Dwork's Proof of Rationality

Let q denote a power of the prime p, and let $V/\mathbb{F}_q$ be a variety. We wish to show that $Z(V/\mathbb{F}_q, t) \in \mathbb{Q}(t)$, following the method of B. Dwork. Moreover, we will give N. Katz' interpretation of this method.

Reductions, and Borel's Theorem

Recall that $Z(V/\mathbb{F}_q, t)$ is defined by the two expressions

$$Z(V/\mathbb{F}_q, t) = \exp\left(\sum_{r \geq 1} \frac{t^r}{r} \#V(\mathbb{F}_{q^r})\right) = \prod_{x \in V \,\text{closed}} \left(1 - t^{-\deg(x)s}\right)^{-1} \in 1 + t\mathbb{Z}[\![t]\!].$$

That the second definition lies in $1 + t\mathbb{Z}[\![t]\!]$ is clear, while the equality of the two is an elementary combinatorial application of Galois theory. For shorthand, we sometimes write $N_r = \#V(\mathbb{F}_{q^r})$, and $Z(V/\mathbb{F}_q, t) = Z(N_r, t)$.

To begin, we make the following observation. If we can write $V = V' \sqcup V''$, even merely on the level of $\overline{\mathbb{F}}_q$-points, then $Z(V/\mathbb{F}_q, t) = Z(V'/\mathbb{F}_q, t)\, Z(V''/\mathbb{F}_q, t)$. Written differently, if $N_r = N_r' + N_r''$, then $Z(N_r, t) = Z(N_r', t)\, Z(N_r'', t)$.

This allows us to make reductions by combinatorially cutting apart V, and proving rationality for its pieces. For example, taking V'' to be a suitable closed subvariety so that with V' affine, and inducting on the dimension, we may replace V with V', assume that V is affine. Moreover, by passing to an even smaller affine, we may assume that V is an affine hypersurface. Additionally, Dwork's method prefers that throw away the intersection of V with the union of the axial hyperplanes, to assume that V is a *toric hypersurface*,

$$V = V(F) = \{x \in \mathbb{A}^n \mid F(x) = 0,\ x_1 x_2 \cdots x_n \neq 0\} \subset \mathbb{G}_m^n, \qquad \text{with } F \in \mathbb{F}_q[x_1, \ldots, x_n].$$

Or next observation is that sending $Z(V/\mathbb{F}_q, t) \mapsto Z(V/\mathbb{F}_q, q^k t)$, with $k \in \mathbb{Z}$, or replacing N_r by $q^{kr} N_r$, preserves rationality. It is easy to check that $N_r' = q^{kr}$ has $Z(N_r', t) = (1 - q^k t)^{-1} \in \mathbb{Q}(t)$, so that adding or subtracting q^{kr} to or from N_r also preserves rationality.

Dwork's proof is based upon the following principle. In order to state this theorem, we say that a Laurent series $F(t) \in K((t))$ over a valued field K is *meromorphic of radius* r (for $r > 0$) if we may write $F(t)$ as a ratio of two series that are convergent on $D_r(0) \subset K$.

Theorem 1 (Borel) *Let $F(t) \in \mathbb{Z}[\![t]\!]$ be a power series, and let S be a finite set of places of $\mathbb{Q}$ containing ∞, with $|\cdot|_v$ meaning normalized absolute value for $v \in S$. If for each $v \in S$, the series $F(t)$ is meromorphic of radius r_v in $\mathbb{Q}_v$, and these radii satisfy*

$$\prod_{v \in S} r_v > 1,$$

then $F(t) \in \mathbb{Q}(t)$.

While the proof of the above theorem is not obvious, it is not very hard. One gives a criterion for rationality in terms of recursive relations on the coefficients of the power series, and shows that these relations are forced by the analytic input. Since this theorem yields little geometric insight, we refer to [2, II.8–II.9] for the proof.

