ARITHMETIC AND COMPLEX BORDISM

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1. Bordism: what is it good for?

Bordism theories are particular sorts of homology theories, built in accordance to the rule

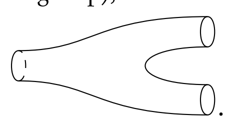
singular homology : simplices :: bordism : manifolds.

Fixing a structure group G, the *geometric chain complex* $GC_*(X;G)$ of a space X is given by

$$GC_n(X;G) = \mathbb{N}\left[f : M \to X \,\middle|\, M \text{ a connected manifold with } G\text{-structure}\right].$$

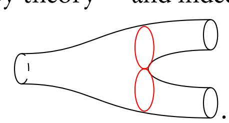
The boundary maps in this complex are induced by the restriction of f to the boundary of M. Hence, the n-cycles of this complex are given by the maps off closed n-manifolds, and the n-boundaries are given by those maps which extend to a map from an $(n+1)$-manifold.

This complex describes the value of the homology theory $\Omega_*^G(X)$ for any space X, but to begin to grapple with it we should first consider the case $X = \text{pt}$. Since equipping a manifold with a map to a point is no extra data at all, the bordism complex of a point is just the complex generated by the manifolds themselves, with boundary maps induced by taking the boundary submanifold. It's instructive to draw at least one picture: in the case where there's no structure group (i.e., when G is the orthogonal group), we can consider the following 2-manifold with boundary:



This can be considered as imposing the relationship $3 \cdot [S^1] = 0$ in the bordism ring $\Omega_*^O(\text{pt})$.[1]

Bordism theory can be found lurking most anywhere that complex manifolds arise in algebraic topology. For example, the intersection of two manifolds is not necessarily a manifold (e.g., the red slice of the figure below), but it can always be made one after small perturbation (e.g., sliding right to $S^1 \sqcup S^1$ or left to S^1). Since sliding is involved, this operation should be captured by homotopy theory — and indeed, the situation is captured by a bordism.



Bordism also supplies a natural domain for homotopically defined algebraic invariants of manifolds. A *genus* is a ring homomorphism $\varphi : \Omega_*^U(\text{pt}) \to R$ to a torsion-free ring R — that is, an algebraic invariant of manifolds which is invariant under bordism, takes disjoint unions to sums of classes in R, and Cartesian products to products of classes in R. Our goal in this talk is to give a description of the data involved in defining a *genus of complex manifolds*, i.e., the study of genera when the structure group is set to the unitary group. This homology theory should be thought of as the bordism theory constructed from complex manifolds — though this can't be taken too literally, as it is not clear what it means to have a complex manifold with boundary, nor one of odd real dimension. This proceeds in three steps, arranged from least to most structure:

(1) Understand rational genera: $\Omega_*^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$.
(2) Understand integral genera: $\Omega_*^U \to R$.
(3) Understand topological genera: $\Omega^U \to E$ for a ring spectrum E.

These notes are a pale summary of Jack Morava's *Forms of K-Theory*.

[1] Thanks, Marius! `http://tex.stackexchange.com/questions/30688/using-tikz-to-draw-cobordisms`

The first of these we can address immediately. The rational coefficient ring $\Omega^U_* \otimes \mathbb{Q}$ is accessible via the rational Hurewicz map and the Pontryagin–Thom construction:

$$\Omega^U_*(\mathrm{pt}) \otimes \mathbb{Q} = \Omega^U_*(\mathbb{S}_{\mathbb{Q}}) \xrightarrow{\cong} \Omega^U_*(H\mathbb{Q}) \cong H\mathbb{Q}_* BU \cong \mathbb{Q}[\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^2, \ldots, \mathbb{C}\mathrm{P}^n, \ldots],$$

and we see it is a polynomial algebra which is rationally generated by the complex projective spaces. Hence, a rationalized genus is described by its action on $\mathbb{C}\mathrm{P}^n$ for all n.

