vec{F}(x, y) = \nabla f(x, y)$ implies $P_y(x, y) = Q_x(x, y)$. If this does not hold at some point, $\vec{F} = \langle P, Q \rangle$ is no gradient field. This is called the component test or Clairot test. We will see later that the condition $\text{curl}(\vec{F}) = Q_x - P_y = 0$ implies that the field is conservative, if the region satisfies a certain property.

4 Let $\vec{F}(x, y) = \langle 2xy^2 + 3x^2, 2yx^2 \rangle$. Find a potential $f$ of $\vec{F} = \langle P, Q \rangle$.

Solution: The potential function $f(x, y)$ satisfies $f_x(x, y) = 2xy^2 + 3x^2$ and $f_y(x, y) = 2yx^2$.

Integrating the second equation gives $f(x, y) = x^2y^2 + h(x)$. Partial differentiation with respect to $x$ gives $f_x(x, y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x, y) = x^2y^2 + x^3$. Find $g, h$ from $f(x, y) = \int_0^x P(x, y) \, dx + h(y)$ and $f_y(x, y) = g(x, y)$.

5 Let $\vec{F}(x, y) = \langle P, Q \rangle = \langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$. It is a gradient field because $f(x, y) = \text{arctan}(y/x)$ has the property that $f_x = (-y/x^2)/(1 + y^2/x^2) = P$, $f_y = (1/x)/(1 + y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F} \, d\vec{r}$, where $\gamma$ is the unit circle is

$$\int_0^{2\pi} \left( \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)} \right) \cdot (-\sin(t), \cos(t)) \, dt$$

which is $\int_0^{2\pi} 1 \, dt = 2\pi$. What is wrong?

Solution: note that the potential $f$ as well as the vector-field $F$ are not differentiable everywhere. The curl of $F$ is zero except at $(0,0)$, where it is not defined.
Remarks: The fundamental theorem of line integrals works in any dimension. You can formulate and check it yourself. The reason is that curves, vector fields, chain rule and integration along curves are easy to generalize in any dimensions. We will see next week that if $R$ is a region “without holes” then $\vec{F}$ is a gradient field if and only if $\text{curl}(F) = 0$ everywhere in $R$. A region $R$ is called simply connected, if every curve in $R$ can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annular region is an example which is not. Any region with a hole is not simply connected. For simply connected regions, the existence of a gradient field is equivalent to the field having curl zero everywhere.

A device which implements a non gradient force field is called a perpetual motion machine. Mathematically, it realizes a force field for which along some closed loops the energy gain is nonnegative. By possibly changing the direction, the energy change is positive. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up with and to see why they don’t work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.

Here is a more recent (of course futile) attempt:
Why does this "perpetual motion machine" not work? The former Harvard professor Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the "antropic principle". "Such a series of motions would receive the technical name of a "perpetual motion" by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".

Nonconservative fields can also be generated by **optical illusion** as M.C. Escher did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher’s pictures "work"?

**Homework**

1. Let $C$ be the space curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $t \in [0, 1]$ and let $\vec{F}(x, y, z) = \langle y, x, 5 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

2. Find the work done by the force field $\vec{F}(x, y) = \langle 2x \sin(y), 2y \rangle$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

3. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = \langle -y, x \rangle / 2$. Compute the line integral of $F$ along an ellipse $\vec{r}(t) = \langle a \cos(t), b \sin(t) \rangle$ with width $2a$ and height $2b$. The result should depend on $a$ and $b$. 
4 After this summer school, you relax in a Jacuzzi and move along curve $C$ which is given by part of the curve $x^{12} + y^{12} = 1$ in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity $\vec{F}(x, y) = \langle x, y^4 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$, the energy you gain from the fluid force when dislocating from $(1, 0)$ to $(0, 1)$.

5 Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x^2 \rangle$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \neq 0$. 
Lecture 21: Greens theorem

Green’s theorem is the second and last integral theorem in two dimensions. In this entire section we do multivariable calculus in 2D, where we have two derivatives, two integral theorems, the fundamental theorem of line integrals and Greens theorem. First two reminders:

If $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field and $C : \vec{r}(t) = \langle x(t), y(t) \rangle, t \in [a, b]$ is a curve, the line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}'(t) \, dt$$

measures the work done by the field $\vec{F}$ along the path $C$.

The curl of a two dimensional vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is the scalar field

$$\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y) \, .$$

The curl$(F)(x, y)$ measures the vorticity of the vector field at $(x, y)$.

One can write $\nabla \times \vec{F} = \text{curl}(\vec{F})$ because the two-dimensional cross product of $(\partial_x, \partial_y)$ with $\vec{F} = \langle P, Q \rangle$ is the scalar $Q_x - P_y$.

1. For $\vec{F}(x, y) = \langle -y, x \rangle$, we have $\text{curl}(F)(x, y) = 2$. For $\vec{F}(x, y) = \langle x^3 + y^2, y^3 + x^2y \rangle$, we have $\text{curl}(F)(x, y) = 2xy - 2y$.

2. If $\vec{F}(x, y) = \nabla f$ is a gradient field then the curl is zero because if $P(x, y) = f_x(x, y), Q(x, y) = f_y(x, y)$ and $\text{curl}(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut’s theorem. The field $\vec{F}(x, y) = \langle x + y, yx \rangle$ for example is not a gradient field because $\text{curl}(F) = y - 1$ is not zero.