For our purposes, we take $S = \{\infty, p\}$. For $v = \infty$, one can use the stupid bound $\#V(\mathbb{F}_{q^r}) \leq \#\mathbb{A}^n(\mathbb{F}_{q^r}) = q^{rn}$ to show that $Z(V/\mathbb{F}_q, t)$ is holomorphic over $\mathbb{R}$ of radius $r_\infty = q^{-n}$. Thus, in order to deduce rationality, we show that $Z(V/\mathbb{F}_q, t)$ is p-adically *entire* meromorphic, i.e., that it is meromorphic over $\mathbb{Q}_p$ for arbitrarily large radius r_p. Finally, we point out that the reductions mentioned above also preserve entire meromorphy, so we will continue to us them.

Analytic Splitting of Characters

Let $V \subset \mathbb{G}_m^n$ and N_r be as in the preceding section. Suppose we can construct a nontrivial character $\chi_r\colon \mathbb{F}_{q^r} \to \mathbb{C}_p^\times$. Then using orthogonality of characters, we easily see that for $x \in \mathbb{G}_m^n(\mathbb{F}_{q^r})$, the quantity

$$\sum_{y \in \mathbb{F}_{q^r}} \chi_r(yf(x))$$

is nonzero precisely when $x \in V(\mathbb{F}_{q^r})$, in which case it equals q^r. Thus, summing over all $x \in \mathbb{G}_m^n(\mathbb{F}_{q^r})$, we get

$$q^r N_r = \sum_{y \in \mathbb{F}_{q^r},\ x \in \mathbb{G}_m^n(\mathbb{F}_{q^r})} \chi_r(yf(x)) = (q^r - 1)^n + \sum_{(y,x) \in \mathbb{G}_m^{n+1}(\mathbb{F}_{q^r})} \chi_r(yf(x)).$$

Since the first term of the right hand side is a $\mathbb{Z}$-linear combination of the functions q^{kr}, the simplifying remarks of the preceding section reduce the entire meromorphy of $Z(V/\mathbb{F}_q, t)$ to the entire meromorphy of $Z(M_r, t)$, where

$$M_r = \sum_{(y,x) \in \mathbb{G}_m^{n+1}(\mathbb{F}_{q^r})} \chi_r(yf(x)).$$

In the remainder of this section, we construct such a χ_r, so that the above work is meaningful, and then proceed to give an analytic expression for it, which will be useful in our proof.

To show that such a χ_r exists, we first let τ denote the Teichmüller lifting. It sends an element of $\mathbb{F}_{q^r}$ to the unique element of $\mu_{q^r-1}(\mathbb{C}_p) \cup \{0\}$ to which it is congruent modulo p. We have easily the following formulas:

$$\tau(xy) = \tau(x)\tau(y), \qquad \text{and} \qquad \tau(x+y) \equiv \tau(x) + \tau(y) \bmod p, \qquad \text{for all } x, y \in \overline{\mathbb{F}}_q.$$

Our other ingredient is a trace. If $q^r = p^s$ and $x \in \mathbb{F}_q$, then one has $\mathrm{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_p}(x) = x + x^p + \cdots + x^{p^{s-1}}$. Then, for any choice ζ of primitive pth root of unity in $\mathbb{C}_p$, we get an additive character by defining

$$\chi_r(x) = \zeta^{\tau(\mathrm{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_p}(x))} = \zeta^{\sum_{i=0}^{s-1} \tau(x)^{p^i}}.$$

It is nontrivial by the surjectivity of the trace on finite fields.

Be warned that although $\tau(\mathrm{Tr}(x)) \in \mathbb{Z}_p$, the individual $\tau(x)^{p^i}$ do not generally lie in $\mathbb{Z}_p$, and ζ^z is only multiplicative for $z \in \mathbb{Z}_p$. Indeed, we often have

$$\chi_r(x) \neq \prod_{i=0}^{s-1} \zeta^{\tau(x)^{p^i}}.$$

To remedy this, we express (some Galois conjugate of) χ_s in terms of a series $\theta(t)$ that allows us to write an identity similar to the failed one above.