2. The arithmetic of genera

By an identical procedure, we can use the integral Hurewicz homomorphism to give a natural map

$$\Omega^U_* = \Omega^U_*(\mathbb{S}) \to \Omega^U_*(H\mathbb{Z}) \cong \operatorname{Sym} \widetilde{H}_* \mathbb{C}\mathrm{P}^\infty,$$

which is an isomorphism over the generic point but is otherwise highly arithmetically interesting. In other words, it is easy to state what a rationalized complex genus is, but which rational genera restrict to integral ones is an arithmetically intricate question. Consider the following examples:

- The *Todd genus* is specified rationally by $\varphi(\mathbb{C}\mathrm{P}^n) = 1$ for all n. This genus exists integrally.
- The genus specified by $\varphi(\mathbb{C}\mathrm{P}^n) = 1$ for $n \leq 1$ and $\varphi(\mathbb{C}\mathrm{P}^n) = 0$ for $n > 1$ exists rationally but not integrally.
- The Rmanuajan τ-function is defined by the generating function $\sum_{n=1}^\infty \tau_n q^n = q \cdot \prod_{j=1}^\infty (1-q^n)^{24}$. (If you haven't seen this before, the first few values are: 1, −24, 252, −1472, 4830, −6048, −16744, 84480, ...) There exists an integral genus φ with $\varphi(\mathbb{C}\mathrm{P}^n) = \tau_{n+1}$.

Something intense must be going on! Luckily for us, Quillen has an explanation:

Theorem (Quillen). *There is a natural bijection between ring maps $\Omega^U_* \to R$ and formal group laws over R.*

I will describe the easy part of this correspondence: given a genus $\varphi : \Omega^U_* \to R$, we can rationalize it to produce a map $\varphi \otimes \mathbb{Q} : \Omega^U_* \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$. We arrange the images of $\mathbb{C}\mathrm{P}^n$ under this map into a generating function as:

$$\log_\varphi(x) := \sum_{n=1}^\infty \frac{\varphi(\mathbb{C}\mathrm{P}^{n-1})}{n} x^n,$$

which is an invertible power series. Thinking of this map as a *logarithm* (a perspective due to Mišcenko), we produce the associated *formal group law* by the formula

$$x +_\varphi y = \exp_\varphi(\log_\varphi x + \log_\varphi y) = x + y + \text{higher order terms.}^2$$

This power series is called a *formal group law* because it satisfies the following properties:

(1) It is symmetric: $x +_\varphi y = y +_\varphi x$.
(2) It has an identity element: $x +_\varphi 0 = x$.
(3) It is associative: $x +_\varphi (y +_\varphi z) = (x +_\varphi y) +_\varphi z$.[3]

Quillen's theorem then asserts that a rational genus is the rationalization of an integral genus if and only if the series $x +_\varphi y$ has integral coefficients. However, these power series are difficult to compute, and so it is not easy to use them to verify the conditions of Quillen's theorem. There is a different rational invariant of a logarithm which makes this slightly easier, called its ζ-function:

$$\zeta_\varphi(s) = \sum_{n=1}^\infty \varphi(\mathbb{C}\mathrm{P}^{n-1}) n^{-s}.$$

Theorem (Honda). *If $\zeta_\varphi(s)$ has an Euler factorization*

$$\zeta_\varphi(s) = \prod_{p \text{ prime}} \left(1 - \sum_{n=1}^\infty b_{n,p} p^{n(1-s)-1}\right)^{-1}$$

[2]Sometimes this is written $x +_! y$, where the "!" is pronounced "surprise", and the surprise is the higher order terms.

[3]These are precisely the properties you would expect of the analytic expansion of a Lie's group multiplication map in a chart at its identity.

with $b_{n,p} \in \mathbb{Z}$ then the formal group law has integral coefficients. Conversely, every integral formal group law is isomorphic to one whose ζ-function has such an Euler factorization.

This more or less answers our question of when a rational genus exists integrally.

