Tmf II — CONSTRUCTION AND COMPUTATION

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ABSTRACT. These are notes for an e-Talbot talk covering Behrens's *Notes on the Construction of tmf* and Bauer's *Computation of the Homotopy of the Spectrum tmf*.

1. INTRODUCTION

This is a service talk. One of the goals of this conference is to discuss Serre duality on derived stacks, and in order to explore this idea we require some interesting examples of derived stacks. We have been continually interested in the moduli stack of elliptic curves, and this is the stack we will seek a topological enrichment for. To begin: what is a topological enrichment?

Definition 1. Let $f\colon \mathscr{N} \to \mathscr{M}_{\mathbf{fg}}$ be a flat, representable morphism of stacks. A *topological enrichment* of $\mathscr{N}$ is a sheaf of E_∞ ring spectra $\mathscr{O}$ on $\mathscr{N}$ such that

$$\pi_n \circ \mathscr{O} \cong \begin{cases} f^*\omega^{\otimes k} & \text{if } n = 2k \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Talk about th complex-orien cohomology o $\mathbb{CP}^1$.

Remark 2. To gain intuition for the above, ignore the structured ring spectrum part and recall that a Landweber-flat even-periodic cohomology theory R is determined by a flat map $\operatorname{Spec} R_0 \to \mathscr{M}_{\mathbf{fg}}$ from an affine, together with a preferred lift to $\mathscr{M}_{\mathbf{fgl}}$. In this case, we have

$$\pi_2 R = [\mathbb{S}^2, R] = [\mathbb{CP}^1, R] = \widetilde{R}^0\mathbb{CP}^1 = \frac{\ker R^0\mathbb{CP}^\infty \to R^0(*)}{(\ker R^0\mathbb{CP}^\infty \to R^0(*))^{\otimes_{R^0\mathbb{CP}^\infty} 2}},$$

which is the cotangent space of $R^0\mathbb{CP}^\infty$ or the invariant 1–forms on $R^0\mathbb{CP}^\infty$.[1] This explains the condition about homotopy groups.

Remark 3. To gain intuition about what a topological enrichment gains us, consider a flat map $\operatorname{Spec} R \to \mathscr{N}$. We recover a complex-orientable homology theory $R_0(-)$ in the same way as Landweber's theorem, and its associated formal group is classified by the composite to $\mathscr{M}_{\mathbf{fg}}$. The data of $\mathscr{O}$ is to compatibly choose lifts to E_∞ ring spectra $\mathscr{O}(R)$ for the étale maps $\operatorname{Spec} R \to \mathscr{N}$.

Fix a fixed map $f\colon \mathscr{N} \to \mathscr{M}_{\mathbf{fg}}$, this has its own associated moduli problem of topological enrichments of f.

Theorem 4 (Goerss–Hopkins–Miller; Behrens; Lurie)**.** *The moduli of topological enrichments of $\mathscr{M}_{\mathbf{ell}}$ is contractible.*

The cohomology theories manufactured from this sheaf are examples of *elliptic spectra*, which are triples (E, C, φ) of an even-periodic cohomology theory E, a elliptic curve C, and an isomorphism φ between $C_0^\wedge$ and the formal group associated to $E^0\mathbb{CP}^\infty$.

We aren't going to prove this theorem—the details are too thick to completely present in an hour, even assuming much more topological technology—but I hope to give enough indications of the proof that you get the gist and are able to slot in the relevant topological inputs as you learn about them. Our approach is to work locally on $\mathscr{M}_{\mathbf{ell}}$: divide it up over primes by passing to the p–completion, then divide the p–complete moduli itself into two further regions by using the following result:

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[1]The identification of these two things is an indirect consequence of the discussion of logarithms from Luca's talk.