Write $E_p(t) \in 1 + t\mathbb{Q}_p[\![t]\!]$ for the Artin–Hasse exponential,

$$E(t) = \exp\left(\sum_{i=0}^{\infty} t^{p^i}/p^i\right) = \prod_{n\geq 1,\ (n,p)=1} (1-t^n)^{-\mu(n)/n}.$$

Fix a root π of the equation $\sum_{i=0}^{\infty} t^{p^i}/p^i = 0$ satisfying $v_p(\pi) = 1/(p-1)$ (by looking at the Newton polygon), and define

$$\theta(t) = E_p(\pi t) \in 1 + \mathbb{Q}_p(\pi)[\![t]\!].$$

The second expression for $E_p(t)$ makes clear that its coefficients are in $\mathbb{Z}_p$ (and hence bounded), from which it follows that $\theta(X)$ converges for $|X| < |\pi|^{-1}$. We must be careful, though, because we cannot calculate $\theta(X)$ by simply substituting straight into the first definition of $E_p(t)$ when $|X| \geq 1$. This essentially boils down to the fact that in the p-adic world, log has a larger radius of convergence than exp, and they are only mutually inverse on a restricted disk.

In spite of the warning, $\log\theta(t)$ does converge for $|X| < |\pi|^{-1}$, and by substituting $t = 1$ we infer that $\theta(1)$ is a pth root of unity. Moreover, a simple estimate shows that $\theta(1) \neq 1$. Thus, choosing ζ from our prior discussion to be $\theta(1)$, we find that $\chi_r(x) = \theta(1)^{x+x^p+\ldots+x^{p^{s-1}}}$. Next we use θ to simplify this expression.

Fix $x \in \mathbb{F}_{q^s}$, and for the moment let $H(t) = E_p(\tau(x)t)E_p(\tau(x)^p t)\cdots E_p(\tau(x)^{p^{s-1}}t)$. Then for all sufficiently small X, there is no problem with convergence when we calculate

$$\begin{aligned} H(X) &= \exp\left(\sum(\tau(x)X)^{p^j}/p^j + \sum(\tau(x)^p X)^{p^j}/p^j + \cdots + \sum(\tau(x)^{p^{s-1}}X)^{p^j}/p^j\right) \\ &= \exp\left(\sum \frac{X^{p^j}}{p^j}(\tau(x)^{p^j} + \tau(x)^{p^{j+1}} + \cdots + \tau(x)^{p^{j+s-1}})\right) \\ &= \exp\left(\sum \frac{X^{p^j}}{p^j}(\tau(x) + \tau(x)^p + \cdots + \tau(x)^{p^{s-1}})\right) \\ &= E_p(X)^{\tau(x)+\tau(x)^p+\cdots+\tau(x)^{p^{s-1}}}. \end{aligned}$$

(The last rearrangement used that $\sum \tau(x)^{p^i} \in \mathbb{Z}_p$.) We deduce that $H(t) = E_p(t)^{\sum \tau(x)^{p^i}}$ *as power series.*

Substituting $t = \pi$ into the preceding result, we conclude that

$$\chi_r(x) = \theta(1)^{x+x^p+\cdots+x^{p^{s-1}}} = \theta(\tau(x))\theta(\tau(x)^p)\cdots\theta(\tau(x)^{p^{s-1}}) \text{ for all } x \in \mathbb{F}_{q^r} = \mathbb{F}_{p^s}.$$

This is our desired identity. It will simplify our manipulations to "relativize" it a little, by defining

$$\theta_q(t) = \theta(t)\theta(t^p)\theta(t^{p^2}) \cdots \theta(t^{q/p}).$$

Note that $\theta_q(t)$ converges for $|X| < |\pi|^{-p/q}$, or $v_p(X) > -p/(p-1)q \stackrel{\text{def}}{=} -\kappa$. We now have

$$\chi_r(x) = \theta_q(\tau(x))\theta_q(\tau(x)^q) \cdots \theta_q(\tau(x)^{q^{r-1}}) \text{ for all } x \in \mathbb{F}_{q^r}.$$

Lifting to Characteristic Zero

We now use the character formula from the preceding section to express M_r in a form that is only involves quantities in $\mathbb{C}_p$. In the next section, we will convert this expression into a "trace formula."

