Additive and multiplicative cocycles and Singer's calculation of the (co)homology of BU's connective covers

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Singer's Calculation (1967)

$$H^*(BU\langle 2k\rangle;\mathbb{Z}_p) = \frac{H^*(BU;\mathbb{Z}_p)}{\mathbb{Z}_p[\theta_{2i} \mid \sigma_p(i-1) < k-1]} \otimes \prod_{t=0}^{p-2} F[M_{2k-3-2t}],$$

where θ_{2i} are particular indecomposables and $F[M]$ is an algebra associated to the closure of a particular element of cohomology under the Steenrod operations.

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Goal: alternative description of this space's homology

Line Bundles and spec $H_* BU\langle 2k\rangle$

- Virtual bundle $\xi_k = \prod_{i=1}^{k}(1 - L_i)$ over $(\mathbb{C}P^\infty)^k$.

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- Adjoint element $g' \in H^*((\mathbb{C}P^\infty)^k; H_* BU\langle 2k\rangle)$

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- Künneth formula, classical calculations: $H^*((\mathbb{C}P^\infty)^k; H_* BU\langle 2k\rangle) = (H_* BU\langle 2k\rangle)[\![x_1, \ldots, x_k]\!]$.

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- $H_* g$ has an interpretation as a power series!

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- Rigid: $g'(\ldots, 0, \ldots) = 1$,
- Multiplicative 2-cocycle:

$$\frac{g(x,y)}{g(z+x,y)} \cdot \frac{g(z,x+y)}{g(z,x)} = 1.$$

(Similar equations for $k > 2$.)

A-H-S Result

- Ring maps $H_* BU\langle 2k\rangle \to A$ send this power series somewhere.
- Symmetric multiplicative 2-cocycles over $A[\![x_1, \dots, x_k]\!]$ are detected by $Rings(C^k, A)$ for a certain ring C^k.

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- (Ando-Hopkins-Strickland) For $k \leq 3$,

$$Rings(C^k, A) \cong Rings(H_* BU\langle 2k\rangle, A),$$

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  as determined by the map's action on g'!
- What about $k > 3$?

- Let f be a multiplicative 2-cocycle.
- $f = 1 + g + o(\mathbf{x}^{n+1})$, g a polynomial of homogenous degree n
- g is an additive 2-cocycle:

$$g(x,y) - g(z+x,y) + g(z,x+y) - g(z,x) = 0.$$

(Again, similar equations for $k > 2$.)

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  (Again, similar equations for $k > 2$.)
- Studying these gives an idea of what we should expect C^k to look like.

Lazard's Cocycles

- $f_n(x, y) = d^{-1}\left((x + y)^n - x^n - y^n\right)$
- Important in formal group laws.

A-H-S's Cocycles

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  - Produced one cocycle for each number of variables k and homogenous degree n, called ζ_k^n
- Modular cocycles (polynomials over $\mathbb{Z}_p$)
  - Classified cocycles up to three variables
  - Bases given by ζ_k^n and $(\zeta_k^{n/p})^p$

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- Basic method: make massive calculations, pray we see something
- We shorthand symmetric polynomials using integer partitions, associating a partition λ to the polynomial

$$\tau\lambda = d^{-1} \sum_{\sigma \in S_k} \mathbf{x}^{\sigma\lambda}.$$

e.g.,

- $\tau(1,1,1) = xyz$,
- $\tau(3,1,1) = x^3yz + xy^3z + xyz^3$.

$\mathbb{Z}_2$	dim 2	3	4	5
dim 5	$\tau(4,1)$	$\tau(2,2,1)$	$\tau(2,1,1,1)$	$\tau(1,1,1,1,1)$
6	$\tau(4,2)$	$\tau(2,2,2)$, $\tau(4,1,1)$	$\tau(2,2,1,1)$	$\tau(2,1,1,1,1)$
7	$\tau(6,1)+$ $\tau(5,2)+$ $\tau(4,3)$	$\tau(4,2,1)$	$\tau(2,2,2,1)$, $\tau(4,1,1,1)$	$\tau(2,2,1,1,1)$
8	$\tau(4,4)$	$\tau(4,2,2)$	$\tau(2,2,2,2)$, $\tau(4,2,1,1)$	$\tau(2,2,2,1,1)$, $\tau(4,1,1,1,1)$
9	$\tau(8,1)$	$\tau(4,4,1)$	$\tau(4,2,2,1)$	$\tau(2,2,2,2,1)$, $\tau(4,2,1,1,1)$
10	$\tau(8,2)$	$\tau(4,4,2)$, $\tau(8,1,1)$	$\tau(4,2,2,2)$, $\tau(4,4,1,1)$	$\tau(2,2,2,2,2)$, $\tau(4,2,2,1,1)$
11	$\tau(10,1)+$ $\tau(9,2)+$ $\tau(8,3)$	$\tau(8,2,1)$	$\tau(4,4,2,1)$, $\tau(8,1,1,1)$	$\tau(4,2,2,2,1)$, $\tau(4,4,1,1,1)$