Green’s theorem: If $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a smooth vector field and $R$ is a region for which the boundary $C$ is a curve parametrized so that $R$ is “to the left”, then

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_G \text{curl}(F) \, dxdy \, .$$

Proof. Consider a square $G = [x, x+h] \times [y, y+h]$ with small $h > 0$. The line integral of $\vec{F} = \langle P, Q \rangle$ along the boundary is $\int_0^h P(x+t, y) \, dt + \int_0^h Q(x+h, y+t) \, dt - \int_0^h P(x+t, y+h) \, dt - \int_0^h Q(x+y+t) \, dt$. It measures the ”circulation” at the place $(x, y)$. Because $Q(x + h, y) - Q(x, y) \sim Q_x(x, y)h$ and $P(x, y+h) - P(x, y) \sim P_y(x, y)h$, the line integral is $(Q_x - P_y)h^2 \sim \int_0^h \int_0^h \text{curl}(F) \, dxdy$. Now take a region $G$ with area $|G|$ and chop it into small squares of size $h$. We need about $|G|/h^2$ such squares. Summing up all the line integrals around the boundaries is the sum of the line integral along the boundary of $G$ because of the cancellations in the interior. On the boundary, it is a Riemann sum of the line integral along the boundary. The sum of the curls of the squares is a Riemann sum approximation of the double integral $\int \int_G \text{curl}(F) \, dxdy$. In the limit $h \to 0$, we obtain Greens theorem.
George Green lived from 1793 to 1841. He was a physicist, a self-taught mathematician as well as a miller. His work greatly contributed to modern physics.

3 If $\vec{F}$ is a gradient field then both sides of Green’s theorem are zero: $\int_C \vec{F} \cdot d\vec{r}$ is zero by the fundamental theorem for line integrals. and $\int_G \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

The already established Clairaut identity

\[
\text{curl}(\text{grad}(f)) = 0
\]

can also be remembered by writing $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{curl}(\nabla f) = \nabla \times \nabla f$. Use now that cross product of two identical vectors is 0. Working with $\nabla$ as a vector is called nabla calculus.

4 Find the line integral of $\vec{F}(x, y) = \langle x^2 - y^2, 2xy \rangle = \langle P, Q \rangle$ along the boundary of the rectangle $[0, 2] \times [0, 1]$. Solution: $\text{curl}(\vec{F}) = Q_x - P_y = 2y - 2y = -4y$ so that $\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \int_0^1 4y \, dy \, dx = 2y^2|_0^1 x^1|_0^2 = 4$.

5 Find the area of the region enclosed by

$\vec{r}(t) = \langle \frac{\sin(\pi t)^2}{t}, t^2 - 1 \rangle$

for $-1 \leq t \leq 1$. To do so, use Green’s theorem with the vector field $\vec{F} = \langle 0, x \rangle$. 
Green’s theorem allows to express the coordinates of the centroid = center of mass

\[
\left( \int \int_{G} x \, dA/A, \int \int_{G} y \, dA/A \right)
\]

using line integrals. With the vector field \( \vec{F} = (0, x^2) \) we have

\[
\int \int_{G} x \, dA = \int_{C} \vec{F} \, \vec{dr}.
\]

An important application of Green is the computation of area. Take a vector field like \( \vec{F}(x, y) = (P, Q) = (-y, 0) \) or \( \vec{F}(x, y) = (0, x) \) which has vorticity \( \text{curl}(\vec{F})(x, y) = 1 \). For \( \vec{F}(x, y) = (0, x) \), the right hand side in Green’s theorem is the area of \( G \):

\[
\text{Area}(G) = \int_{C} x(t) \dot{y}(t) \, dt.
\]

Let \( G \) be the region under the graph of a function \( f(x) \) on \([a, b]\). The line integral around the boundary of \( G \) is 0 from \((a, 0)\) to \((b, 0)\) because \( \vec{F}(x, y) = (0, 0) \) there. The line integral is also zero from \((b, 0)\) to \((b, f(b))\) and \((a, f(a))\) to \((a, 0)\) because \( N = 0 \). The line integral along the curve \((t, f(t))\) is

\[
- \int_{a}^{b} (-y(t), 0) \cdot (1, f'(t)) \, dt = \int_{a}^{b} f(t) \, dt.
\]

Green’s theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If \( \vec{F} \) is a gradient field then \( \text{curl}(F) = 0 \) everywhere.

Is the converse true? Here is the answer:

A region \( R \) is called **simply connected** if every closed loop in \( R \) can continuously be pulled together within \( R \) to a point inside \( R \).

If \( \text{curl}(\vec{F}) = 0 \) in a simply connected region \( G \), then \( \vec{F} \) is a gradient field.

Proof. Given a closed curve \( C \) in \( G \) enclosing a region \( R \). Green’s theorem assures that \( \int_{C} \vec{F} \, \vec{dr} = 0 \). So \( \vec{F} \) has the closed loop property in \( G \). This is equivalent to the fact that line integrals are path independent. In that case \( \vec{F} \) is therefore a gradient field: one can get \( f(x, y) \) by taking the line integral from an arbitrary point \( O \) to \((x, y)\). In the homework, you look at an example of a not simply connected region where the \( \text{curl}(\vec{F}) = 0 \) does not imply that \( \vec{F} \) is a gradient field.

An engineering application of Greens theorem is the **planimeter**, a mechanical device for measuring areas. We will demonstrate it in class. Historically it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field \( \vec{F} \) associated to a planimeter which is obtained by placing a unit vector perpendicular to the arm).

One can prove that \( \vec{F} \) has vorticity 1. The planimeter calculates the line integral of \( \vec{F} \) along a given curve. Green’s theorem assures it is the area.
Homework

1. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$ with $\vec{F} = \langle 2y + x \sin(y), x^2 \cos(y) - 3y^{200}\sin(y) \rangle$ along a triangle $C$ which traverses the vertices (0, 0), $(\pi, 0)$ and $(\pi, \pi)$ in this order.

2. Evaluate the line integral of the vector field $\vec{F}(x, y) = \langle 2xy^2, 2x^2 \rangle$ along the rectangle with vertices (0, 0), (2, 0), (2, 3), (0, 3) again in the order given.

3. Find the area of the region bounded by the hypocycloid $\vec{r}(t) = \langle 2\cos^3(t), 2\sin^3(t) \rangle$ using Green’s theorem. The curve is parameterized by $t \in [0, 2\pi]$.