Suppose that our original polynomial F has degree d, and that it can be written as

$$F(x_1, \ldots, x_n) = \sum_i a_i x^i, \text{ with the } a_i \in \mathbb{F}_q.$$

In the above expression, i ranges over a finite set of length-n multi-indices. Henceforth, T denotes the collection of indeterminates $T_0, T_1, \ldots, T_n$. For brevity, we let $\mathbb{Q}_q$ denote the field obtained by adjoining the $(q-1)$st roots of unity, i.e. the Teichmüller lifts of $\mathbb{F}_q$. We define a series $G(T)$ and its coefficients A_j by

$$G(T) = \sum_j A_j T^j = \prod_i \theta_q(\tau(a_i) T_0 T^i) \in \mathbb{Q}_q(\pi)[\![T_0, T_1, \ldots, T_n]\!].$$

In this equation, we consider the i to be the same as those appearing in the equation that precedes it, in the sense that they have $i_0 = 0$ and T^i does not involve T_0. The multi-indices j, however, range over all $n+1$ components.

It is important to point out that since each $T_0 T^i$ has $i_1 + \cdots + i_n \leq d$, the j for which $A_j \neq 0$ all satisfy $j_1 + \cdots + j_n \leq dj_0$. Another thing to notice about $G(T)$ is that we have a sharp knowledge about the growth of its coefficients. In each term $\theta_q(\tau(a_i)T_0T^i)$, we have $|\tau(a_i)|_p = 1$ or 0, and moreover the coefficient of $T_0^b T^{bi}$ is the $\tau(a_i)^b$ times the bth coefficient of θ_q, which has $v_p \geq b\kappa$. Thus, the exponent of T_0 captures which term of θ_q we get, and after multiplying through, we are left with

$$v_p(A_j) \geq j_0 \kappa.$$

Thus $G(T)$, as well as $G(T^{p^i})$, has a radius of convergence > 1. This is sufficient to justify all our manipulations with it that follow.

We now completely remove all characteristic p geometry from our considerations. We observe that for $(y, x) \in \mathbb{G}_m^{n+1}(\mathbb{F}_{q^r})$,

$$\begin{aligned} \chi_r(yf(x)) &= \prod_i \chi_r(a_i y x^i) = \prod_i \theta_q(\tau(a_i y x^i))\theta_q(\tau(a_i y x^i)^q) \cdots \theta_q(\tau(a_i y x^i)^{q^{r-1}}) \\ &= \prod_i \theta_q(\tau(a_i y x^i)) \prod_i \theta(\tau(a_i y x^i)^q) \cdots \prod \theta(\tau(a_i x y^i)^{q^{r-1}}) \\ &= G(\tau(y), \tau(x)) G(\tau(y)^q, \tau(x)^q) \cdots G(\tau(y)^{q^{r-1}}, \tau(x)^{q^{r-1}}), \end{aligned}$$

where τ acts on x by acting on each coordinate. Comparing with our equation for M_r, and writing $G^{(r)}(T) = G(T)G(T^q)\cdots G(T^{q^{r-1}})$, we find that

$$M_r = \sum_{z \in \mu_{q^r-1}^{n+1}(\mathbb{C}_p)} G^{(r)}(z).$$

A "Trace Formula"

In this section we express M_r in terms of the trace of an operator on a p-adic Banach vector space. Compiling these traces into a zeta function, the result will be the reciprocal of the operator's Fredholm series.

