Math 155: Designs and groups  
Handout #5:  
The 3-(8,4,1) Steiner system and the isomorphism $\text{PSL}_2(\mathbb{F}_7) \cong \text{GL}_3(\mathbb{F}_2)$

Let $\mathcal{D}$ be the 3-(8,4,1) Steiner system. It exists because for instance the affine planes in $\mathbb{F}_2^3$ give a suitable system of 14 blocks; and it is unique because the derived design is the unique $\Pi_2$, and from the intersection triangle of $\mathcal{D}$ we know that $\mathcal{D}$ contains the complement of each of its blocks, so that accounts for the $7 + 7 = 14$ blocks of $\mathcal{D}$.

Let $G = \text{Aut}(\mathcal{D})$. This contains the affine linear group $\{v \mapsto Av + b : A \in \text{GL}_3(\mathbb{F}_2), b \in \mathbb{F}_2^3\}$. In particular $G$ permutes $\mathbb{F}_3^2$ transitively so its order is $8 \cdot \#\text{Aut}(\Pi_2) = 8 \cdot 168$. Since that is the number of affine linear transformations, $G$ is identified with that affine linear group. We obtain a surjective homomorphism $G \rightarrow \text{GL}_3(\mathbb{F}_2)$ by mapping $v \mapsto Av + b$ to $A$. The kernel of this homomorphism is the 8-element group of translations $x \mapsto x + b$.

But we can also obtain $\mathcal{D}$ as follows: the 8 points are $\mathbb{P}^1(\mathbb{F}_7)$, and the blocks are the images under $\text{PSL}_2(\mathbb{F}_7)$ of $\{0, 1, 3, \infty\}$. Since the stabilizer of this in $\text{PSL}_2(\mathbb{F}_7)$ is $A_4$, there are $\#\text{PSL}_2(\mathbb{F}_7)/\#A_4 = 168/12 = 14$ blocks. Since $\text{PSL}_2(\mathbb{F}_7)$ acts transitively on 3-element subsets of $\mathbb{P}^1(\mathbb{F}_7)$ [even though it does not act 3-transitively!], the blocks constitute a 3-design. So we get a 3-(8,4,1) design with automorphisms by $\text{PSL}_2(\mathbb{F}_7)$. Since our design must be isomorphic with $\mathcal{D}$, this means $\text{PSL}_2(\mathbb{F}_7)$ is contained in $G$.

Composing the inclusion $\text{PSL}_2(\mathbb{F}_7) \hookrightarrow G$ with the homomorphism $G \rightarrow \text{GL}_3(\mathbb{F}_2)$ we obtain an isomorphism $\text{PSL}_2(\mathbb{F}_7) \rightarrow \text{GL}_3(\mathbb{F}_2)$ whose kernel has order at most 8. But $\text{PSL}_2(\mathbb{F}_7)$ is simple of order 168 so the kernel is trivial. Since $\text{GL}_3(\mathbb{F}_2)$ also has order 168 our map is in fact an isomorphism.

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1\footnote{Since $\text{PSL}_2(\mathbb{F}_7)$ is of index 2 in $\text{PGL}_2(\mathbb{F}_7)$, we can prove this transitivity by showing that the stabilizer in $\text{PGL}_2(\mathbb{F}_7)$ of a three-point set is not contained in $\text{PSL}_2(\mathbb{F}_7)$. Since all three-point sets are equivalent under $\text{PGL}_2(\mathbb{F}_7)$ we choose $\{0, 1, \infty\}$ and note that the involution $x \leftrightarrow 1 - x$ permutes it but is not in $\text{PSL}_2(\mathbb{F}_7)$ because $-1$ is not a square. This argument shows that for any field $F$ the group $\text{PSL}_2(F)$ acts transitively on three-point subsets of $\mathbb{P}^1(F)$ if and only if $-1$ is not a square in $F$. When $F$ is finite this means $|F|$ is not 1 mod 4. So the first counterexample is $\mathbb{F}_5$, when $\text{PSL}_2(\mathbb{F}_7)$ has two orbits on the 3-point subsets. Here the complement of such a subset is of the same size; is it in the same orbit?}