Characteristic 2 Data

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7	$\tau(6,1)+$ $\tau(5,2)+$ $\tau(4,3)$	$\tau(4,2,1)$	$\tau(2,2,2,1)$, $\tau(4,1,1,1)$	$\tau(2,2,1,1,1)$
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9	$\tau(8,1)$	$\tau(4,4,1)$	$\tau(4,2,2,1)$	$\tau(2,2,2,2,1)$, $\tau(4,2,1,1,1)$
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9	$\tau(8,1)$	$\tau(4,4,1)$	$\tau(4,2,2,1)$	$\tau(2,2,2,2,1)$, $\tau(4,2,1,1,1)$
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$\mathbb{Z}_3$	dim 2	3	4	5
deg 12	$\tau(9,3)$	$\tau(6,3,3)$,	$\tau(3,3,3,3)$,	$\tau(6,3,1,1,1)-$
		$\tau(9,2,1)-$	$\tau(9,1,1,1)$	$\tau(4,3,3,1,1)+$
		$\tau(10,1,1)$		$\tau(3,3,3,2,1)$
13	$\tau(12,1)+$	$\tau(9,3,1)$	$\tau(4,3,3,3)-$	$\tau(3,3,3,3,1)$,
	$\tau(10,3)+$		$\tau(6,3,3,1)$,	$\tau(9,1,1,1,1)$
	$\tau(9,4)$		$\tau(9,2,1,1)-$	
			$\tau(10,1,1,1)$	
14	$\tau(12,2)-$	$\tau(9,3,2)-$	$\tau(9,3,1,1)$	$\tau(6,3,3,1,1)-$
	$\tau(13,1)+$	$\tau(12,1,1)-$		$\tau(4,3,3,3,1)+$
	$\tau(11,3)-$	$\tau(10,3,1)-$		$\tau(3,3,3,3,2)$,
	$\tau(10,4)+$	$\tau(9,4,1)$		$\tau(9,2,1,1,1)-$
	$\tau(9,5)$			$\tau(10,1,1,1,1)$

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		$\tau(9,2,1)-$	$\tau(9,1,1,1)$	$\tau(4,3,3,1,1)+$
		$\tau(10,1,1)$		$\tau(3,3,3,2,1)$
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deg 12	$\tau(9,3)$	$\tau(6,3,3)$, $\tau(9,2,1)-$ $\tau(10,1,1)$	$\tau(3,3,3,3)$, $\tau(9,1,1,1)$	$\tau(6,3,1,1,1)-$ $\tau(4,3,3,1,1)+$ $\tau(3,3,3,2,1)$
13	$\tau(12,1)+$ $\tau(10,3)+$ $\tau(9,4)$	$\tau(9,3,1)$	$\tau(4,3,3,3)-$ $\tau(6,3,3,1)$, $\tau(9,2,1,1)-$ $\tau(10,1,1,1)$	$\tau(3,3,3,3,1)$, $\tau(9,1,1,1,1)$
14	$\tau(12,2)-$ $\tau(13,1)+$ $\tau(11,3)-$ $\tau(10,4)+$ $\tau(9,5)$	$\tau(9,3,2)-$ $\tau(12,1,1)-$ $\tau(10,3,1)-$ $\tau(9,4,1)$	$\tau(9,3,1,1)$	$\tau(6,3,3,1,1)-$ $\tau(4,3,3,3,1)+$ $\tau(3,3,3,3,2)$, $\tau(9,2,1,1,1)-$ $\tau(10,1,1,1,1)$