Lemma 5 (Serre–Tate)**.** *The p–divisible group of an elliptic curve is either formal of height 2 or an extension of an étale p–divisible group of height 1 by a formal group of height 1.* □

Elliptic curves over a p–complete ring with connected p–divisible groups (i.e., those of the first form) are called *supersingular*, whereas those of the second form are called *ordinary*. The most important fact about supersingular elliptic curves is that they are uncommon:

Lemma 6. *The supersingular locus of $\mathcal{M}_{\mathbf{ell}}$ is 0–dimensional. In fact, it is the zero-locus of a polynomial of degree $\lfloor (p-1)/12 \rfloor + \{0,1,2\}$.* □

Write $i\colon \mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}} \to \mathcal{M}_{\mathbf{ell}}$ for the open inclusion of the ordinary locus. We then plan to recover a topological enrichment by constructing the pieces of the following pullback:[2]

$$\begin{tikzcd}
 & \mathcal{O}_{\mathrm{top}} \arrow[r] \arrow[d] & (\mathcal{O}_{\mathrm{top}})^{\wedge}_{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}} \arrow[r, equal] \arrow[d] & \mathcal{O}^{\mathrm{ss}} \\
\mathcal{O}^{\mathrm{ord}} \arrow[r, equal] & i_* i^* \mathcal{O}_{\mathrm{top}} \arrow[r] & i_* i^* \left((\mathcal{O}_{\mathrm{top}})^{\wedge}_{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}} \right). &
\end{tikzcd}$$

2. The supersingular locus

Our task in this section is to define $\mathcal{O}^{\mathrm{ss}}$ on $\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}}$, the supersingular part of the topological enrichment. In order to make this definition, we need to specify its behavior on formal étale affines. Since the moduli is itself 0–dimensional, these are exactly the formal affine covers of the deformation spaces of the supersingular curves in the larger moduli $\mathcal{M}_{\mathbf{ell}}$. The following arithmetic result gives us a crucial reduction:

Theorem 7 (Serre–Tate)**.** *The map $\mathcal{M}_{\mathbf{ell}} \to \mathcal{M}_{\mathbf{pdiv}}(2)$ is formally étale.*[3] □

Lemma 8. *The deformation theory of a connected p–divisible group of height d as a p–divisible group is isomorphic to the deformation theory of the associated formal group of height d as a formal group.* □

This reduces us to a case from JD's talk: the inclusion of a deformation neighborhood of a height 2 point on the moduli of formal groups, which he showed had an associated homology theory called *Morava E–theory*. A *very* extravagant application of the tools from Ben's talk yields the following theorem, essentially owing to the very nice (i.e., formally smooth) deformation space and very nice (i.e., formally smooth) space of operations:

Theorem 9 (Goerss–Hopkins–Miller)**.** *Let Γ be a finite height formal group over a perfect field of positive characteristic. The moduli of topological enrichments of $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$ is homotopy equivalent to $B\operatorname{Aut}\Gamma$, and these operations are distinguished by their behavior on the cohomology of $\mathbb{CP}^{\infty}$.* □

In particular, this gives a topological enrichment of $\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}}$: it is the pullback of the Goerss–Hopkins–Miller sheaf along the Serre–Tate map

$$\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}} = \coprod_{\text{supersingular } C} (\mathcal{M}_{\mathbf{ell}})^{\wedge}_{C} \xrightarrow{\text{f.é.}} \coprod_{\text{supersingular } C} (\mathcal{M}_{\mathbf{pdiv}}(2))^{\wedge}_{C[p^{\infty}]} \xleftarrow{\cong} \coprod_{\text{supersingular } C} (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\widehat{C}}.$$

Remark 10. This buys you more than just a bouquet of Morava E–theories, or even the global sections

$$\mathcal{O}^{\mathrm{ss}}\left(\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}}\right) = \prod_{\text{supersingular } C} E_{\widehat{C}}^{h\operatorname{Aut} C}.$$

For instance, the moduli $\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}(N)$ of supersingular elliptic curves C equipped with specified isomorphisms $C[N] \cong (\mathbb{Z}/N)^{\times 2}$ forms an étale cover of $\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}$ whenever $p \nmid N$, and hence this sheaf produces spectra $\mathit{TMF}(N)^{\mathrm{ss}} = \mathcal{O}^{\mathrm{ss}}(\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}(N))$ satisfying $(\mathit{TMF}(N)^{\mathrm{ss}})^{hGL_2(\mathbb{Z}/N)} \simeq \mathit{TMF}^{\mathrm{ss}}$.