In all, there is an equivalence of data between logarithms, formal group laws, and ζ-functions over torsion-free rings — but formal group laws deserve special attention, as they're the only version of this data that survives to positive characteristic. An interesting arithmetical question is: what obstructs the transfer of the other data to positive characteristic? For an example of this phenomenon, the logarithm satisfies the following integral equation:

$$\log_\varphi(x) = \int \left(\left. \frac{\partial(x +_\varphi y)}{\partial y} \right|_{y=0} \right)^{-1} dx.$$

Performing this power series integration requires occasional division by p, and it is a nontrivial theorem that the first obstructions to the existence of these coefficients lie in degrees of the form p^h. If such a nontrivial obstruction exists in degree p^h, then h is called the *height* of the formal group law (at p). Height is also something that the ζ-function can detect: an integral formal group law has height 1 at p if and only if $p \nmid b_{1,p}$.

2.1. **Example: The Todd genus.** Let's compute all of these invariants for the Todd genus. From the prescribed action on projective spaces, we can produce the arithmetic invariants:

Arithmetic invariant	Value for the Todd genus
Logarithm	$\log_{\mathrm{Td}}(x) = \sum_{n=1}^\infty \frac{1}{n} x^n = -\ln(1-x)$,
Exponential	$\exp_{\mathrm{Td}}(x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!} x^n = 1 - \exp(-x)$,
ζ-function	$\zeta_{\mathrm{Td}}(s) = \sum_{n=1}^\infty n^{-s} = \zeta(s)$,
Euler factorization	$\zeta_{\mathrm{Td}}(s) = \prod_{p \text{ prime}} (1-p^{-s})^{-1}$.

Since all the $b_{n,p}$ coefficients are either 0 or 1, Honda's theorem asserts that the associated formal group law lies over $\mathbb{Z}$, and Quillen's theorem then confirms that the rationalized Todd genus arises from an integral Todd genus. We can also verify the result of Honda's theorem by manually computing the formal group law. Namely,

$$x +_{\mathrm{Td}} y = \log_{\mathrm{Td}}^{-1}(\log_{\mathrm{Td}} x + \log_{\mathrm{Td}} y) = 1 - \exp(-(-\ln(1-x) - \ln(1-y))) = 1 - (1-x)(1-y) = x + y - xy.$$

Sure enough, it's integral! Remarking that is the analytic expansion at 1 of the multiplication law on $\mathbb{G}_m$, we'll refer to it as the "formal multiplicative group law". Finally, we see (either from the logarithm or from the ζ-function) that it is of height 1 for all primes.

3. K-THEORIES FROM ORDINARY GENERA

Having discussed the move from rational algebra to integral algebra, the next enrichment we can seek is to move from algebra to topology: given a genus $\varphi : \Omega^U_* \to R_*$, when can we produce a homology theory R and a natural transformation $\Omega^U \to R$ inducing φ on coefficients?[4] There is a general (and pretty) result along these lines called the *Landweber exact functor theorem*, which gives conditions for the following formula to be a homology functor:

$$R_*(X) = \Omega^U_*(X) \otimes_{\Omega^U_*} R_*.$$

However, a proper explanation of this theorem requires knowledge not developed here, so we will instead avoid the LEFT and discuss Morava's alternative approach to the question for height 1 genera.

Theorem (Morava)**.** *There is a family of cohomology theories K_u parametrized by $u \in \mathbb{Z}_p^\times$, each with coefficient ring $(K_u)_* = \mathbb{Z}_p[t^\pm]$, all of them distinct, and all of them isomorphic when tensored up to* $\operatorname{colim}_n \mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p(\zeta_n : p \nmid n)$.

[4]This does not fit into my narrative here, but I have found this too enlightening to leave out: the spectrum of *framed bordism* coincides with the sphere spectrum. Hence, the unit map $\mathbb{S} \to K$ of any ring spectrum K can be thought of as a genus for framed manifolds. However, it is unlikely that all of the data of a framing is necessary to compute this genus, and a factorization $\mathbb{S} \to \Omega^U \to K$ of the unit (i.e., one of the conditions of a multiplicative map $\Omega^U \to K$) is a witness to that fact that only the residual structure of the unitary group is necessary.