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13	$\tau(12,1)+$ $\tau(10,3)+$ $\tau(9,4)$	$\tau(9,3,1)$	$\tau(4,3,3,3)-$ $\tau(6,3,3,1)$, $\tau(9,2,1,1)-$ $\tau(10,1,1,1)$	$\tau(3,3,3,3,1)$, $\tau(9,1,1,1,1)$
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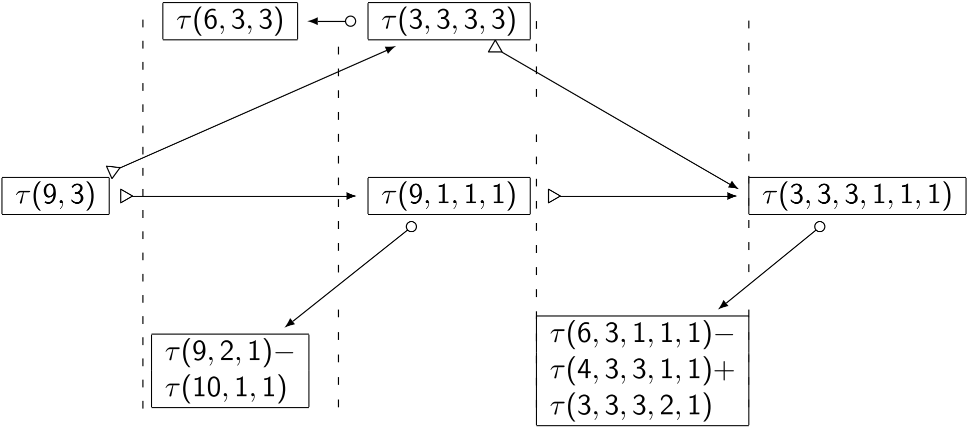
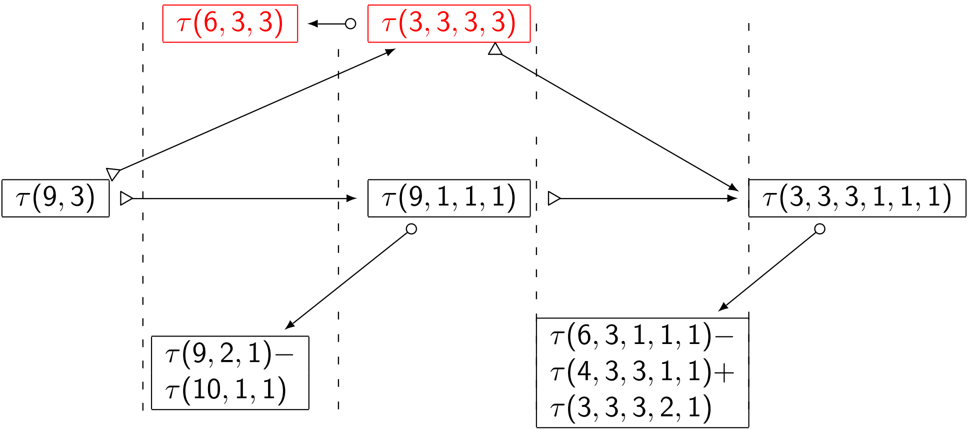
	$\tau(9,5)$			$\tau(10,1,1,1,1)$

Degree 12, Characteristic 3



Degree 12, Characteristic 3



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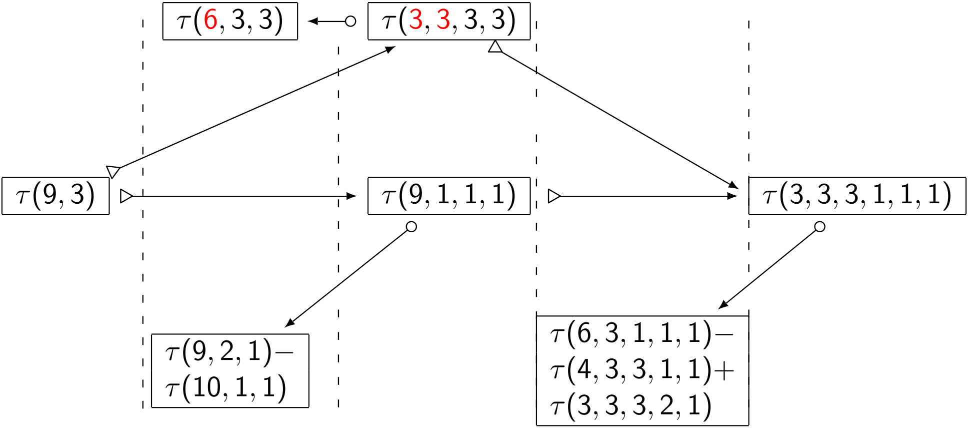