[2] This is an incarnation of the chromatic fracture square. The top-right node is the $K(2)$–local component, the bottom-left is the $K(1)$–local component, and the bottom-right is the gluing data: the $K(1)$–localization of the $K(2)$–local component.

[3] In general, the Serre–Tate theorem states that $\mathcal{M}^{d}_{\mathbf{ab}} \to \mathcal{M}_{\mathbf{pdiv}}(2d)$ is formally étale.

3. The ordinary locus

Before addressing the ordinary locus in earnest, where our goal is to manufacture a *lot* of cohomology theories, we spend a moment thinking about a particular example: p–adic K–theory, a familiar cohomology theory whose formal group looks like one associated to an ordinary elliptic curve. (Generally, we will call such things *ordinary* E_∞ ring spectra.) Using the Goerss–Hopkins–Miller theorem stated above, p–adic K–theory appears as the E_∞ ring spectrum[4] of sections assigned to the étale cover

$$\operatorname{Spf} \mathbb{Z}_p \to (\mathcal{M}_{\mathbf{fg}})^\wedge_{\widehat{\mathbb{G}}_m}.$$

Using $\operatorname{Aut} \widehat{\mathbb{G}}_m = \mathbb{Z}_p^\times$, the second half of the Goerss–Hopkins–Miller theorem endows this with a $\mathbb{Z}_p^\times$–indexed family of E_∞ ring maps, denoted ψ^k for $k \in \mathbb{Z}_p^\times$. Their effect on the cohomology of $\mathbb{CP}^\infty$ is given by

$$\begin{aligned} \psi^k \colon K_p^0 \mathbb{CP}^\infty &\to K_p^0 \mathbb{CP}^\infty, \\ x &\mapsto 1-(1-x)^k, \end{aligned}$$

where $x \in K_p^0 \mathbb{CP}^\infty$ is the "usual" power series generator.

On the other hand, the E_∞ ring structure itself gives another map

$$P^p \colon K_p^0 \to K_p^0 B\Sigma_p,$$

the p^{th} power operation. There is a calculation $K_p^0 B\Sigma_p = K_p^0\{1,\theta\}$, and in these terms the effect of the p^{th} power operation on the cohomology of $\mathbb{CP}^\infty$ is

$$\begin{aligned} P^p \colon K_p^0 \mathbb{CP}^\infty &\to K_p^0 \mathbb{CP}^\infty \otimes K_p^0\{1,\theta\}, \\ x &\mapsto (1-(1-x)^p) \otimes \theta. \end{aligned}$$

This extra operation describes an extension of the $\mathbb{Z}_p^\times$ action on K_p to an action of the monoid $\mathbb{Z}_p$.[5]

In general, the p–adic K–theory of an E_∞ ring spectrum also carries a $\mathbb{Z}_p$–action (and perhaps a little more, if the homotopy groups are not torsion-free), called a *θ–algebra*. A different version of the Goerss–Hopkins–Miller theorem gives a reverse-engineering tool that converts information about θ–algebras into information about the space of possible ordinary E_∞ ring spectra yielding them on evaluation of p–adic K–theory, or the mapping space between two such. A tool from JD's talk lets us guess which algebra should be associated to the p–adic K–theory of the global sections of $\mathcal{O}^{\mathrm{ord}}$, pictured in the first pullback square below:

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Spf} W_1 \arrow[r] \arrow[d] & \operatorname{Spf} V_\infty^\wedge \arrow[r] \arrow[d] & \operatorname{Spf} \mathbb{Z}_p \arrow[d, "\text{f.é.}"] \\ \cdots \arrow[r] & \mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}(p^1) \arrow[r] & \mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}} \arrow[r] & \mathcal{M}_{\mathbf{fg}}. \end{tikzcd}$$

