Second countability and paracompactness

(Addendum for Math 230a Fall 2014)

Recall we define an n-manifold to be any space which is paracompact, Hausdorff, locally homeomorphic to $\mathbb{R}^n$ (aka locally Euclidean), and equipped with a smooth atlas.

Here we prove

THEOREM 0.1. Assume X is a topological space which is Hausdorff, locally Euclidean, and connected. Then the following are equivalent:

(1) X is second countable
(2) X is paracompact.
(3) X admits a compact exhaustion.

COROLLARY 0.2. If X is not connected, we have the following equivalences:

(1) X is second countable
(2) X is paracompact, and has only countably many connected components.
(3) X admits a compact exhaustion.

REMARK 0.3. In fact, if X is not locally Euclidean nor connected, but locally compact, then second countability implies paracompactness. Conversely, if X admits a cover by precompact, path-connected opens, then paracompactness implies second countability.

Being locally homeomorphic to $\mathbb{R}^n$ is a nice way to satisfy both these cases.

REMARK 0.4. So the class of manifolds we consider is larger than second countable manifolds.

REMARK 0.5. In fact, in the course of the proof, we will see that there is a fourth condition that is equivalent to all others in the Corollary: (4) X admits a cover by countably many compact sets.

1. Definitions

Definition 1 (Paracompact). *An open cover* $\{V_\beta\}$ *is called* locally finite *if for all* $x \in X$*, there exists* $U \subset X$ *open with* $x \in U$ *such that* $V_\beta \cap U = \emptyset$ *for all but finitely many* β*.*

A space X *is called* paracompact *if every open cover* $\{U_\alpha\}$ *admits a locally finite refinement.*

Definition 2. *A space* X *is called* second countable *if it admits a countable base for its topology.*

Definition 3. *A* compact exhaustion *of a space* X *is a sequence of compact sets* $\{K_i\}_{i \in \mathbb{Z}_{\geq 0}}$ *such that* $X = \cup_i K_i$*, and*

$$K_i \subset int(K_{i+1}).$$

In particular, after some i_0*, all the* K_i *must contain some open set.*

2. Lemma

Lemma 2.1. Let X be locally Euclidean and Hausdorff. It X can be written as a union of countably many compact subsets, then X is second countable.

3. Proof of theorem assuming Lemma.

(2) $\implies$ (1). Let $\{U_\alpha\}$ be a cover of X by all open sets that are homeomorphic to $\mathbb{R}^n$. (Since X is connected, the n is fixed.) We let $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ be a choice of homeomorphism for each α. Let $\mathcal{U} = \{\phi^{-1}(B)\}$ be the collection of preimages of open balls of finite radius. $\mathcal{U}$ forms another open cover of X (in fact, a subcover of $\{U_\alpha\}$). All elements of $\mathcal{U}$ have compact closure.

Since X is paracompact, we choose a locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. Note since $\mathcal{V}$ is a refinement, $\overline{V}$ is compact for each $V \in \mathcal{V}$.

Fix $V_1 \in \mathcal{V}$. For every $V \in \mathcal{V}$, there is some integer n such that one can find a string of open sets $V_1, \ldots, V_n = V$ for which $V_i \cap V_{i+1} \neq \emptyset$.[1] By sending each V to the minimal such integer, one has a map $\mathcal{V} \to \mathbb{Z}_{\geq 1}$. We now show that the fibers of this map are finite. Then we will have shown that $\mathcal{V}$ is countable. Since X can be written as the union of $\overline{V_i}$, it is second countable (see Lemma 2.1).

Let $\mathcal{V}_{\leq n+1}$ be the preimage of $\{1, \ldots, n+1\} \in \mathbb{Z}$. Assume by induction that $\mathcal{V}_{\leq n}$ is finite, so $K_n := \overline{\cup_{V \in \mathcal{V}_{\leq n}} V}$ is compact.[2] By the property of being

[1] For instance, for $x \in V_1$ and $x' \in V$, take a path $\gamma : [0,1] \to X$ from x to x'. By compactness of $[0,1]$, there is a finite collection of V_i satisfying this property. Note one can take such a path because X is locally Euclidean—connectedness implies path-connectedness.