For any real number $R > 0$, write W_R for the vector space

$$W_R = \{f \in \mathbb{Q}_q(\pi)[T] \mid f = \sum_{\substack{j_1+\cdots+j_n \le dj_0,\\ j_0 \le R}} f_j T^j\}.$$

We define a couple operators on these spaces. First, for fixed $H \in W_{R_0}$, we have a mapping

$$H_R\colon W_R \to W_{R+R_0}, \qquad f \mapsto H \cdot f.$$

Second, we have a map $\psi_q\colon W_R \to W_{R/q}$ given on monomials by $\psi_q(T^i) = T^{i/q}$, this quantity being interpreted as zero when q does not divide all the components of i.

When $R \ge R_0/(q-1)$, we may form an *endomorphism* of W_R via the composition

$$\psi_q \circ H_R\colon W_R \to W_{R+R_0} \to W_{(R+R_0)/q} \subseteq W_R.$$

This endomorphism is important because its trace can be related to M_r.

Here is our main tool to relate M_r to linear algebra. Assume briefly that $H(T) = T^i$, a monomial, with $T^i \in W_{R_0}$. Then H_R sends monomials to monomials, $H_R T^j = T^{j+i}$. The map ψ_q also sends monomials to monomials (or zero). Therefore, we can compute the trace $\psi_q \circ H_R$ as the number of monomials fixed by $\psi_q \circ H_R$. But $\psi_q \circ H_R(T^j) = T^j$ if and only if $i + j = qj$, meaning that $j = i/(q-1)$. Thus the trace is 1 if and only if $(q-1)$ divides every component of i, and is 0 otherwise. On the other hand, notice that, by orthogonality of characters, the sum

$$\sum_{z \in \mu_{q-1}^{n+1}(\mathbb{C}_p)} z^i$$

is equal to $(q-1)^{n+1}$ if $(q-1)$ divides every component of i, and is 0 otherwise. Finally, both the trace of $\psi_q \circ H_R$ and the sum $\sum_{z \in \mu_{q-1}^{n+1}} H(z)$ are linear in $H \in W_{R_0}$, from which we deduce for *all* such H that

$$\sum_{z \in \mu_{q-1}^{n+1}(\mathbb{C}_p)} H(z) = (q-1)^{n+1}\mathrm{Tr}(\psi_q \circ H_R).$$

We can massage this a little more to get an expression that is useful for all powers q^r. Consider the product $H^{(r)}(T) = H(T)H(T^q)\cdots H(T^{q^{r-1}})$. One easily finds that keeping

$R \geq R_0/(q-1)$ still, $\psi_{q^r} \circ H_R^{(r)}$ takes $W_R \to W_R$. Moreover, using the obvious fact that $\psi_q \circ H(T^q)_R = H_R \circ \psi_q$, one can prove by induction that

$$\psi_{q^r} \circ H_R^{(r)} = (\psi_q \circ H_R)^r.$$

Putting this equation together with our preceding result, we get

$$\sum_{z \in \mu_{q^r-1}^{n+1}(\mathbb{C}_p)} H^{(r)}(z) = (q^r - 1)^{n+1} \mathrm{Tr}(\psi_q \circ H_R)^r.$$

Note in particular that, as a consequence, the right hand side does not depend on R, for R sufficiently large. Therefore, we can (and will) drop the subscript R from this expression without introducing any ambiguity.