4. Let $G$ be the region $x^6 + y^6 \leq 1$. Compute the line integral of the vector field $\vec{F}(x, y) = \langle x^{800} + \sin(x), y^{12} \rangle$ along the boundary.

5. This is a classic: let $\vec{F}(x, y) = \langle -y/(x^2 + y^2), x/(x^2 + y^2) \rangle$. Let $C : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle$, $t \in [0, 2\pi]$.
   a) Compute $\int_C \vec{F} \cdot d\vec{r}$.
   b) Show that curl($\vec{F}$) = 0 everywhere for $(x, y) \neq (0, 0)$.
   c) Let $f(x, y) = \arctan(y/x)$. Verify that $\nabla f = \vec{F}$.
   d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does a) and b) not contradict Green’s theorem?
Lecture 22: Curl and Divergence

The curl in two dimensions was the scalar field \( \text{curl}(F) = Q_x - P_y \). By Greens theorem, it had been the average work of the field done along a small circle of radius \( r \) around the point in the limit when the radius of the circle goes to zero. Greens theorem has explained what the curl is. In three dimensions, the curl is a vector:

\[
\text{curl} \left( \langle P, Q, R \rangle \right) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.
\]

In Nabla calculus notation this is \( \nabla \times \vec{F} \). Note that the third component of the curl is for fixed \( z \) just the two dimensional vector field \( \vec{F} = \langle P, Q \rangle \) is \( Q_x - P_y \). While the curl in 2 dimensions is a scalar field, it is a vector in 3 dimensions. In \( n \) dimensions, it would have \( n(n-1)/2 \) components, the number of 2-dimensional coordinate planes in \( n \) dimensions. The curl measures the "vorticity" of the field.

If a field has zero curl everywhere, the field is called **irrotational**.

The curl is often visualized using a "paddle wheel". If you place such a wheel into the field into the direction \( \vec{v} \), its rotation speed of the wheel measures the quantity \( \vec{F} \cdot \vec{v} \). Consequently, the direction in which the wheel turns fastest, is the direction of \( \text{curl}(\vec{F}) \). Its angular velocity is the length of the curl. The wheel could actually be used to measure the curl of the vector field at any point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl near the vortex center.

In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives, the **gradient**, the **curl** and the **divergence**.

The **divergence** of \( \vec{F} = \langle P, Q, R \rangle \) is the scalar field \( \text{div}(\langle P, Q, R \rangle) = \nabla \cdot \vec{F} = P_x + Q_y + R_z \).
The divergence can also be defined in two dimensions, but it is there not fundamental. We want in \( n \) dimensions to have \( n \) fundamental derivatives and for each a fundamental theorem.

The **divergence** of \( \vec{F} = \langle P, Q \rangle \) is \( \text{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y \).

In two dimensions, the divergence is just the curl of a \(-90\) degrees rotated field \( \vec{G} = \langle Q, -P \rangle \) because \( \text{div}(\vec{G}) = Q_x - P_y = \text{curl}(\vec{F}) \). The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

With the "vector" \( \nabla = \langle \partial_x, \partial_y, \partial_z \rangle \), we can write \( \text{curl}(\vec{F}) = \nabla \times \vec{F} \) and \( \text{div}(\vec{F}) = \nabla \cdot \vec{F} \). Formulating formulas using the "Nabla vector" and using rules from geometry is called **Nabla calculus**. This works both in 2 and 3 dimensions even so the \( \nabla \) vector is not an actual vector but an operator.

The following combination of divergence and gradient often appears in physics:

\[
\Delta f = \text{div}(\text{grad}(f)) = f_{xx} + f_{yy} + f_{zz}.
\]

It is called the Laplacian of \( f \). We can write \( \Delta f = \nabla^2 f \) because \( \nabla \cdot (\nabla f) = \text{div}(\text{grad}(f)) \).

We can extend the Laplacian also to vector fields with

\[
\Delta \vec{F} = \langle \Delta P, \Delta Q, \Delta R \rangle \text{ and write } \nabla^2 \vec{F}.
\]

Here are some identities:

- \( \text{div(curl}(\vec{F})) = 0 \).
- \( \text{curl(grad}(\vec{F}) = \vec{0} \).
- \( \text{curl(curl}(\vec{F})) = \text{grad(div}(\vec{F}) - \Delta(\vec{F})) \).

**Proof.** \( \nabla \cdot \nabla \times \vec{F} = 0 \).
\( \nabla \times \nabla \vec{F} = \vec{0} \).
\( \nabla \times \nabla \times \vec{F} = \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla) \vec{F} \).

**1 Question:** Is there a vector field \( \vec{G} \) such that \( \vec{F} = \langle x + y, z, y^2 \rangle = \text{curl}(\vec{G}) \)?

**Answer:** No, because \( \text{div}(\vec{F}) = 1 \) is incompatible with \( \text{div(curl}(\vec{G})) = 0 \).

**2** Show that in simply connected region, every irrotational and incompressible field can be written as a vector field \( \vec{F} = \text{grad}(f) \) with \( \Delta f = 0 \). Proof. Since \( \vec{F} \) is irrotational, there exists a function \( f \) satisfying \( F = \text{grad}(f) \). Now, \( \text{div}(F) = 0 \) implies \( \text{divgrad}(f) = \Delta f = 0 \).

**3** Find an example of a field which is both incompressible and irrotational. Solution. Find \( f \) which satisfies the Laplace equation \( \Delta f = 0 \), like \( f(x, y) = x^3 - 3xy^2 \), then look at its gradient field \( \vec{F} = \nabla f \). In that case, this gives

\[
\vec{F}(x, y) = \langle 3x^2 - 3y^2, -6xy \rangle.
\]
If we rotate the vector field $\vec{F} = \langle P, Q \rangle$ by 90 degrees = $\pi/2$, we get a new vector field $\vec{G} = \langle -Q, P \rangle$. The integral $\int_C \vec{F} \cdot ds$ becomes a flux $\int_\gamma \vec{G} \cdot dn$ of $G$ through the boundary of $R$, where $dn$ is a normal vector with length $|r'|dt$. With $\text{div}(\vec{F}) = (P_x + Q_y)$, we see that $\text{curl}(\vec{F}) = \text{div}(\vec{G})$.