[2] It is a closed set contained in the finite union $\cup_{V \in \mathcal{V}_{\leq n}} \overline{V}$.

locally finite, for every $x \in K_n$, we have a neighborhood U_x for which only finitely many $V \in \mathcal{V}$ intersect U_x. Taking a finite subcover of $\{U_x\}_{x \in K_n}$, we see that the set of all V that intersects K_n is finite. But $\mathcal{V}_{n+1}$ is a subset of those V intersecting K_n, so $\mathcal{V}_{n+1}$ is finite. The base case of the induction is $n = 1$, where the preimage of 1 is precisely $\{V_1\}$. □

(1) $\implies$ (3). For a countable base $\mathcal{B} = \{V_i\}$, we can assume that each V_i has compact closure.[3] Since $\{V_i\}$ is countable, we put an ordering on its elements: $V_1, V_2, \ldots$.

Let $K_1 = \overline{V_1}$. By induction on n, let i_n be the smallest integer for which

$$K_n \subset U_1 \cup \ldots \cup U_{i_n}.$$

Then we let

$$K_{n+1} := \overline{U_1 \cup \ldots \cup U_{i_n}}.$$

□

(3) $\implies$ (1). Consider a compact exhaustion $\{K_i\}$. Let us take a collection of open sets $W_i = int(K_i)$. Then $\overline{W_i} \subset K_i$ is compact for each i, and the W_i form an open cover of X with the property that $\overline{W_i} \subset W_{i+1}$.

Fix an open cover $\{U_\alpha\}$. Let $x \in X$. Then there is a smallest i for which $x \in W_i$ but $x \notin W_{i-1}$. Then $\overline{W_i} - W_{i-1}$ is compact.

If $i \geq 3$, this compact set is covered by the open sets

$$V_{\alpha,i} := U_\alpha \cap W_{i+1} \cap (\overline{W_{i-2}})^C.$$

We let $\mathcal{V}_i \subset \{V_{\alpha,i}\}$ be an open subcover guaranteed by compactness.

If $i = 1$ or 2, we let

$$V_{\alpha,i} := U_\alpha \cap W_3$$

and the collection of these open sets is an open cover for $\overline{W_i}$. We choose a finite subcover $\mathcal{V}_i$.

We claim that $\bigcup_{i \geq 1} \mathcal{V}_i$ is a locally finite refinement. It is a refinement because $V_{\alpha,i} \subset U_\alpha$ for all α, i. It is locally finite because, for any x such that $x \in W_i - W_{i-1}$, we can consider the open set $U = W_i - \overline{W_{i-2}}$. By construction, only elements of $\mathcal{V}_{i+1}$, $\mathcal{V}_i$, and $\mathcal{V}_{i-1}$ intersect this open set, and all these sets are finite. Finally, the collection is a cover by construction.

So in fact, we have demonstrated that any cover admits a locally finite, countable refinment. □

[3] For instance, by restricting to those V_i for which this property is satisfied. We lose no generality since we are assuming X is locally Euclidean—and in particular, locally compact—and Hausdorff.

4. Proof of Lemma

PROOF OF LEMMA 2.1. Let $X = \bigcup L_i$ where each L_i is compact. We know $\mathbb{R}^n$ is second countable.[4] For each L_i, choose an open cover by neighborhoods homeomorphic to $\mathbb{R}^n$, then take a finite subcover. Taking the union of a countable base for each element of this subcover, we obtain a countable base for L_i. Now take the union, over all i, of the countable base for each L_i. A countable union of countable sets is again countable. $\square$

[4]For instance, by taking all balls of rational radius centered at rational points.