\[ \chi(G) = \sum K(x) \]

\[ K(x) = \sum (-1)^k V_{k-1} \]

\[ = 1 - \frac{V_0}{2} + \frac{V_1}{3} - \ldots \]

\[ V_k = \# K_{k+1} \text{ subgraphs in sphere } S(x) \]
total curvature = 2
\[ \chi(G) = b_0 - b_1 + b_2 \]
\[ \chi = 4 \]
\[ x = 2 - 3 = -1 \]
\[ \sum_{x \in V} V_{k-1}(x) = (k+1) V_k \]

\( k=0: \) vertex count

\( k=1: \) Euler:

\( k=2: \) every triangle gets counted three times
PROOF OF GAUSS-BONNET

\[ \sum_{x \in V} K(x) = \sum_{x \in V} \sum_{k=0}^{\infty} (-1)^k V_{k-1}^V(x) \]

\[ = \sum_{k=0}^{\infty} (-1)^k \sum_{x \in V} V_{k-1}^V(x) \]

\[ = \sum_{k=0}^{\infty} (-1)^k v_k \]

\[ = \chi(G) \]

**definition**

**order of summation**

**handshake**

**definition**
S^\prime(x) = S(x) \cap \{f(y) < f(x)\}

S^\prime(x) \text{ not contractible}

\chi(S^\prime(x)) = i_f(x)

i_f(x), is zero at regular points
POINCARE HOPF

\[ \chi(G) = \sum_x i_f(x) \]

\[ i_f(x) = 1 - \chi(S^-) \]

\[ S^- = S(x) \cap \{ f(y) < f(x) \} \]
PROOF

induction. Start at minimum, then add points

\[
\chi(G \cup B(x)) = \chi(G) + \chi(B(x)) - \chi(S(x))
\]

\[
= \chi(G) + 1 - \chi(S(x))
\]

\[
= \chi(G) + \imath(f(x))
\]

\[
\chi(G \cup H) = \chi(G) + \chi(H) - \chi(G \cap H)
\]
Morse function $f$

saddles

max

min
INTEGRAL GEOMETRY

Blaschke 1885-1962  Chern 1911-2004  Banchoff
\[ \chi(G) = \sum_i \chi_f(x) \]

averaging over \( f \)

\[ \chi(G) = \sum K(x) \]
\[ K(x) = \mathbb{E}\left[ i_f(x) \right] \]
\[ S \cap \{ f(y) < f(x) \} = S^- \]

positive curvature

\[ S \cap \{ f(y) < f(x) \} = \text{two arcs} \]

negative curvature

\[ S \cap \{ f(y) < f(x) \} = \emptyset \]

positive curvature
$\chi(S \cap \{f(y)<f(x)\}) = -1$

$1 - \chi(S \cap \{f(y)<f(x)\}) = 1$

$1 - \chi(S \cap \{f(y)<f(x)\}) = -1$
RIEMANN    ROCH

1826-1866    1839-1866
RIEMANN ROCH


G graph triangle free  D divisor
K canonical divisor  L=Laplacian

$K = \sum (\text{deg } (x)-2) (x)$

$r(D) - r(K-D) = \chi(G) + \text{deg}(D)$

principal divisors:  $(L f)$

Jacobi Group=Zero divisors/Principal divisors

Linear system: effective divisors $\sim D$
\(r(D) = \text{dimension}\)
\( \chi(G) = 1 \)
\( \chi(D) = 6 \)

\( D = (5,1) \)

\( K = (-1,-1) \) principal divisor

\( K - D = (-6,-2) \)

\( r(D) = 6 \)

\( r(K - D) = -1 \)

\( \chi(G) + \deg(D) = 7 \)

\( \chi(G) = 1 - g = 1 - 0 = 1 \)

\( \deg(D) = 5 + 1 = 6 \)
RIEMANN-HURWITZ

Work with Thomas Tucker

\[ \chi(G) = n \chi(H) - \sum_{x} (e_{x} - 1) \]

\[-4 = 5 \times 0 - (5 - 1)\]

\[ e_{x}^{-1} = \sum_{a(x) = x} (-1)^{k(x)} \]