Recall that $G(T)$ is the series representing θ of a lifting of T_0 times the equation defining V, and that we mentioned briefly during our analysis of G that all its nonzero terms $A_j T^j$ have $j_1 + \cdots + j_n \leq d j_0$, where d is the degree of the hypersurface V. Therefore, if we let ${}_{R_0}G(T)$ be the sum of those terms of G with $j_0 \leq R_0$, then ${}_{R_0}G$ is a polynomial in W_{R_0}. The above forumla then applies, and we have

$$M_r = \sum_{z \in \mu_{q^r-1}^{n+1}(\mathbb{C}_p)} G^{(r)}(z) = \lim_{R_0 \to \infty} \sum_{z \in \mu_{q^r-1}^{n+1}(\mathbb{C}_p)} ({}_{R_0}G)^{(r)}(z) = (q^r - 1)^{n+1} \lim_{R_0 \to \infty} \mathrm{Tr}(\psi_q \circ {}_{R_0}G)^r.$$

This leads us to define

$$L_r = \lim_{R_0 \to \infty} {}_{R_0}L_r, \text{ with } {}_{R_0}L_r = \mathrm{Tr}(\psi_q \circ {}_{R_0}G)^r,$$

so that $M_r = (q^r - 1)^{n+1} L_r$. By the reductions we discussed in the first section, it suffices to prove that $Z(L_r, t)$ is entire meromorphic.

To treat $Z(L_r, t)$, we begin by pointing out that for any operator A on a characteristic zero, finite-dimensional vector space S, one has the identity of power series

$$Z(\mathrm{Tr}(A^r), t) = \exp\left(\sum_{r \geq 1} \frac{t^r}{r} \mathrm{Tr}(A^r)\right) = \frac{1}{\det(1 - tA)}.$$

Since both sides only depend on the *list* of generalized eigenvalues of A, this becomes a simple combinatorial exercise. (A. Weil was no doubt mindful of this fact when he put forth his conjectures.)

The upshot is that then we have

$$Z(L_r, t) = \lim_{R_0 \to \infty} Z({}_{R_0}L_r, t) = \lim_{R_0 \to \infty} \frac{1}{\det(1 - t(\psi_q \circ {}_{R_0}G))},$$

the limits being taken independently coefficient-by-coefficient. It thus suffices to prove that $\lim_{R_0 \to \infty} \det(1 - t(\psi_q \circ {}_{R_0}G))$ is entire meromorphic.

Nonarchimedean Fredholm Theory

The final result is the basic theorem of p-adic Fredholm theory, due to J.-P. Serre, generalizing Dwork's original work. This theory is elegantly exposited in [4]. For any real number $b > 0$, let $L(b)$ be the infinite-dimensional space

$$\begin{aligned} L(b) &= \{\sum f_j T^j \mid f_j \in \mathbb{Q}_q(\pi),\ j_1 + \cdots + j_n \le dj_0,\ \text{and } |f_j \pi^{-bj_0}|_p \text{ is bounded}\} \\ &= \{\sum f_j T^j \mid f_j \in \mathbb{Q}_q(\pi),\ \sum_{\ell=1}^{n} j_\ell \le dj_0,\ \text{and } f(X) \text{ converges for } |X|_p < |\pi|_p^{-b}\}. \end{aligned}$$

The space $L(b)$ is a completion of the union of the W_R, and it is an orthonormalizable p-adic Banach space under the norm $\sup |f_j \pi^{-bj_0}|_p$.

One easily verifies that for b sufficiently small ($b < \kappa$, say), the operators $\psi_q \circ {}_{R_0}G$ on $L(b)$ have a limit, which we denote by $\psi_q \circ G$. Looking at the entries of its "matrix," one reads off two properties. First, this operator is completely continuous. (The point is that this operator acting on a power series does not increase the size of the coefficients, and even with finite exceptions decreases the degree of the monomial they are attached to.) The second is that

$$\det(1 - t(\psi_q \circ G)) = \lim_{R_0 \to \infty} \det(1 - t(\psi_q \circ {}_{R_0}G)).$$

By complete continuity, Serre's p-adic Fredholm theory asserts that this power series is entire. Since $\det(1 - t(\psi_q \circ G))^{-1} = Z(L_r, t)$, this completes the proof of rationalty.