Green’s theorem now becomes

$$\int\int_R \text{div}(\vec{G}) \, dxdy = \int_C \vec{G} \cdot \vec{dn} ,$$

where $dn(x, y)$ is a normal vector at $(x, y)$ orthogonal to the velocity vector $\vec{r}'(x, y)$ at $(x, y)$. This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don’t treat this however as a different theorem in two dimensions. It is just Green’s theorem in disguise.

This result shows:

The divergence at a point $(x, y)$ is the average flux of the field through a small circle of radius $r$ around the point in the limit when the radius of the circle goes to zero.

We have now all the derivatives we need. In dimension $d$, there are $d$ fundamental derivatives.

These derivatives are all examples of exterior derivatives. Let us stress that it is important to honor the dimensions. Many books treat two dimensional situations using terminology from three dimensions. This leads to confusion. Geometry in two dimensions should be treated like a ”flatlander” and ignoring the three dimensional space. It is a modern point of view is that geometry should be done with intrinsic notions which do not assume that geometry is part of a larger space. Integral theorems become more transparent if you look at them in the appropriate dimension. In one dimension, we had one theorem, the fundamental theorem of calculus. In two dimensions, there is the fundamental theorem of line integrals and Greens theorem. In three dimensions there are three theorems: the fundamental theorem of line integrals, Stokes theorem and the divergence theorem. We will look at the remaining two theorems in the next class.

---

1 A. Abbott, Flatland, A romance in many dimensions, 1884
Homework

1. Find your own nonzero vector field \( \vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle \) in each of the following cases:
   a) \( \vec{F} \) is irrotational but not incompressible.
   b) \( \vec{F} \) is incompressible but not irrotational.
   c) \( \vec{F} \) is irrotational and incompressible.
   d) \( \vec{F} \) is not irrotational and not incompressible.

2. The vector field \( \vec{F}(x, y, z) = \langle x, y, -2z \rangle \) satisfies \( \text{div}(\vec{F}) = 0 \). Can you find a vector field \( \vec{G}(x, y, z) \) such that \( \text{curl}(\vec{G}) = \vec{F} \)? Such a field \( \vec{G} \) is called a vector potential.
   \textbf{Hint.} Write \( \vec{F} \) as a sum \( \langle x, 0, -z \rangle + \langle 0, y, -z \rangle \) and find vector potentials for each of the parts using a vector field you have seen on the blackboard in class.

3. Evaluate the flux integral \( \iint_S \langle 0, 0, yz \rangle \cdot \vec{dS} \), where \( S \) is the surface with parametric equation \( x = uv, y = u + v, z = u - v \) on \( R : u^2 + v^2 \leq 4 \).

4. Evaluate the flux integral \( \iint_S \text{curl}(\vec{F}) \cdot \vec{dS} \) for \( \vec{F}(x, y, z) = \langle xy, yz, zx \rangle \), where \( S \) is the part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the square \([0, 1] \times [0, 1]\) and has an upward orientation.

5. a) What is the relation between the flux of the vector field \( \vec{F} = \nabla g / |\nabla g| \) through the surface \( S : \{ g = 1 \} \) with \( g(x, y, z) = x^6 + y^4 + 2z^8 \) and the surface area of \( S \)?
   b) Find the flux of the vector field \( \vec{G} = \nabla g \times \langle 0, 0, 1 \rangle \) through the surface \( S \).

\textbf{Remark} This problem, both part a) and part do not need any computation. You can answer each question with one sentence. In part a) compare \( \vec{F} \cdot \vec{dS} \) with \( dS \) in that case.
Lecture 23: Stokes Theorem

Given a surface $S$ parametrized as $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over a domain $G$ in the $uv$-plane.

The **flux integral** of $\vec{F}$ through $S$ is defined as the double integral

$$\int \int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv.$$

We use the short hand notation $d\vec{S} = (\vec{r}_u \times \vec{r}_v) \, du \, dv$ and think of $d\vec{S}$ as an infinitesimal normal vector to the surface. The flux integral can be abbreviated as $\int \int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if $\vec{F} =$ fluid velocity field, then $\int \int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through $S$ in unit time.

Because $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$ is a unit vector normal to the surface and on the surface, $\vec{F} \cdot \vec{n}$ is the normal component of the vector field with respect to the surface. One could write therefore also $\int \int_S \vec{F} \cdot d\vec{S} = \int \int \vec{F} \cdot \vec{n} \, dS$, where $dS$ is the surface element we know from when we computed surface area. The function $\vec{F} \cdot \vec{n}$ is the scalar projection of $\vec{F}$ in the normal direction. Whereas the formula $\int \int 1 \, dS$ gave the area of the surface with $dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv$, the flux integral weights each area element $dS$ with the normal component of the vector field with $\vec{F}(\vec{r}(u, v)) \cdot \vec{n}(\vec{r}(u, v))$. It is important that we do not want to use this formula for computations (even so it appears in books) because finding $\vec{n}$ gives additional work. We just determine the vectors $\vec{F}(\vec{r}(u, v))$ and $\vec{r}_u \times \vec{r}_v$ and integrate its dot product over the domain $G$.

1. Compute the flux of $\vec{F}(x, y, z) = (0, 1, z^2)$ through the upper half sphere $S$ parametrized by $\vec{r}(u, v) = \langle \cos(u) \sin(v), \sin(u) \sin(v), \cos(v) \rangle$.

