VISTAS IN MATHEMATICS
A JANUARY @ GSAS MINICOURSE

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10 Sessions
Mon January 11 - Fri January 22
1pm-2pm, Mon-Fri
Science Center, Room 232
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Course Summary.
Most people, of all walks of life, have had a bad relationship with mathematics. At least in their own exposure to it, it has probably seemed either boring, doctrinal, too abstract, or too difficult. Harvard graduate students are no exception. Probably, this is because mathematics is generally taught in most pre-college settings and many college settings in a rigid way: either you are right or wrong; either you get it, or you do not. If you weren’t good at problem solving, you couldn’t be on the math team and so you thought you may as well not pursue math anymore.

This course is intended to expose anyone interested in giving mathematics another chance. Mathematics isn’t about getting the right answer, but rather about asking the right question. There is not just one way to solve a problem; rather, understanding the problem better is an art. You can run into something interesting or maybe even beautiful whether you find a complete solution or not. While mathematics is well-grounded in logic and proof, it is also playful and creative. And while it is indeed abstract, it is wholly grounded in our human capacity to think about the reality that surrounds us. Of course not everyone should be a mathematician, but why not take some time to see how much fun a more artistic approach to mathematics can be?

The course will not discuss this philosophy directly. Rather, it will attempt to do this philosophy in ten sessions, each meant to explore and experiment with a particular problem or object - a ‘vista.’ There will be some lecture and discussion, along with some time experimenting less formally with the mathematical ideas that arise. The first week will consist of various easily accessible examples of what a mathematician thinks about, how, and why it is so interesting. The second week will give overviews of some of the most important open problems in mathematics, namely, the Clay Institute Millenium Prize Problems in mathematics. You too can compete for the $1 million prize attached to each problem! For more information on the overall layout of the course, see ‘Course Arc’ below.

Date: January 10, 2010.
Prerequisites.
This course will be as self-contained as possible and no mathematical experience will be assumed. No readings will be required, but further resources for the interested will be given. Some high school level mathematical concepts (e.g. polynomials) will be reintroduced with questions welcomed. The course will attempt to be minimally cumulative so that attendees can sample various sessions at will, but the third session on “Groups” will be very helpful for several of the later sessions. Also, the second week will feature slightly more technical material than the first week, so those who have an especially dicey history with mathematics are encouraged to come to the first week of courses.

Session Format.
I imagine a session to consist of 30 to 45 minutes of lecture, depending on the topic. Discussion will be mixed in and questions constantly welcomed. In most sessions, the remaining time will give attendees a chance to experiment with the session’s concepts and discuss their observations. We will attempt to come to a good stopping point after an hour, but I will make myself available for additional discussion for quite a while, all depending on the attendees’ interest. In particular, those interested in asking technical questions that are accessible to only some of the students will have a chance to discuss technical topics after the formal end of class. Handouts will include additional resources and questions for interested attendees to investigate afterwards.

Course Arc.
After discussing what mathematics is in the first session, the first week will consist of a very basic mathematical object in each session. These objects are: 1) the counting numbers 2) truth 3) the notion of a “group” and 4) 2-dimension geometry. The main goal of the first week will be to introduce how a mathematician thinks and what a mathematician studies by looks at these maximally elementary mathematical objects. Mathematical perspectives can be a bit mind-bending at first, but bear great fruit once one gets used to them. These four simple objects have a great deal of mathematics in them!

The second week will consist of a presentation of four of the seven Millenium Prize problems: the Poincaré Conjecture, the P vs NP problem, the Birch and Swinnerton-Dyer Conjecture, and the Riemann Hypothesis. Each of these problem represents the cutting edge of a field of mathematics, and were popularized by their selection by the Clay Mathematics Institute as problems that will earn the solver a $1 million prize. For those who attend many of the session, will build on the notions introduced in the first week. Specifically, the Poincaré Conjecture builds on topics 4 and somewhat on 3, the P vs NP problem builds on 2, and the Birch and Swinnerton-Dyer Conjecture and the Riemann Hypothesis build on 1. Obviously it is not possible to give a true overview of each problem - there is too much technical detail. Therefore the sessions in the second week will consist of background to each problem, illustrating their significance within the history of mathematics. The final session will cover a topic of student interest.

Anyone interested is welcome to attend any number of the sessions, and to email me questions.

Session Summaries.
• Monday January 11: What is Mathematics?
We will discuss what we perceive mathematics to be, introduce a notion of mathematics as “the science of patterns,” and perhaps compare these two. We’ll then overview the overall arc of the mini-course: in the first week we’ll learn about interesting examples of mathematical ideas in order to get a more intimate understanding of what mathematics is and how mathematicians think. In the second week we’ll do our best to overview a few cutting edge mathematical problems known as “Millenium Problems” in order to understand what the current frontiers of mathematics are. To conclude this session, we’ll introduce the notion of proof and give an example of a mathematical proof.

• Tuesday January 12: The Counting Numbers.
  • The counting numbers 1, 2, 3, 4, 5, and so forth have a great deal of mathematics in them. We’ll spend some time in discussion, listing observations about the counting numbers. Then we’ll discuss how a mathematician thinks about and characterizes the integers. Our aim will be to discuss as much of Euclid’s original investigations of the counting numbers as possible.