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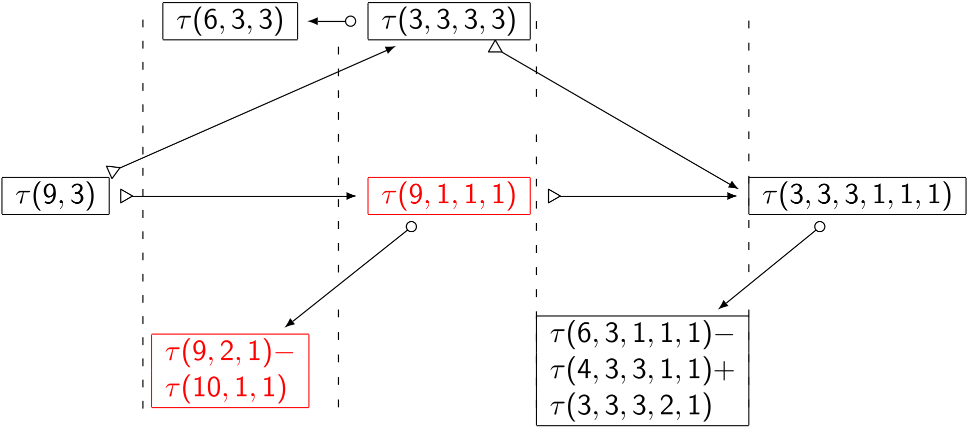

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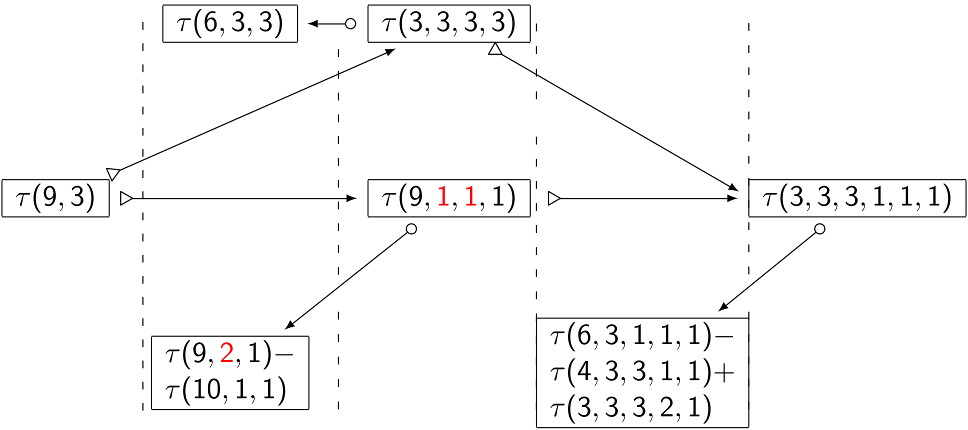

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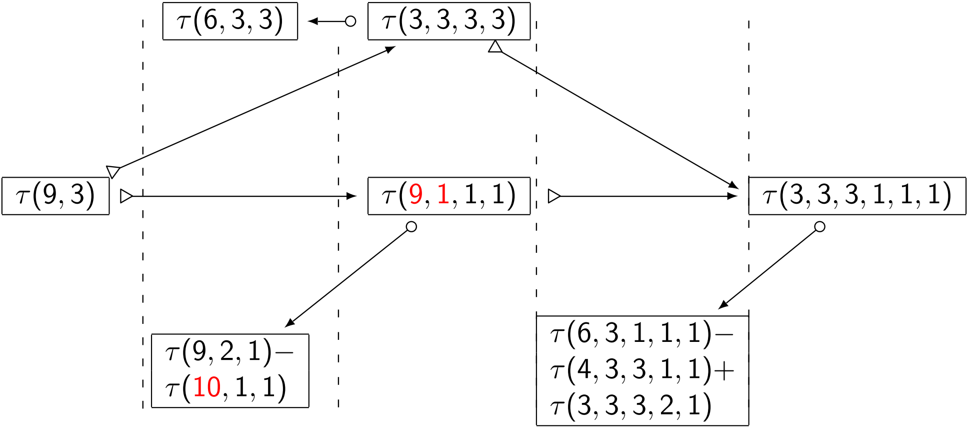
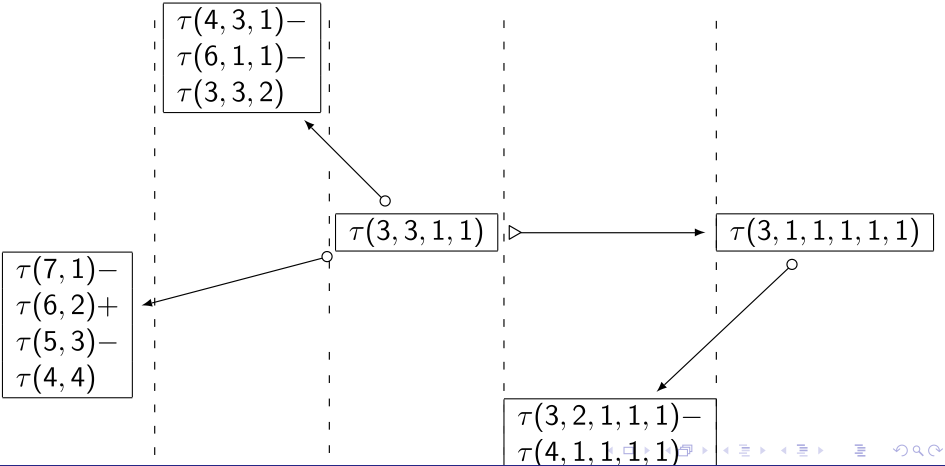