Epilogue: Katz' Thesis

N. Katz' Ph.D. thesis [3] took part of the above analytic constructions and gave them a geometric meaning. We close this note by trying to give clear statements of his main theorems, which make two comparisons to bona-fide cohomology theories. The first is to the de Rham cohomology of a lifting, and the second is to the Monsky–Washnitzer "formal" cohomology.

We lift F to $\widetilde{F}$ over $\mathbb{Q}_q(\pi)$ in a manner that preserves its degree, and let $\widetilde{V} = V(\widetilde{F}) \subset \mathbb{G}_m^n$. Denote by $\widetilde{V}^{\mathrm{cl}}$ the closure of $\widetilde{V}$ in $\mathbb{P}^n$. In this section, we assume that $\widetilde{V}^{\mathrm{cl}}$ is regular and in general position.

Let W denote the union of the W_R. (Katz uses $\mathcal{L}^+$ for our W.) We distinguish a couple important subspaces:

$$\begin{aligned} W^0 &= \{\sum A_j T^j \in W \mid A_j \ne 0 \text{ only for } j_0 \ge 1\} \text{ and} \\ W^{00} &= \{\sum A_j T^j \in W \mid A_j \ne 0 \text{ only for all } j_k \ge 1\}. \end{aligned}$$

These spaces have a number of "diffential operators" acting on them, including:

$$D_{T_0} = T_0 \frac{\partial}{\partial T_0} + \pi T_0 \widetilde{F}, \qquad \text{and } D_{T_k} = T_k \frac{\partial}{\partial T_k} + \pi T_0 \left(T_k \frac{\partial \widetilde{F}}{\partial T_k} \right) \text{ for } k \ne 0.$$

The space W^0 is generated over the operators D_{T_k} ($0 \le k \le n$) by its elements that are T_0-homogeneous of degree one, for a generic lifting $\widetilde{F}$. Let ω be the top-degree differential form

$$\omega = \frac{1}{T_n \frac{\partial F}{\partial T_n}} \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_{n-1}}{T_{n-1}},$$

which is regular on $\widetilde{V}$. Then by our generation claim, for generic $\widetilde{F}$ there is at most one mapping

$$W^0 \to H^{n-1}_{\mathrm{dR}}(\widetilde{V}/\mathbb{Q}_q(\pi)) \tag{1}$$

that is defined on elements T_0-homogeneous, degree-one elements by the formula

$$f \mapsto \left[\frac{f}{T_0} \cdot \omega\right].$$

Katz' theorem is that this setup does indeed provide a well-defined map, which in turn induces (commuting) isomorphisms

$$\begin{array}{ccc} \dfrac{W^0}{\sum\limits_{k\neq 0} D_{T_k} W^0 + D_{T_0} W} & \cong & H^{n-1}_{\mathrm{dR}}(\widetilde{V}/\mathbb{Q}_q(\pi)) \\ \bigcup & & \bigcup \\ \text{image of } W^{00} & \cong & \text{image of } H^{n-1}_{\mathrm{dR}}(\widetilde{V}^{\mathrm{cl}}/\mathbb{Q}_q(\pi)) \end{array}.$$

Moreover, the cokernel of the above inclusions is one dimensional for n odd, and is zero for $n \geq 2$ and even. Thus Dwork's operator can be thought of as acting on the cocycles representing the (nontrivial part of the) de Rham cohomology of a generic lifting of V. This gives the first interpretation.

Let $A^\dagger$ denote a "weak liftt" of the coordinate ring of V, and write Frob for a lifting of the Frobenius morphism to $A^\dagger \otimes \mathbb{Q}$. Then the map of (1) above extends by continuity to a map

$$\{f \in L(b) \mid f(0) = 0\} \to A^\dagger \otimes \mathbb{Q},$$

which intertwines $q\beta$ with Frob, where β is a certain right inverse β to $\psi_q \circ G$. This shows how Dwork's operator can be thought of as acting on cocycles representing the Monsky–Washnitzer cohomology of V; in fact, the Reich trace formula can be recovered from Dwork's trace formula via this construction.

References

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