**Solution.** We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u, v)) = \langle 0, 1, \cos^2(v) \rangle$ so that

$$\int_0^{2\pi} \int_0^{\pi} -\langle 0, 1, \cos^2(v) \rangle \cdot \langle \cos(u) \sin^2(v), \sin(u) \sin^2(v), \cos(v) \sin(v) \rangle \, du \, dv.$$

The flux integral is $\int_0^{2\pi} \int_{\pi/2}^{\pi} -\sin^2(v) \sin(u) - \cos^3(v) \sin(v) \, dv \, du$ which is $-\int_{\pi/2}^{\pi} \cos^3 v \sin v \, dv = \cos^4(v)/4|_{\pi/2}^{\pi/2} = -1/4$. 

2 Calculate the flux of the vector field \( \vec{F}(x, y, z) = (1, 2, 4z) \) through the paraboloid \( z = x^2 + y^2 \) lying above the region \( x^2 + y^2 \leq 1 \). **Solution:** We can parametrize the surface as \( \vec{r}(r, \theta) = (r \cos(\theta), r \sin(\theta), r^2) \) where \( \vec{r} \times \vec{r}_\theta = (-2r^2 \cos(\theta), -2r^2 \sin(\theta), r) \) and \( \vec{F}(\vec{r}(u, v)) = (1, 2, 4r^2) \). We get \( \int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2 \cos(v) - 4r^2 \sin(v) + 4r^3) \, dr \, d\theta = 2\pi. \)

3 Compute the flux of \( \vec{F}(x, y, z) = (2, 3, 1) \) through the torus parameterized as \( \vec{r}(u, v) = ((2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v)) \), where both \( u \) and \( v \) range from 0 to \( 2\pi \). **Solution:** There is no computation is needed. Think about what the flux means.

The following theorem is the second fundamental theorem of calculus in three dimensions:

**Stokes theorem:** Let \( S \) be a surface bounded by a curve \( C \) and \( \vec{F} \) be a vector field. Then

\[
\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.
\]

Proof. Stokes theorem can be proven in the same way than Greens theorem. Chop up \( S \) into a union of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along the boundaries adds up to the line integral of the boundary of \( S \). Stokes theorem for a small triangle can be reduced to Greens theorem because with a coordinate system such that the triangle is in the \( xy \) plane, the flux of the field is the double integral of \( \text{curl} \vec{F} \cdot d\vec{S} = \text{curl} \vec{F}(\vec{r}) \cdot \vec{n} \, dudv = (Q_x - P_y) \cos(\theta) \, dudv \) where \( \theta \) is the angle between the normal vector and \( \vec{F} = (P, Q, R) \). On the other hand, since the power \( \vec{F}(\vec{r}) \cdot r'(t) dt = (P(\vec{r}) \cos(\theta) x'(t) + Q(\vec{r}) \cos(\theta) y'(t)) dt \) also has everything multiplied by \( \cos(\theta) \), the result for the space triangle follows from Green.

4 Let \( \vec{F}(x, y, z) = (-y, x, 0) \) and let \( S \) be the upper semi hemisphere, then \( \text{curl}(\vec{F})(x, y, z) = (0, 0, 2) \). The surface is parameterized by \( \vec{r}(u, v) = (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \) on \( G = [0, 2\pi] \times [0, \pi/2] \) and \( \vec{r}_u \times \vec{r}_v = \sin(v) \vec{r}(u, v) \) so that \( \text{curl}(\vec{F})(x, y, z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v) \sin(v) 2 \). The integral \( \int_0^{2\pi} \int_0^{\pi/2} \sin(2v) \, dv \, du = 2\pi. \)

The boundary \( C \) of \( S \) is parameterized by \( \vec{r}(t) = (\cos(t), \sin(t), 0) \) so that \( d\vec{r} = r'(t) dt = (-\sin(t), \cos(t), 0) \, dt \) and \( \vec{F}(\vec{r}(t)) \cdot r'(t) dt = \sin(t)^2 + \cos^2(t) = 1 \). The line integral \( \int_C \vec{F} \cdot d\vec{r} \) along the boundary is \( 2\pi \).
If \( S \) is a surface in the \( xy \)-plane and \( \vec{F} = \langle P, Q, 0 \rangle \) has zero \( z \) component, then \( \text{curl}(\vec{F}) = \langle 0, 0, Q_x - P_y \rangle \) and \( \text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y \, dxdy \). We see that for a surface which is flat, Stokes theorem is a consequence of Green’s theorem. If we put the coordinate axis so that the surface is in the \( xy \)-plane, then the vector field \( F \) induces a vector field on the surface such that its 2D curl is the normal component of \( \text{curl}(\vec{F}) \). The reason is that the third component \( Q_x - P_y \) of \( \text{curl}(\vec{F}) \) is the two dimensional curl:

\[
\vec{F}(\vec{r}(u, v)) \cdot \langle 0, 0, 1 \rangle = Q_x - P_y.
\]

If \( C \) is the boundary of the surface, then

\[
\int \int_S \vec{F}(\vec{r}(u, v)) \cdot (0, 0, 1) \, dudv = \int_C \vec{F}(\vec{r}(t)) \, \vec{r}'(t) \, dt.
\]

Calculate the flux of the curl of

\[
\vec{F}(x, y, z) = \langle -y, x, 0 \rangle
\]

through the surface parameterized by

\[
\vec{r}(u, v) = \langle \cos(u) \cos(v), \sin(u) \cos(v), \cos^2(v) + \cos(v) \sin^2(u + \pi/2) \rangle.
\]

Because the surface has the same boundary as the upper half sphere, the integral is again \( 2\pi \) as in the above example.

For every surface bounded by a curve \( C \), the flux of \( \text{curl}(\vec{F}) \) through the surface is the same. Proof. The flux of the curl of a vector field through a surface \( S \) depends only on the boundary of \( S \). Compare this with the earlier statement that for every curve between two points \( A, B \) the line integral of \( \text{grad}(f) \) along \( C \) is the same. The line integral of the gradient of a function of a curve \( C \) depends only on the end points of \( C \).