• Wednesday January 13: Truth and Paradoxes.
  • The first half of the 20th century saw a revolution in the understanding of mathematical truth, building on the work and beliefs of Archimedes, Boole, Cantor, Frege, and Hilbert, and culminating in the work of Russell, and Gödel. This may sound fancy, but among the most important of these ideas is really nothing more deep than the ancient Greek’s thoughts on the paradox of the speaker who says “I am lying.” Is that statement true or false? This lesson will give an overview of this history of truth in mathematics and will hopefully conclude with some examples to think about.

• Thursday January 14: Groups.
  • This lesson will attempt to give attendees an introduction to the notion of a “group,” one of the most important notions in the part of mathematics known as abstract algebra. The group is an object that encapsulates the relation between various objects. The main examples of such objects we will consider are the symmetries of geometric shapes such as squares and pyramids. Having dealt with symmetries as a relatively concrete example of a group, we will then introduce the role that groups played in the search for solutions for $x$ in polynomial equations such as $x^4 - 4x^3 + 7x^2 + x - 13$.

• Friday January 15: Alternate Geometries.
  • For centuries, mathematicians had speculated as to the necessity of the parallel postulate in Euclid’s original 2-dimensional geometry, nowadays known as “Euclidean geometry.” That is, they wondered if the parallel postulate, which is the statement that “given a line and a point not on the line, there exists a unique line through the point not intersecting the original line,” instead of being assumed as an axiom for the geometry, could be established from more basic principles. In the 19th century, non-Euclidean geometries were discovered: they obeyed the first axioms of Euclidean geometry but not the parallel postulate, showing that the parallel postulate must be assumed as an axiom of Euclidean geometry. We will discuss Euclidean and the alternate 2-dimensional geometries, carefully saying what a “geometry” is along the way.
• Monday January 18: **Millenium Problems: The Poincaré Conjecture.**
  - With the second week begins the discussion of a few of the Millenium Prize Problems (see course summary for more information on these). The Poincaré Conjecture is the only one of the seven problems to have been completed to date. The conjecture concerns the characterization of three dimensional manifolds. As in most of the Millenium Problem sessions, our time will be spent giving background to the problem to see how the problem represents a current frontier of mathematical research. In this case, we will discuss 2-dimensional manifolds such as the surface of a doughnut or surface of the sphere, and discuss how mathematicians study them. Then we will discuss what a 3-dimensional manifold is, and how the Poincaré Conjecture is a certain extension of these ideas that we observe in 2-dimensions to 3-dimensions.

• Tuesday January 19: **Millenium Problems: P vs. NP.**
  - Is the internet secure? Well, internet security is not 100% bullet proof: it depends on certain problems being so hard for a computer to solve that they are essentially intractable. But if you had a powerful enough computer, you could crack the main current internet security devices. But current computers are so very slow relative to the difficulty of these problems that the internet is basically secure. The P vs NP is a problem in computing theory that has significant ramifications for internet security and many other computing issues. Computing problems can be grouped into 3 groups, P, NP, and E. P problems are relatively easy to solve while E problems are very difficult. Loosely, NP problems are problems for which it is hard to find a solution, but for which the correctness of a potentially correct solution can be be verified easily. The hard problem securing the internet is of class NP. The P vs NP problem is a conjecture that all problems that are of class NP are actually of class P. If this is the case, we would have to find a new way to secure the internet! This session will consist of a gentle introduction to this problem.

• Wednesday January 20: **Millenium Problems: The Birch and Swinnerton-Dyer Conjecture.**
  - In algebra class, we learned the quadratic formula: the solutions $x$ to the polynomial equation
    \[ ax^2 + bx + c = 0 \]
  are given by
    \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]
  For more complicated polynomial equations, it is much more difficult to characterize their solutions. For examples, Fermat’s Last Theorem states that there are no integer solutions to the polynomial equation
    \[ x^n + y^n = z^n \]
  for $n$ an integer greater than 2, other than trivial ones (like $x = 0, y = 1, z = 1$). The mathematics of Andrew Wiles’ 1994 proof of Fermat’s Last Theorem is one of the great achievements of 20th century mathematics. The Birch and
Swinnerton-Dyer conjecture proposes a way to tell if a certain class of polynomial equations called elliptic curves have infinitely many integer solutions. This session will essentially consist of an introduction to the question of solving polynomial equations, culminating in an overview of the Birch and Swinnerton-Dyer conjecture.

• Thursday January 21: Millenium Problems: The Riemann Hypothesis.
  – Is there music in the prime numbers? The Riemann Hypothesis proposes that this is so in a quite direct way! And it is generally considered to be the most important open problem in mathematics. When someone plucks a string, a sound of a certain frequency is given, which we can picture as a wave. If other strings are plucked, the total sound that we hear corresponds to the sum of those wave functions. If the Riemann Hypothesis is true, then the prime counting function

\[ f(x) = \#\{\text{the prime numbers no more than } x\}, \]

that is, the function that takes in a positive real number \( x \) and spits out the number of prime numbers no larger than \( x \) (so for example \( f(6) = 3, f(16) = 6 \)), is given as the sum of such waves. The overview above sacrifices the main idea for accuracy, but this session will attempt to paint this picture as fully as possible.

• Friday January 22: Final Session.
  – This session will cover a topic of student interest. I can propose topics for you to select from if you like. If something goes terribly wrong, this could be an overflow session.

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