Before turning to a proof, I would like to explore what this theorem means in terms of the "forms" of algebraic geometry. First, there is only one isomorphism type of a height 1 formal group law over $\bar{\mathbb{F}}_p$; in a separably closed setting, we are always able to solve for the coefficients of a power series exhibiting an isomorphism. On the other hand, over $\mathbb{F}_p$ the nonexistence of solutions to polynomial equations may become a problem. Indeed, it does: there are many nonisomorphic formal group laws over $\mathbb{F}_p$, fully detected by the invariant $u \in \mathbb{Z}_p^\times$ in the equation

$$\exp_\varphi(u \cdot p \cdot \log_\varphi(x)) = x^p \pmod{p}.$$

Moreover, each such invariant is realizable: there is a formal group law with given invariant u specified by the ζ-function $\zeta_u(s) = (1 - \alpha p^{-s} + p^{1-2s})^{-1}$, $\alpha = u^{-1} + pu$.

This sort of phenomenon is generic. A *form* of an object Y is an object X over a base k such that X becomes isomorphic to Y when passing to the algebraic closure $\bar{k}$ of k. Forms of an object are detected by Galois cohomology along the lines of the following recipe: let h be a multiplicative homology theory over $k = \mathbb{F}_p$, and consider the base-changed theory $\bar{h} = h \otimes_k \bar{k}$. Set $\operatorname{Aut} \bar{h}$ to be the group of automorphisms of $\bar{h}$, on which the absolute Galois group acts by conjugation. There is then an injection

$$\{\text{forms of } h\} \to H^1(\operatorname{Gal}(\bar{k}/k); \operatorname{Aut} \bar{h}).$$

Theorem (Morava)**.** *There is the following calculation:*

$$H^1(\operatorname{Gal}(\bar{k}/k); \operatorname{Aut} \bar{h}) \cong \begin{cases} 0 & \textit{when } h = H\mathbb{F}_p, \\ 0 & \textit{when } h = MU/p, \\ \mathbb{Z}_p^\times & \textit{when } h = KU/p. \end{cases}$$

So, there are no nontrivial forms of ordinary homology or of complex bordism to be had, but there are plenty of nontrivial forms of K-theory. Morava's theorem shows that every nontrivial form of K-theory is possible and even that they coincide with the possible nontrivial forms of the formal multiplicative group.

I would like to outline how the proof of the existence of the family K_u goes. To begin, we enlarge our coefficient ring in two ways: first, we may freely replace $\mathbb{F}_p$ with its ring of Witt vectors $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$, as Hensel's lemma guarantees we introduce no new formal group laws nor isomorphisms by doing so. As $\mathbb{Z}_p$ is torsion-free, we now have access to the tools developed in the previous section. Then, consider the algebra $W = \operatorname{colim}_n \mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p(\zeta_n : p \nmid n)$ of Witt vectors over $\bar{\mathbb{F}}_p$: this is a faithfully flat $\mathbb{Z}_p$-algebra, and hence it is sufficient to check that a genus $\varphi : \Omega_*^U[x_{p-1}^{-1}] \to \mathbb{Z}_p[t^\pm]$ produces a homology theory by the formula

$$K_\varphi(X) \otimes W = \left(\Omega_*^U[x_{p-1}^{-1}] \otimes_{\Omega_*^U} \mathbb{Z}_p[t^\pm]\right) \otimes_{\mathbb{Z}_p} W$$

to deduce the same for K_φ.