Moreover, $V_\infty^\wedge$ has a natural structure as a θ–algebra: the interesting map ψ^p acts by

$$\psi^p \colon (C, \eta \colon \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}) \mapsto \left(\begin{tikzcd} \widehat{\mathbb{G}}_m[p] \arrow[r] \arrow[d] & \widehat{\mathbb{G}}_m \arrow[r, "p"] \arrow[d, "\eta"] & \widehat{\mathbb{G}}_m \arrow[d, dotted, "\eta^{(p)}"] \\ C[p] \arrow[r] & C \arrow[r, "p"] & C^{(p)} \end{tikzcd} \right).$$

Unfortunately, this θ–algebra is not nice enough to apply the Goerss–Hopkins–Miller theorem. At this point, it becomes convenient to work at $p \geq 5$ for simplicity, where introducing a formal $(\mathbb{Z}/p)$–level structure fixes this:

Lemma 11 (Igusa). *For $p \geq 5$, the moduli $\mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}(p)$ is affine.* □

[4] K–theory can be given the structure of an E_∞ ring spectrum directly from the geometry of vector bundles, and it is a further theorem that these two structures agree.

[5] An alternative description of the effect of the power operation is that it encodes quotienting by an order p subgroup of the formal group associated to E_Γ, and it so happens that $\widehat{\mathbb{G}}_m[p]$ is the *unique* such subgroup for $\widehat{\mathbb{G}}_m$.

Corollary 12. *The associated θ–algebra W_1 has vanishing Goerss–Hopkins–Miller obstruction groups, hence realizes uniquely to an ordinary E_∞ ring spectrum $TMF(p)^{\mathrm{ord}}$, and the action of $(\mathbb{Z}/p)^\times$ on the level structure enhances to a coherent $(\mathbb{Z}/p)^\times$–action on $TMF(p)^{\mathrm{ord}}$.* □

We define TMF^{ord}, our candidate for $\Gamma(\mathscr{O}^{\mathrm{ord}})$, to be the $(\mathbb{Z}/p)^\times$–fixed points of $TMF(p)^{\mathrm{ord}}$, and indeed its p–adic K–theory is $V_\infty^\wedge$. More than this, it turns out that the θ–algebra associated to any formal étale affine over $\mathscr{M}_{\mathbf{ell}}^{\mathrm{ord}}$ has a unique realization *as an algebra under $TMF(p)^{\mathrm{ord}}$*, and maps between such also lift uniquely—and this gives us the desired sheaf $\mathscr{O}^{\mathrm{ord}}$. This is a common strategy: find a topological enrichment of an affine cover of your stack of interest, descend it to the stack itself, and use it to govern the rest of the affines.

4. Gluing data

The last thing we have to do to construct the pullback square is to manufacture a map of sheaves

$$i_* i^* \mathscr{O}_{\mathrm{top}} \to i_* i^* \left((\mathscr{O}_{\mathrm{top}})^\wedge_{\mathscr{M}_{\mathbf{ell}}^{\mathrm{ss}}} \right).$$

This is rather similar to the construction of $\mathscr{O}^{\mathrm{ord}}$ itself: we construct a candidate map $TMF^{\mathrm{ord}} \to (TMF^{\mathrm{ss}})^{\mathrm{ord}}$ of global sections, and then we use this to control the map of sheaves using relative Goerss–Hopkins obstruction theory.

The main result that marries algebra to topology are the following two results about $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$. The first is that $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$ counts as an elliptic spectrum:

Lemma 13. *There is an elliptic curve C^{alg} over an affine $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}})$ such that $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$ is an elliptic spectrum for this curve.* □

This gives two candidates for a θ–algebra structure on the p–adic K–theory of TMF^{ss}: there is the θ–algebra structure coming from the geometric map $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}}) \to \operatorname{Spf} V_\infty^\wedge$, and there is the θ–algebra structure coming from the fact that TMF^{ss} is an E_∞ ring spectrum, and hence $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$ is an (ordinary) E_∞ ring spectrum.

Theorem 14. *The natural θ–algebra structure on $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}})$ induced by the map $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}}) \to \operatorname{Spf} V_\infty^\wedge$ agrees with the Goerss–Hopkins–Miller θ–algebra structure on $\pi_*(TMF^{\mathrm{ss}})^{\mathrm{ord}}$.* □

This is to be read as a recognition theorem for the θ–algebra structure on the topological object $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$: it matches the algebraic model. Once this is established, the Goerss–Hopkins–Miller obstructions can be shown to vanish, and it follows that the above map lifts to a map of the E_∞ rings of global sections, and then one proceeds to produce the map of sheaves by further applications of obstruction theory.

