Math 155: Designs and Groups

Homework Assignment #7 (31 March 2010):
The Tutte 8-cage; transvections; about the representation $A_7 \hookrightarrow A_8 \cong \text{GL}_4(F_2)$

This problem set is due Friday, Apr. 9 in class.

1. The Tutte 8-cage is the bipartite graph $T$ whose vertices are labeled by the pairs and synthemes of a 6-element set, with each syntheme adjacent to its component pairs. Show that $T$ has girth 8 and that no cubic graph of girth 8 can have less than $T$’s 30 points. (It is known that $T$ is the unique cubic graph of girth 8 on 30 vertices but you need not prove this.) Show that $\text{Aut}(T)$ is the 1440-element group $\text{Aut}(S_6)$. [This is easier than it may seem once you note that $\text{Aut}(T)$ contains that group. Every automorphism either preserves the two parts of $T$ or switches them; it is enough to prove that every automorphism preserving the parts is in $S_6$. Do this by relating each part with the triangle graph $T_2(6)$. Cf. problem 1a of the fifth homework set.]

2. Show that $T$ can also be described thus: the vertices are the blocks of the 3-(10,4,1) design, each of which is adjacent to the three blocks disjoint from it.

3. i) Let $k$ be a field of characteristic 2. Show that the only involutions in $\text{PGL}_3(k)$ are the transvections. For $n > 3$ find an involution in $\text{PGL}_n(k)$ that is not a transvection.
   ii) Prove that the finite simple group $\text{PSL}_3(F_4)$ contains no element of order 6, and thus is not isomorphic with the group $\text{GL}_4(F_2) \cong A_8$ of the same size.

For the last two problems: since $\text{GL}_4(F_2) \cong A_8$ there is an index-8 subgroup of $\text{GL}_4(F_2)$ isomorphic with $A_7$. In other words, $A_7$ acts on the 15 nonzero vectors in $F_2^4$. It turns out that the action is doubly transitive. In particular the point stabilizer has size $(A_7)/15 = 168$, which is suggestive…

4. Let $\Sigma$ be an unstructured 7-element set. A “$\Pi_2$-structure” is a choice of 2-(7,3,1) design of subsets of $\Sigma$ (i.e. an identification of $\Sigma$ with the points of $\Pi_2$). Show that there are 30 such structures, and that the action of $A_7$ on them has two orbits of size 15. Give a combinatorial definition, similar to what we did for hyperovals and subplanes of $\Pi_4$, of an equivalence relation on the $\Pi_2$-structures for which the equivalence classes are the two $A_7$ orbits; and prove directly that it is an equivalence relation.

5. Let $P$ be one of the orbits. Since we expect to identify $P$ with the nonzero vectors of $F_2^4$, there should be a distinguished set of three-element subsets of $P$, namely the sets of nonzero vectors in 2-dimensional subsets of $F_2^4$, which are the blocks of a (2,3,15) Steiner system. Give a combinatorial construction of such a Steiner system of subsets of $P$. Use this to provisionally define the structure of an $F_2$ vector space on $P \cup \{0\}$. What must you check to verify that it works?

This can be used to give an alternative approach to the isomorphism $\text{GL}_4(F_2) \cong A_8$. 