Let $\Lambda_1(W)$ be the space $\operatorname{Hom}(\Omega_*^U[v_1^{-1}], W)$, topologized with the Zariski topology. Then we have a sheaf $\mathscr{K}$ of functors over $\Lambda_1(W)$, where the stalk at any given genus φ is K_φ, and the value of the sheaf on an open set determined by the subbasic $\Omega_*^U[v_1^{-1}][M_1^{-1}, \ldots, M_k^{-1}]$ is given by the functor associated to the further localized genus $\Omega_*^U[v_1^{-1}][M_1^{-1}, \ldots, M_k^{-1}] \to W$. Because all height 1 formal group laws over $W = \mathbb{W}(\mathbb{F}_p^{\text{sep}})$ are isomorphic, the group $\Gamma(W)$ of invertible power series over W acts transitively on the base. The sheaf of homology theories is equivariant against this action, and there is the following theorem:

Theorem. *If $\Lambda_1(W)$ is a space carrying a transitive action by a group $\Gamma(W)$ and $\mathscr{K}$ is a $\Gamma(W)$-equivariant sheaf on $\Lambda_1(W)$ satisfying some finiteness properties, then $\mathscr{K}$ is locally trivial.*

Since $KU_p^\wedge$ is the stalk of this sheaf at the Todd genus, $KU_p^\wedge$ is a homology theory, and the sheaf is locally trivial, it follows that all of these functors are in fact homology theories.

4. Elliptic cohomology near the cuspidal curve

Having introduced the idea of a sheaf of homology functors and given a successful application, we can now look for more naturally occuring instances of the same idea. Specifically: where do height 1 formal group laws come from? One answer to this question is the theory of elliptic curves. A complex elliptic curve can be specified by a quotient $\mathbb{C}/\Lambda$, where Λ is a rank 2 lattice $\Lambda = (1, \tau)$ generated by the complex numbers 1 and $\tau = a + bi$, $b > 0$. An equivalent presentation with somewhat nicer properties is given by exponentiation: the sublattice generated by 1 gives the kernel of the exponential $z \mapsto \exp(2\pi i z)$, and the element τ is sent to some $q = \exp(2\pi i \tau) \in \mathbb{C}^\times$ with $0 < |q| < 1$. Hence, the same elliptic curve can be expressed as $C_q = \mathbb{C}^\times / q^{\mathbb{Z}}$.

Theorem (Tate–Jacobi)**.** *There is a canonical formal group law associated to the complex elliptic curve C_q whose coefficients lie in the subring $\mathbb{Z}[E_4(q), E_6(q)]$ of $\mathbb{C}$, where the Eisenstein series E_4 and E_6 are convergent power series in q. After adjoining q, this formal group law becomes canonically isomorphic to the multiplicative one.*

Analysis of the ϑ-functions involved in this construction show that actually we can form an elliptic curve, called the *Tate curve*, over the ring $\mathbb{Z}((q))$ of formal Laurent series. The same techniques as in the previous section forms a sheaf of homology theories over $\operatorname{Hom}(\mathbb{Z}_p((q)), W)$. Its global sections form a homology theory with coefficient ring $\mathbb{Z}_p((q))[t^\pm]$, called *Tate K-theory*. This is a kind of *elliptic cohomology* near the cuspidal curve at $q = 0$.

This motivates all kinds of interesting questions. Can we extend this sheaf further out on the moduli of elliptic curves? What happens at stacky points? What happens at points classifying supersingular elliptic curves (i.e., those whose formal group is of height 2 rather than 1)? Can we make the sheaf into a sheaf of spectra — or of structured spectra? Can features of the moduli of elliptic curves be used to predict behavior in homotopy theory? What about using other moduli of abelian varieties? How does this connect to geometry, like the Todd genus connects to Riemann–Roch and the index theorem? Attempting to answer questions like these has occupied a large number of homotopy theorists for the past two decades, and their results have become extremely rich and intricate.

Appendix A. Forms of the informal multiplicative group

Having spent this talk considering nontrivial forms of the formal multiplicative group, it seems prudent to mention that one can also study nontrivial forms of the informal multiplicative group. This should be thought of as a warm-up problem to the study of elliptic cohomology, and so turns out to have structure of interest to topologists. The 1-dimensional group $\mathbb{G}_m$ can be thought of (up to Möbius transformation) as the unique group structure on $\mathbb{P}^1 \setminus \{0, 1\}$ with identity point at ∞. In general, $\mathbb{P}^1$ punctured at any two distinct points carries a unique group structure, and over the separable closure of the ground ring, all of these punctured spheres become biholomorphic to the standard one.