Arithmetic fracture is dealt with similarly, but it is *far* simpler. Because $\mathbb{Q} \otimes TMF$ has a smooth $\mathbb{Q}$–algebra as its homotopy, the obstructions governing the version of Goerss–Hopkins–Miller for commutative $H\mathbb{Q}$–algebras vanish, letting us lift algebraic results into homotopy theory wholesale.

5. Variations on these results

Remark 15. At the prime 3, the proof of Igusa's theorem needs amplification, but the statement remains the same and the rest of the story goes through smoothly.

Remark 16. At the prime 2, two further things go wrong: one must pass to the Igusa cover $\mathscr{M}_{\mathbf{ell}}^{\mathrm{ord}}(4)$ before it becomes affine, but then the Galois group of this cover is C_2, which has infinite cohomological dimension at 2. Appealing to the equivalence $KO = KU^{hC_2}$, one works with 2–adic *real* K–theory instead, which somehow pre-computes the Galois action.

Remark 17. There is another way to construct TMF^{ord} at low primes, given by a complex consisting of two E_∞ cells attached to $\mathbb{S}$. The way this is done, essentially, is by constructing a complex whose p–adic K–theory matches the expected value: first it must have the right dimension, and then the action of θ must be corrected.

Remark 18. There is an analogous (and much easier) picture for the moduli of forms of the multiplicative group: any ordered pair of puncture points in $\mathbb{A}^1$ can be used to give $\mathbb{P}^1$ the unique structure of a group with identity at ∞, and the associated formal group is classified by a map $\mathscr{M}_{\mathbb{G}_m} \to \mathscr{M}_{\mathbf{fg}}$; there is an equivalence $\mathscr{M}_{\mathbb{G}_m} \simeq BC_2$; and KU forms the global sections of a topological enhancement of $\operatorname{Spec}\mathbb{Z} \to \mathscr{M}_{\mathbf{fg}}$ which descends using the complex-conjugation action to $BC_2 \to \mathscr{M}_{\mathbf{fg}}$.

Remark 19. With some effort[6], the construction of $\mathscr{O}_{\mathrm{top}}$ outlined here extends to the compactified moduli $\overline{\mathscr{M}_{\mathbf{ell}}}$ where Weierstrass curves with nodal singularities are allowed, i.e., where Δ is *not* inverted (as in $y^2 + xy = x^3$). The resulting global sections yields a spectrum *Tmf*, which is *not* a periodic ring spectrum. The connective truncation of that spectrum is denoted *tmf*, and it is expected to arise as the global sections of a topological enrichment of a stack of generalized cubics, i.e., where cuspidal singularities are also allowed (as in $y^2 = x^3$). The existence proof for a topological enrichment of $\mathscr{M}_{\mathbf{cub}}$ remains elusive.

6. Descent on homotopy

One of the main upsides of finding one of these topological enrichments is that it comes a equipped with a spectral sequence computing the homotopy of its global sections, coming from recovering $\mathscr{O}(\mathscr{N})$ as the homotopy limit of finer and finer covers of $\mathscr{N}$:

Lemma 20. *For $\mathscr{O}$ a topological enrichment of an appropriate map $\mathscr{N} \to \mathscr{M}_{\mathbf{fg}}$, there is a spectral sequence*

$$E_2^{s,t} = H^s(\mathscr{N}; \pi_t \circ \mathscr{O}) \Rightarrow \pi_{t-s}\Gamma(\mathscr{N}; \mathscr{O}). \quad \square$$

Lemma 21. *This spectral sequence is isomorphic to the MU–Adams spectral sequence for $\mathscr{O}(\mathscr{N})$.*

Main observation. Consider the Čech complex associated to the affine cover $\mathscr{M}_{\mathbf{Weier}} \to \mathscr{M}_{\mathbf{ell}}$. We claim that the complex making up the E_1–term of the descent spectral sequence is isomorphic to the complex making up the E_1–term of the MU–Adams spectral sequence. To illustrate, we compute the first two terms of each and compare them.