Electric and magnetic fields are linked by the **Maxwell equation** \( \text{curl}(\vec{E}) = -\frac{1}{c} \vec{B} \). If a closed wire \( C \) bounds a surface \( S \) then \( \int \int_S \vec{B} \cdot d\vec{S} \) is the flux of the magnetic field through \( S \). Its change can be related with a voltage using Stokes theorem:

\[
d/dt \int \int_S \vec{B} \cdot d\vec{S} = \int \int_S \vec{B} \cdot d\vec{S} = \int \int_S -c \, \text{curl}(\vec{E}) \cdot d\vec{S} = -c \int_C \vec{E} \cdot d\vec{r} = U, \]

where \( U \) is the voltage. If we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator like an alternator in a car. Stokes theorem explains why we can generate electricity from motion.

**Stokes theorem was found by Ampère in 1825.** George Gabriel Stokes (1819-1903) (probably inspired by work of Green) rediscovers the identity around 1840.

George Gabriel Stokes

André Marie Ampère
1 Find \( \int_C \vec{F} \cdot d\vec{r} \), where \( \vec{F}(x, y, z) = \langle 2x^2y, 2x^3/3, 2xy \rangle \) and \( C \) is the curve of intersection of the hyperbolic paraboloid \( z = y^2 - x^2 \) and the cylinder \( x^2 + y^2 = 1 \), oriented counterclockwise as viewed from above.

2 If \( S \) is the surface \( x^8 + y^8 + z^6 = 1 \) and assume \( \vec{F} \) is a smooth vector field in space. Explain why \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0 \).

3 Evaluate the flux integral \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \), where 
\[
\vec{F}(x, y, z) = \langle xe^{y^2}z^3 + 2xyz e^{x^2+z}, x + z^2 e^{x^2+z}, ye^{x^2+z} + ze^x \rangle
\]
and where \( S \) is the part of the ellipsoid \( x^2 + y^2/4 + (z + 1)^2 = 2, \) \( z > 0 \) oriented so that the normal vector points upwards.

4 Find the line integral \( \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is the circle of radius 3 in the \( xz \)-plane oriented counter clockwise when looking from the point \((0, 1, 0)\) onto the plane and where \( \vec{F} \) is the vector field
\[
\vec{F}(x, y, z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z)) \rangle.
\]
Use a convenient surface \( S \) which has \( C \) as a boundary.

5 Find the flux integral \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \), where \( \vec{F}(x, y, z) = \langle 2 \cos(\pi y)e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y)e^{2x}, 2xz \rangle \)
and \( S \) is the surface parametrized by
\[
\vec{r}(s, t) = \langle (1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s \rangle
\]
with \( 0 \leq t \leq 2\pi, 0 \leq s \leq 1 \) and oriented so that the normal vectors point to the outside of the thorn.
Lecture 24: Divergence theorem

There are three integral theorems in three dimensions. We have seen already the fundamental theorem of line integrals and Stokes theorem. Here is the divergence theorem, which completes the list of integral theorems in three dimensions:

Divergence Theorem. Let $E$ be a solid with boundary surface $S$ oriented so that the normal vector points outside. Let $\mathbf{F}$ be a vector field. Then

$$\int \int \int_E \text{div}(\mathbf{F}) \, dV = \int \int_S \mathbf{F} \cdot dS.$$

To prove this, one can look at a small box $[x, x + dx] \times [y, y + dy] \times [z, z + dz]$. The flux of $\mathbf{F} = \langle P, Q, R \rangle$ through the faces perpendicular to the $x$-axes is $[\mathbf{F}(x + dx, y, z) \cdot \langle 1, 0, 0 \rangle + \mathbf{F}(x, y, z) \cdot \langle -1, 0, 0 \rangle]dydz = P(x + dx, y, z) - P(x, y, z) = P_x \, dx \, dy \, dz$. Similarly, the flux through the $y$-boundaries is $P_y \, dy \, dx \, dz$ and the flux through the two $z$-boundaries is $P_z \, dz \, dx \, dy$. The total flux through the faces of the cube is $(P_x + P_y + P_z) \, dx \, dy \, dz = \text{div} (\mathbf{F}) \, dx \, dy \, dz$. A general solid can be approximated as a union of small cubes. The sum of the fluxes through all the cubes consists now of the flux through all faces without neighboring faces and fluxes through adjacent sides cancel. The sum of all the fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\text{div} (\mathbf{F}) \, dx \, dy \, dz$ is a Riemann sum approximation for the integral $\int \int \int_G \text{div} (\mathbf{F}) \, dx \, dy \, dz$. In the limit, where $dx, dy, dz$ goes to zero, we obtain the divergence theorem.
The theorem explains what divergence means. If we average the divergence over a small cube is equal the flux of the field through the boundary of the cube. If this is positive, then more field exists the cube than entering the cube. There is field "generated" inside. The divergence measures the expansion of the field.

1. Let \( \vec{F}(x,y,z) = \langle x, y, z \rangle \) and let \( S \) be sphere. The divergence of \( \vec{F} \) is the constant function \( \text{div}(\vec{F}) = 3 \) and \( \int \int \int_\Omega \text{div}(\vec{F}) \, dV = 3 \cdot 4\pi/3 = 4\pi \). The flux through the boundary is \( \int \int S \vec{r} \cdot (\vec{r}_u \times \vec{r}_v) \, dudv = \int \int S |\vec{r}(u,v)|^2 \sin(v) \, dudv = \int_0^{2\pi} 4\pi \sin(v) \, dv = 4\pi \) also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

2. What is the flux of the vector field \( \vec{F}(x,y,z) = \langle 2x, 3z^2 + y, \sin(x) \rangle \) through the solid \( G = [0,3] \times [0,3] \times [0,3] \setminus ([0,3] \times [1,2] \times [0,3] \cup [1,2] \times [0,3] \times [0,3] \cup [0,3] \times [1,2] \times [0,3]) \) which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the divergence theorem: \( \text{div}(\vec{F}) = 2 \) and so \( \int \int \int_G \text{div}(\vec{F}) \, dV = 2 \int \int \int_G dV = 2\text{Vol}(G) = 2(27 - 7) = 40 \). Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

3. Find the flux of \( \text{curl}(\vec{F}) \) through a torus if \( \vec{F} = \langle yz^2, z + \sin(x) + y, \cos(x) \rangle \) and the torus has the parametrization \( \vec{r}(\theta, \phi) = \langle (2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi) \rangle \).