We consider the stack of such objects, presented as a Hopf algebroid as follows. Let $X_0 = \operatorname{Spec} \mathbb{Z}[b, c, \Delta^{-1}]$. A map $\operatorname{Spec} R \to X_0$ then selects two points, given as the zeroes of the quadratic $x^2 + bx + c$, together with a guarantee that they are distinct since the discriminant $\Delta = b^2 - 4c$ is invertible. The group multiplication associated to this punctured $\mathbb{P}^1$ is given by the formula

$$(x_1, x_2) \mapsto \frac{x_1 x_2 - c}{x_1 + x_2 - b}.$$

Then, we wanted to consider these up to Möbius transformation, but the only Möbius transformations fixing ∞ are the translations, and so we set $X_1 = \operatorname{Spec} \mathcal{O}_{X_0}[r]$. These two rings have a few structure maps:

$$\begin{aligned}
X_0 &\xrightarrow{\text{identity}} X_1 : & \varepsilon(r) &= 0, & & \\
X_1 &\xrightarrow{\text{domain}} X_0 : & \eta_L(b) &= b, & \eta_L(c) &= c, \\
X_1 &\xrightarrow{\text{codomain}} X_0 : & \eta_R(b) &= 2r + b, & \eta_R(c) &= br + c, \\
X_1 \times_{X_0} X_1 &\xrightarrow{\circ} X_1 : & \Delta(r) &= r \otimes 1 + 1 \otimes r. & &
\end{aligned}$$

Altogether, these makes (X_0, X_1) into a groupoid scheme, and by correcting descent we produce a stack $\mathcal{M}_{\mathbb{G}_m}$.

The joy of working with stacks is that we can modify our presentation without modifying the stack. We slightly rearrange the above data: first, factor the quadratic, so that its two puncture points are given by α and β. Then, use the translation isomorphism to set $\beta = 0$; this choice of a canonical representative removes r from our presentation of the isomorphisms. However, we have introduced a new isomorphism by choosing α and β: we could have chosen them in a different order. Algebraically encoding this presentation results in the schemes $Y_0 = \operatorname{Spec} \mathbb{Z}[\alpha^{\pm}]$ and $Y_1 = \operatorname{Spec} \mathscr{O}_{Y_0}[s]/(s^2 - \alpha s)$. The structure maps this time are

$$\begin{aligned}
Y_0 &\xrightarrow{\text{identity}} Y_1 : & \varepsilon(s) &= 0, \\
Y_1 &\xrightarrow{\text{domain}} Y_0 : & \eta_L(\alpha) &= \alpha, \\
Y_1 &\xrightarrow{\text{codomain}} Y_0 : & \eta_R(\alpha) &= 2s - \alpha, \\
Y_1 \times_{Y_0} Y_1 &\xrightarrow{\circ} Y_1 : & \Delta(s) &= s \otimes 1 + 1 \otimes s.
\end{aligned}$$

The cohomology of this latter stack is well-known to algebraic topologists as the E_2-page of a homotopy descent spectral sequence which computes $\pi_* KU^{hC_2} \cong \pi_* KO$. The real work begins, then, by trying to transport this situation to the world of homology theories and spectra: can we find a sheaf of homology theories over $\mathscr{M}_{\mathbb{G}_m}$ which reduces to the structure sheaf on points and whose derived sections gives KO?

As a further remark, if we allow the degenerate case of the twice-punctured origin (i.e., we do not invert Δ) and compute the cohomology of that stack, we produce the E_2-page of a spectral sequence computing the connective $\pi_* ko$ instead. Interestingly, this does not correspond to the descent spectral sequence for ku^{hC_2}.

Finally, a further-further remark: this stack and its connection to algebraic topology is discussed from a different viewpoint in Mike Hopkins's talk *From Spectra to Stacks*, given at the Talbot workshop in 2007.