Degree 8, Characteristic 3



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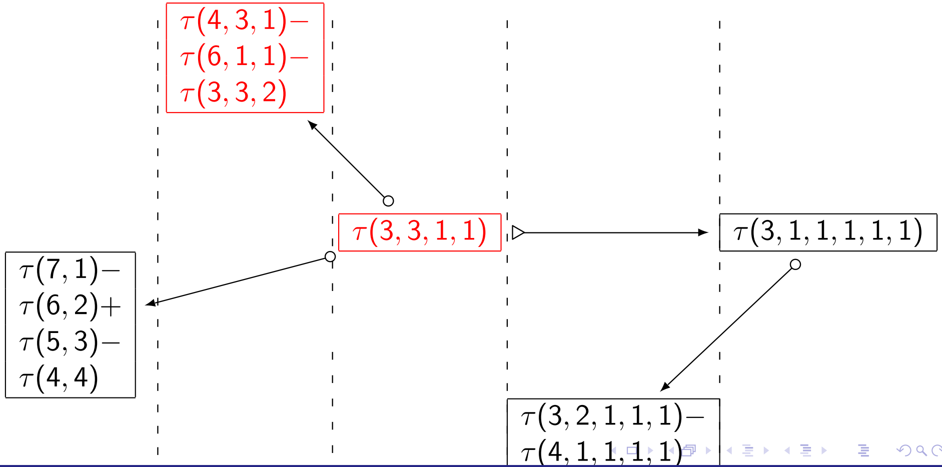
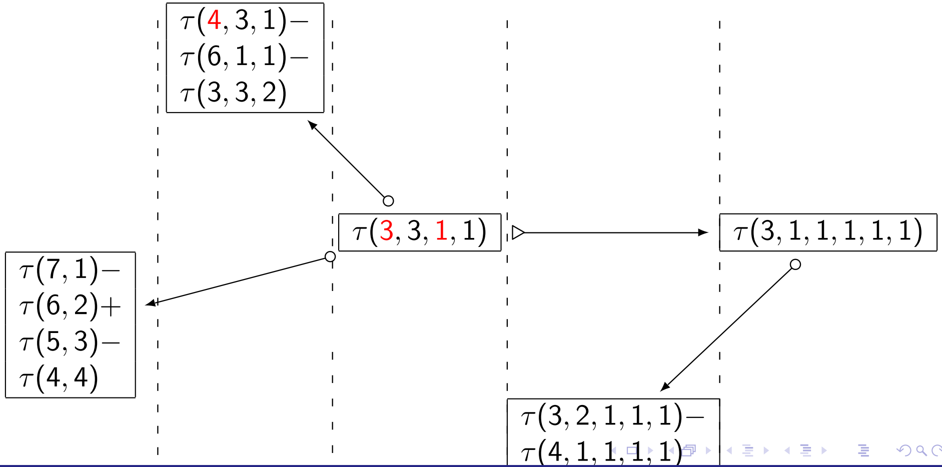