A = \mathbb{Z}_{5}^{n}

order n
\[ A = \mathbb{Z}_2 \]

\[ \chi(G) = 2 \chi(H) - \sum_{x \text{ fixed}} (e_x - 1) \]

\[ e_x = 2 \]

\[ H = G/A \]
A COROLLARY OF RH

Assume G has no edges.

\[ |G| = n \ |G/A| - \sum e_x - 1 \]

\[ 0 = n \ |G/A| - \sum e_x \]

\[ |G/A| = \frac{1}{n} \sum e_x \]

\[ |G/A| = \frac{1}{n} \sum |X^a| \]

\[ \sum_{x} \sum_{a} X^a (x) \]

\[ \sum_{a} \sum_{x} X^a (x) \]

\[ \sum_{a} |X^a| \]

Burnside Lemma follows from Riemann Hurwitz!
A DEEPER APPLICATION OF RH

$A = \mathbb{Z}_2$

$H = G / A$

intersection points: ramifications

$\chi(G) = 2 \chi(H) - \chi(U)$
HURWITZ

1859-1919
A HOMEOMORPHISM:
THE END
There had been no time to deliver this part in 20 minutes also.
FOR GRAPHS?
G planar if and only if G contains no subgraph H 1-homeomorphic to K(5) or K(3,3)
WHAT DO WE WANT?

\[ \chi = 1 \quad \text{dim}=2 \]

\[ \chi = 0 \quad \text{dim}=1 \]

\[ \chi = 0 \quad \text{dim}=1 \]

\[ \chi = 0 \quad \text{dim}=2 \]
$O$ is finite topology:

$\emptyset, X$ are in $O$

$\bigcap B$ are in $O$

$\bigcup B$ are in $O$

$B \in \mathcal{B}$
ChALLENGE:

\( \emptyset \) is often discrete topology: graph is completely disconnected.
GRAPH TOPOLOGY

$G = (V,E)$ finite simple graph

$\mathcal{O}$ is graph topology:

- subbase $\mathcal{B}$ of contractible sets
- intersections in $\mathcal{B}$ are contractible if $\dim(A \cap B) \geq \dim(A)$ or $\dim(B)$
- every edge is contained in an $A$ in $\mathcal{B}$
- nerve graph is homotopic to $G$
INDISCRETE
OPTIMAL
NERVE NOT HOMOTOPIC
There exist graph topologies defined by subbase $B$ and $C$ such that

$$\varphi : B \rightarrow C$$

is lattice isomorphism

$\varphi$ preserves dimension
EXAMPLE

6 unit balls

6 balls
FACTS

1. Homeomorphic graphs are homotopic.
2. Homeomorphic graphs have the same cohomologies.
3. Isomorphic graphs are homeomorphic.
4. Triangularizations of M are homeomorphic.
5. 1-homeomorphisms are homeomorphisms.
6. Homeomorphism with nonzero L has contractible fixed set.
A graph endomorphism of a contractible graph has a fixed clique.

Generalizes edge theorem of Nowakowski and Rival from 1979.

Add a loop at every vertex. Then every graph endomorphism has a fixed edge.
Lefschetz number: super trace on cohomology

Example: \( T = (14)(23) \)  
\( L(T) = 1 - 0 + 0 \)  
\( x = (2,3) \in \text{Fix} \) with \( i_T(x) = 1 \)
KAKUTANI
EQUIVALENCE RELATIONS

ISOMORPHIC
  Cyclic(5)  Cyclic(6)

HOMEOMORPHIC
  Disc  Point

HOMOTOPIC
  Sphere  Homology sphere

COHOMOLOG

METRIC

TOPOLOGICAL

ALGEBRAIC
HOMOTOOPY

\[ X \times K_2 \xrightarrow{F} X \]

\[ Y \times K_2 \xrightarrow{G} Y \]

\[ f \quad g \]
PROBLEM

If $G$ is homeomorphic to $H$ and $G$ is planar, is $H$ planar?
THE END