**Solution:** The answer is 0 because the divergence of \( \text{curl}(\vec{F}) \) is zero. By the divergence theorem, the flux is zero.

4. Similarly as Green’s theorem allowed to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field \( \vec{F}(x,y,z) = \langle x, 0, 0 \rangle \) which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: \( \int \int \delta G \langle x, 0, 0 \rangle \cdot d\vec{S} = \text{Vol}(G) \).

5. How heavy are we, at distance \( r \) from the center of the earth?

**Solution:** The law of gravity can be formulated as \( \text{div}(\vec{F}) = 4\pi \rho \), where \( \rho \) is the mass density. We assume that the earth is a ball of radius \( R \). By rotational symmetry, the gravitational force is normal to the surface: \( \vec{F}(\bar{x}) = \vec{F}(r)\bar{x}/||\bar{x}|| \). The flux of \( \vec{F} \) through a ball of radius \( r \) is \( \int \int_S \vec{F}(x) \cdot d\vec{S} = 4\pi r^2 \vec{F}(r) \). By the **divergence theorem**, this is
\[ 4\pi M_r = 4\pi \int \int f_{B_r} \rho(x) \ dV, \] where \( M_r \) is the mass of the material inside \( S_r \). We have \((4\pi)^2 \rho r^3/3 = 4\pi r^2 \hat{F}(r)\) for \( r < R \) and \((4\pi)^2 \rho R^3/3 = 4\pi r^2 \hat{F}(r)\) for \( r \geq R \). Inside the earth, the gravitational force \( \hat{F}(r) = 4\pi \rho r/3 \). Outside the earth, it satisfies \( \hat{F}(r) = M/r^2 \) with \( M = 4\pi R^3 \rho/3 \).

To the end we make an overview over the integral theorems and give an other typical example in each case.

The fundamental theorem for line integrals, Green’s theorem, Stokes theorem and divergence theorem are all incarnation of one single theorem \( \int_A dF = \int_{\partial A} F \), where \( dF \) is a exterior derivative of \( F \) and where \( \partial A \) is the boundary of \( A \). They all generalize the fundamental theorem of calculus.

**Fundamental theorem of line integrals**: If \( C \) is a curve with boundary \( \{A, B\} \) and \( f \) is a function, then

\[
\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)
\]

**Remarks.**

1) For closed curves, the line integral \( \int_C \nabla f \cdot \vec{dr} \) is zero.
2) Gradient fields are path independent: if \( \vec{F} = \nabla f \), then the line integral between two points \( P \) and \( Q \) does not depend on the path connecting the two points.
3) The theorem holds in any dimension. In one dimension, it reduces to the fundamental theorem of calculus \( \int_a^b f'(x) \ dx = f(b) - f(a) \)
4) The theorem justifies the name conservative for gradient vector fields.
5) The term ”potential” was coined by George Green who lived from 1783-1841.

**Example.** Let \( f(x, y, z) = x^2 + y^4 + z \). Find the line integral of the vector field \( \vec{F}(x, y, z) = \nabla f(x, y, z) \) along the path \( \vec{r}(t) = \langle \cos(5t), \sin(2t), t^2 \rangle \) from \( t = 0 \) to \( t = 2\pi \).

**Solution.** \( \vec{r}(0) = \langle 1, 0, 0 \rangle \) and \( \vec{r}(2\pi) = \langle 1, 0, 4\pi^2 \rangle \) and \( f(\vec{r}(0)) = 1 \) and \( f(\vec{r}(2\pi)) = 1 + 4\pi^2 \). The fundamental theorem of line integral gives \( \int_C \nabla f \ d\vec{r} = f(r(2\pi)) - f(r(0)) = 4\pi^2 \).

**Green’s theorem.** If \( R \) is a region with boundary \( C \) and \( \vec{F} \) is a vector field, then

\[
\int \int_R \text{curl}(\vec{F}) \ dxdy = \int_C \vec{F} \cdot \vec{dr}.
\]
Remarks.
1) Greens theorem allows to switch from double integrals to one dimensional integrals.
2) The curve is oriented in such a way that the region is to the left.
3) The boundary of the curve can consist of piecewise smooth pieces.
4) If $C : t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$, the line integral is $\int_C \langle P(x(t), y(t)), Q(x(t), y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$.
5) Green’s theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If curl($\vec{F}$) = 0 in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
7) Taking $\vec{F}(x, y) = \langle -y, 0 \rangle$ or $\vec{F}(x, y) = \langle 0, x \rangle$ gives area formulas.

Example. Find the line integral of the vector field $\vec{F}(x, y) = \langle x^4 + \sin(x) + y, x + y^3 \rangle$ along the path $\vec{r}(t) = \langle \cos(t), 5\sin(t) + \log(1 + \sin(t)) \rangle$, where $t$ runs from $t = 0$ to $t = \pi$.

Solution. curl($\vec{F}$) = 0 implies that the line integral depends only on the end points $(0, 1), (0, -1)$ of the path. Take the simpler path $\vec{r}(t) = \langle -t, 0 \rangle, -1 \leq t \leq 1$, which has velocity $\vec{r}'(t) = \langle -1, 0 \rangle$. The line integral is $\int_{-1}^1 (-t^4 - \sin(-t), -t) \cdot \langle -1, 0 \rangle \, dt = -t^5/5|_{-1}^1 = -2/5$.

Remark We could also find a potential $f(x, y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that grad($f$) = $\vec{F}$. Again, we get $f(0, -1) - f(0, 1) = -1/5 - 1/5 = -2/5$.