(1) Consider the pullback diagram of stacks

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \mathscr{M}_{\mathbf{fgl}} \\ \downarrow & & \downarrow \\ \mathscr{M}_{\mathbf{ell}} & \longrightarrow & \mathscr{M}_{\mathbf{fg}}. \end{array}$$

In the same stroke, this is also the pullback diagram computing $\operatorname{Spec} MU_* TMF$.

(2) Now consider the two iterated pullback cubes pictured in Figure 1. That they compute equivalent pullbacks begets an isomorphism $\mathscr{M}_{\mathbf{quad.trans.}} \cong (\mathscr{M}_{\mathbf{fg}} \times \mathscr{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \times_{\mathscr{M}_{\mathbf{fgl}}} \mathscr{M}_{\mathbf{Weier}}$.

(n) The general case is similar, but requires stomaching iterated pullbacks in n–cubes. $\square$

We now appeal to Katharine's and Dominic's talks in order to compute $\pi_* TMF[1/6]$. Since 2 and 3 are both inverted, we can use scaling and translation to complete both the cube and the square to replace an arbitrary Weierstrass curve with *unique* one of the form $y^2 = x^3 + c_4 x + c_6$, with $\Delta = -24(4c_4^3 + 27c_3^2)$ and in fact the map

$$\operatorname{Spec}\mathbb{Z}[c_4, c_6, \Delta^{-1}][1/6] \to \mathscr{M}_{\mathbf{ell}} \times \operatorname{Spec}\mathbb{Z}[1/6]$$

is an equivalence of stacks. Since the quasicoherent sheaf cohomology of affines is always amplitude 0, this spectral sequence is concentrated on the 0–line, and we recover

$$\pi_* TMF[1/6] \cong \mathbb{Z}[c_4, c_6, \Delta^{-1}][1/6], \quad |\Delta| = 24.$$

Remark 22. This is all a rather elaborate way of recovering the homotopy of the *complex-orientable* ring spectrum *TMF*$[1/6]$. The joy is that the machinery works at low primes too, where the homotopy is *much* harder to compute. The moduli of elliptic curves has infinite cohomological dimension at the primes 2 and 3, and the descent spectral sequence is riddled with differentials—enough to give a horizontal vanishing line.

[6]Actually, a significant chunk of the trouble is already present in the details of this construction, since dealing with the gluing data requires working with an algebraized curve, which is only classified by a map to the compactified moduli.

$$
\begin{tikzcd}
 & (\mathscr{M}_{\mathbf{fgl}} \times \mathscr{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \times_{\mathscr{M}_{\mathbf{fgl}}} \mathscr{M}_{\mathbf{Weier}} \arrow[rr] \arrow[dl] & & \mathscr{M}_{\mathbf{Weier}} \arrow[dl] \arrow[dd] \\
\mathscr{M}_{\mathbf{fgl}} \times \mathscr{M}_{\mathbf{ps}}^{\mathrm{gpd}} \arrow[rr] \arrow[dd] & & \mathscr{M}_{\mathbf{fgl}} \arrow[dd] & \\
 & & & \mathscr{M}_{\mathbf{ell}} \arrow[dl] \\
\mathscr{M}_{\mathbf{fgl}} \arrow[rr] & & \mathscr{M}_{\mathbf{fg}}, &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathscr{M}_{\mathbf{quad.trans.}} \arrow[rr] \arrow[dd] & & \mathscr{M}_{\mathbf{Weier}} \arrow[dl] \arrow[dd] \\
 & & \mathscr{M}_{\mathbf{fgl}} \arrow[dd] & \\
 & \mathscr{M}_{\mathbf{Weier}} \arrow[rr, crossing over] \arrow[dl] & & \mathscr{M}_{\mathbf{ell}} \arrow[dl] \\
\mathscr{M}_{\mathbf{fgl}} \arrow[rr] & & \mathscr{M}_{\mathbf{fg}}. &
\end{tikzcd}
$$

FIGURE 1. Two equivalent homotopy pullback cubes.