Degree 8, Characteristic 3



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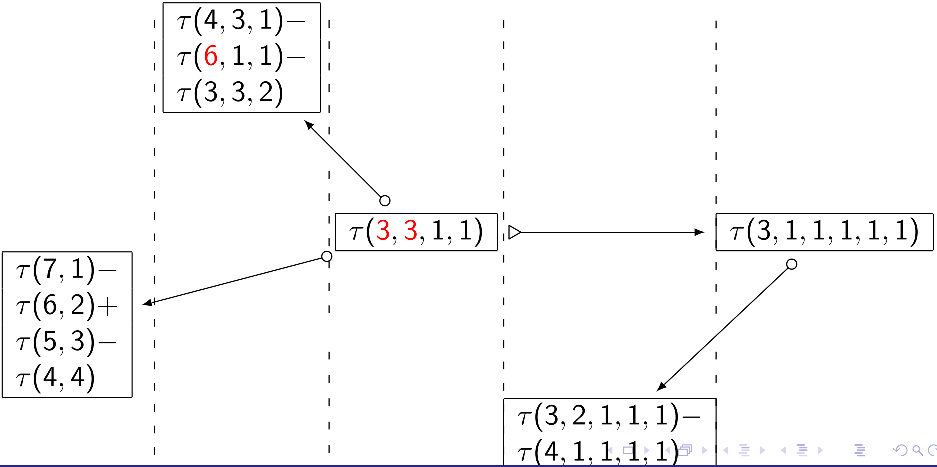

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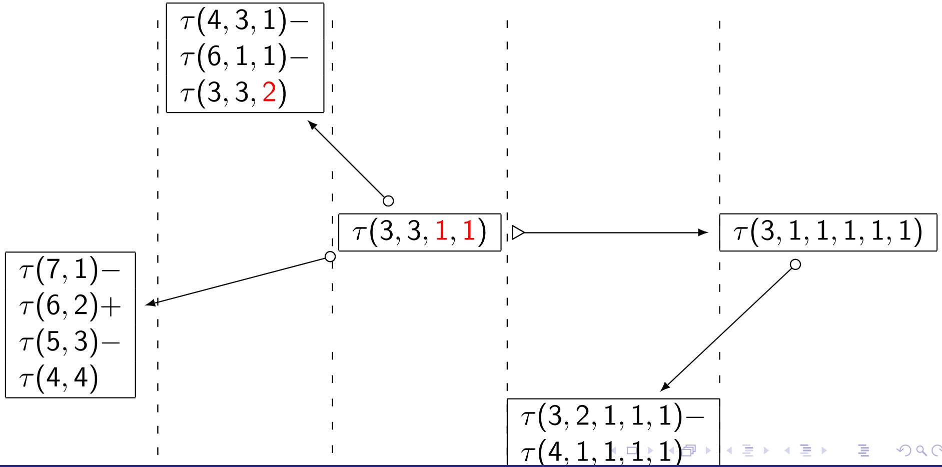

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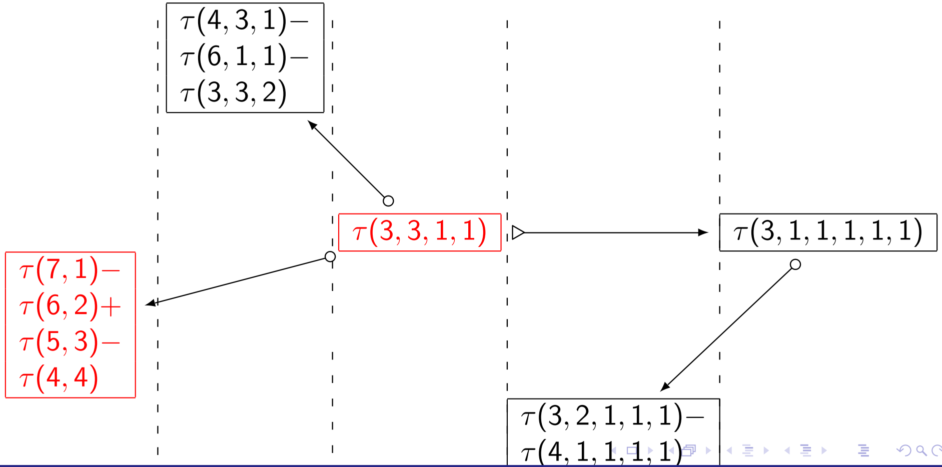

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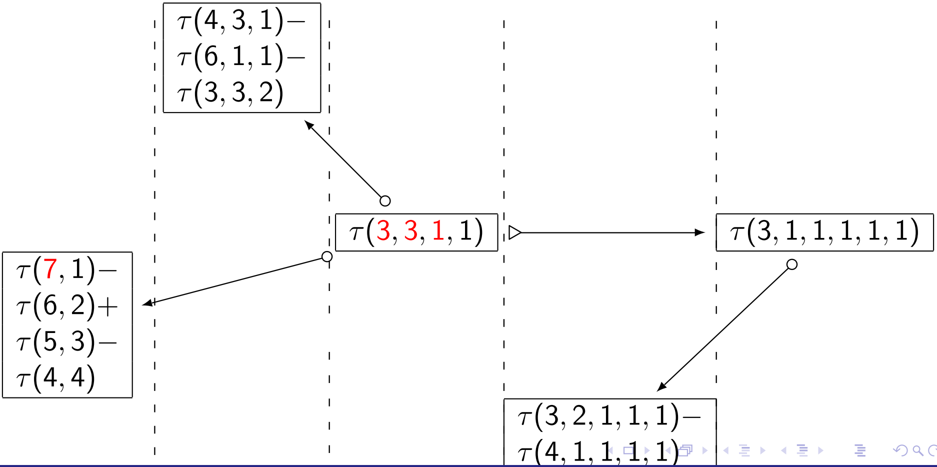