Stokes theorem. If $S$ is a surface with boundary $C$ and $\vec{F}$ is a vector field, then

$$\int \int_S \text{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r}.$$ 

Remarks.
1) Stokes theorem allows to derive Greens theorem: if $\vec{F}$ is $z$-independent and the surface $S$ is contained in the $xy$-plane, one obtains the result of Green.
2) The orientation of $C$ is such that if you walk along $C$ and have your head in the direction of the normal vector $\vec{n} \times \vec{r}$, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).
4) The flux of the curl of a vector field does not depend on the surface $S$, only on the boundary of $S$.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

Example. Compute the line integral of $\vec{F}(x, y, z) = \langle x^3 + xy, y, z \rangle$ along the polygonal path $C$ connecting the points $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.

Solution. The path $C$ bounds a surface $S: \vec{r}(u, v) = \langle u, v, 0 \rangle$ parameterized by $R = [0, 2] \times [0, 1]$.

By Stokes theorem, the line integral is equal to the flux of curl($\vec{F}$)($x, y, z$) = $\langle 0, 0, -x \rangle$ through $S$. The normal vector of $S$ is $\vec{n} \times \vec{r}_v = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$ so that $\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 \langle 0, 0, -u \rangle \cdot \langle 0, 0, 1 \rangle \, du \, dv = \int_0^2 \int_0^1 -u \, du \, dv = -2$.

Divergence theorem: If $S$ is the boundary of a region $E$ in space and $\vec{F}$ is a vector field, then

$$\int \int \int_E \text{div}(\vec{F}) \, dV = \int \int_S \vec{F} \cdot d\vec{S}.$$
Remarks.
1) The divergence theorem is also called **Gauss theorem**.
2) It can be helpful to determine the flux of vector fields through surfaces.
3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

**Example.** Compute the flux of the vector field $\vec{F}(x, y, z) = \langle -x, y, z^2 \rangle$ through the boundary $S$ of the rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.

**Solution.** By Gauss theorem, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box:

$$
\int_0^3 \int_{-1}^2 \int_1^2 2z \, dx \, dy \, dz = (3 - 0)(2 - (-1))(4 - 1) = 27.
$$

How do these theorems fit together? In $n$-dimensions, there are $n$ theorems. We have here seen the situation in dimension $n=2$ and $n=3$, but one could continue. The fundamental theorem of line integrals generalizes directly to higher dimensions. Also the divergence theorem generalizes directly since an $n$-dimensional integral in $n$ dimensions. The generalization of curl and flux is more subtle, since in 4 dimensions already, the curl of a vector field is a 6 dimensional object. It is a $n(n-1)/2$ dimensional object in general.

![Diagram of theorems]

In one dimensions, there is one derivative $f(x) \rightarrow f'(x)$ from scalar to scalar functions. It corresponds to the entry $1 \rightarrow 1$ in the Pascal triangle. The next entry $1 \rightarrow 1$ corresponds to differentiation in two dimensions, where we have the gradient $f \rightarrow \nabla f$ mapping a scalar function to a vector field with 2 components as well as the curl, $F \rightarrow \text{curl}(F)$ which corresponds to the transition $2 \rightarrow 1$. The situation in three dimensions is captured by the entry $1 \rightarrow 1 \rightarrow 1$ in the Pascal triangle. The first derivative $1 \rightarrow 1$ is the gradient. The second derivative $3 \rightarrow 3$ is the curl and the third derivative $3 \rightarrow 1$ is the divergence. In $n = 4$ dimensions, we would have to look at $1 \rightarrow 4 \rightarrow 6 \rightarrow 4 \rightarrow 1$. The first derivative $1 \rightarrow 4$ is still the gradient. Then we have a first curl, which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps an object with 6 components back to a vector field, we would have to look at $1 \rightarrow 4 \rightarrow 6 \rightarrow 4 \rightarrow 1$. When setting up calculus in dimension $n$, one talks about **differential forms** instead of scalar fields or vector fields. Functions are 0 forms or $n$-forms. Vector fields can be described by 1 or $n-1$ forms. The general formalism defines a derivative $d$ called **exterior derivative** on differential forms as well as integration of such $k$ forms on $k$ dimensional objects.

There is a **boundary operation** $\delta$ which maps a $k$-dimensional object into a $k-1$ dimensional object. This boundary operation is dual to differentiation. They both satisfy the same relation $dd(F) = 0$ and $\delta \delta G = 0$. Differentiation and integration are linked by the general Stokes theorem:

$$
\int_{\delta G} F = \int_G dF
$$
which becomes a single theorem called **fundamental theorem of multivariable calculus**. The theorem becomes much simpler in quantum calculus, where geometric objects and differential forms are on the same footing. There are various ways how one can generalize this. One way is to write it as $< \delta G, F > = < G, dF >$ which in linear algebra would be written in the form $< A^T v, w > = < v, Aw >$, where $A$ is a matrix and $< v, w >$ is the dot product. Since in our traditional calculus we deal with "smooth" functions and fields that we have to pay a prize and consider in turn "singular" objects like points or curves and surfaces. These are idealized objects which have zero diameter, radius or thickness. Nature likes simplicity and elegance \(^1\): and has chosen quantum mathematics to be more fundamental. But the symmetry in which **geometry and fields become indistinguishable** manifests only in the very small. While it is already well understood mathematically, it will take a while until such formalisms will enter calculus courses.

**Homework**

1. Compute using the divergence theorem the flux of the vector field $\vec{F}(x, y, z) = \langle 5y, xy, 2yz \rangle$ through the unit cube $[0, 1] \times [0, 1] \times [0, 1]$.

2. Find the flux of the vector field $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ through the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.

3. Use the divergence theorem to calculate the flux of $\vec{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ through the sphere $S : x^2 + y^2 + z^2 = 1$, where the sphere is oriented so that the normal vector points outwards.

4. Assume the vector field

$$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int_S \vec{F} \cdot d\vec{S}$.

Find $\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and $S$ is the boundary of the solid built with 9 unit cubes shown in the picture.

---

\(^1\)Leibniz: 1646-1716