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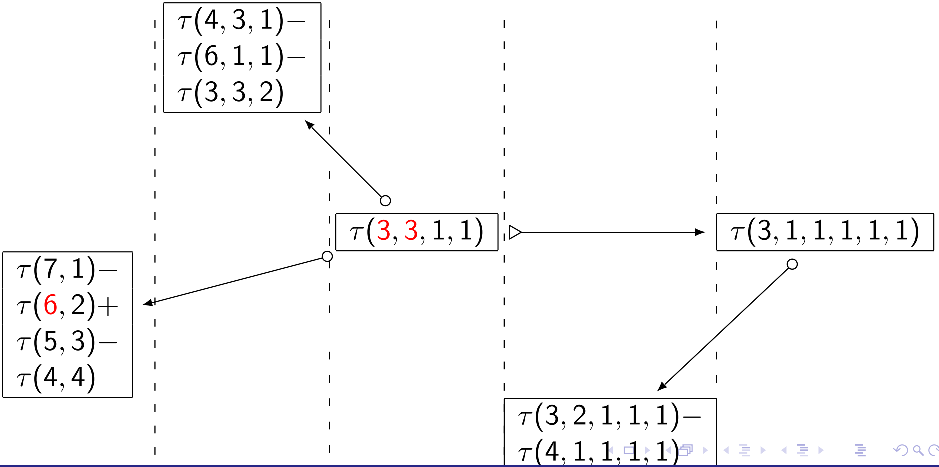

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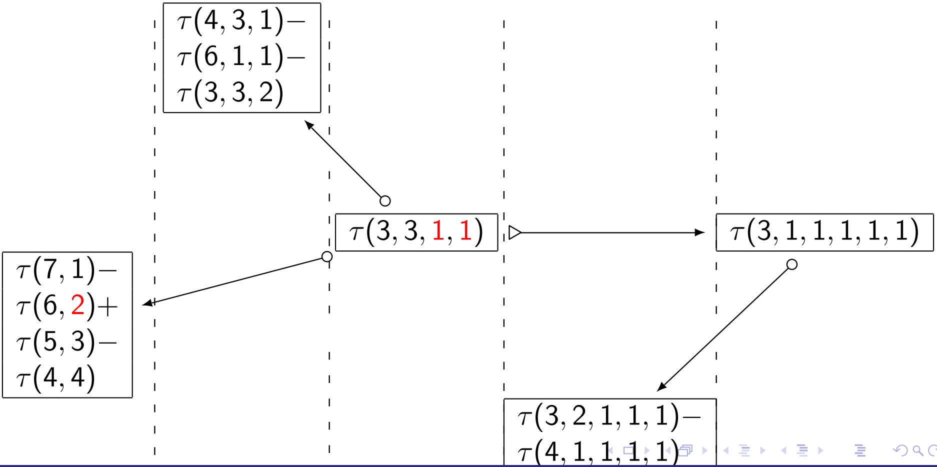

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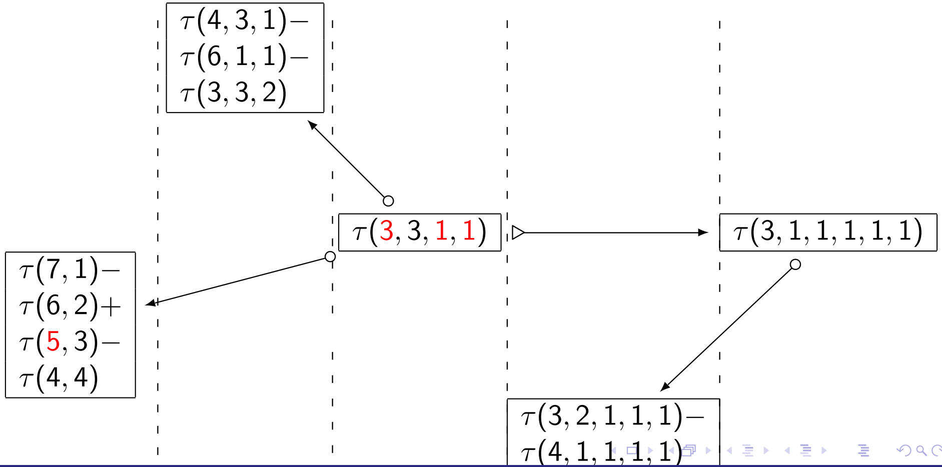

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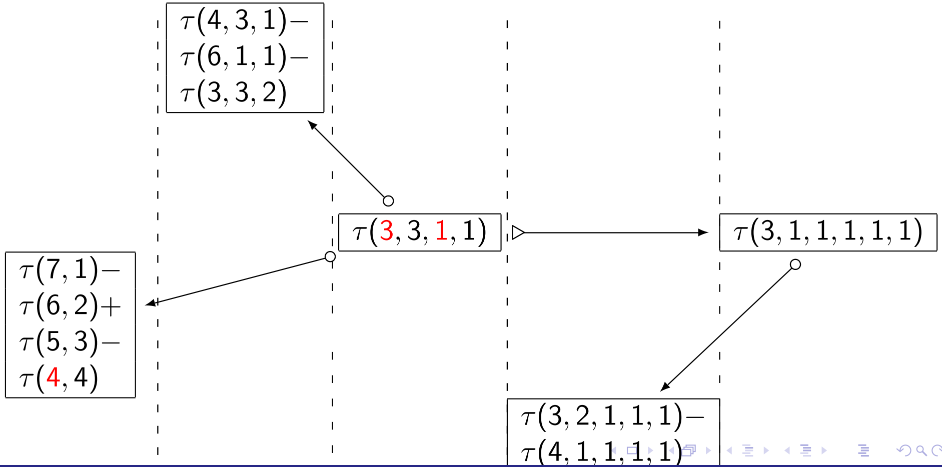

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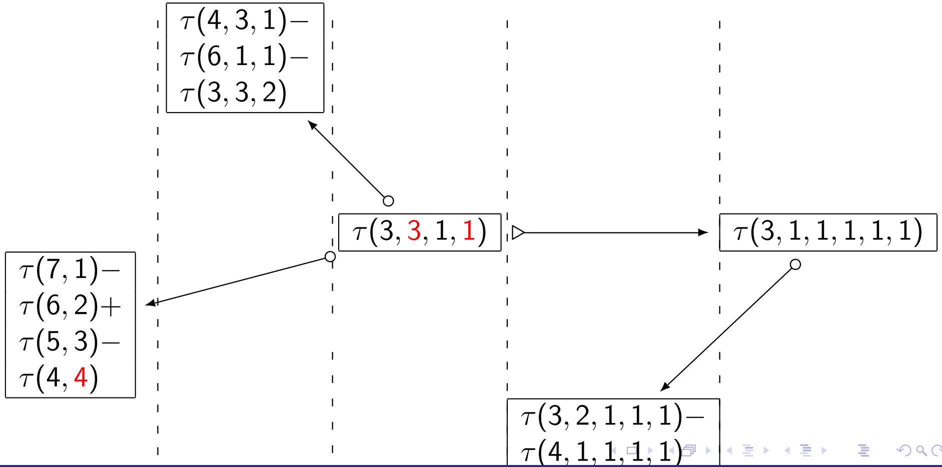

- Coefficients come from ζ_k^n.
- To state this formally, we introduce gathering operators:

$$G_{i,j}\lambda = (\lambda \setminus (\lambda_i, \lambda_j)) \cup (\lambda_i + \lambda_j).$$

The Big Theorem

Theorem

Select a power-of-p partition λ of n with length k. Let $T^m\lambda$ denote the set of all possible partitions of the form $G_{i_1,j_i}\cdots G_{i_m,j_m}\lambda$. Then, if either $m \leq p-2$ or if λ is the shortest power-of-p partition of n, the polynomial

$$\sum_{\mu \in T^m\lambda} c_\mu \cdot (\tau\mu)$$

will be a cocycle, where c_μ is the coefficient of $\tau\mu$ in $\pi_p\zeta^n_{k-m}$. In addition, cocycles formed in this manner give a basis for the space of modular cocycles.

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- Useful bound on number of distinct multiplicative cocycles

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- Our initial motivation – what does spec $H_* BU\langle 2k\rangle$ really look like?