DIFFERENTIALS IN A MAY SPECTRAL SEQUENCE ARE TOPOLOGICAL

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Before we do any mathematics, I want to take a moment to thank the organizers for putting all the pieces of this workshop together. I think everyone's learned a lot, and I think they did a great job exemplifying the claim that algebraic topology is a wonderfully friendly field. We should also take a moment to congratulate all of you: not only have you survived to the end, which is commendable, but you've all given wonderful talks. In particular, you've made my talk a lot easier, since I'm in the extremely privileged position of getting to build off of all of your hard work at introducing these complex ideas. So, hats off.

1. Introduction: The May spectral sequence

OK, with that out of the way, let's get to work. I want to start by reminding you about the May spectral sequence from this morning, though it will take us a while to come back and see why I'm bringing it up. Mark's whole sequence of talks have been about computing stable homotopy groups, and the centerpiece of his lectures has been the Adams spectral sequence:

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathscr{A}^*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}\mathbb{S}^0 \otimes \mathbb{Z}_2^\wedge.$$

Mark even made some remark about how wonderful this reduction was because "machines can compute these Ext groups" — and that's true, but what he really means is that it's always possible, not that it's easy.

But we know what to do when we're faced with a hard problem: separate it into easy problems and then push all the hard parts into the differentials of a spectral sequence which reassembles the data. Peter May constructed a spectral sequence of algebras of the form

$$E_1^{*,*,*} = \mathbb{F}_2[h_{ij} \mid i, j \geq 1] \Rightarrow \operatorname{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{F}_2, \mathbb{F}_2),$$

where h_{ij} has degree $(1, 2^j(2^i-1), i)$ and corresponds to the element $\xi_i^{2^j} \in \mathscr{A}_*$.

That's cool and really helpful, and I'd like to demonstrate this method in a small example. Anne-Marie and Mark both talked about the subalgebra $\mathscr{A}(1)^*$ spanned by Sq^1 and Sq^2, which had something to do with kO, connective real K-theory. This subalgebra is really tiny, and so the corresponding filtration is really short. There ends up being a version of May's theorem which identifies the E_1 page as $\mathbb{F}_2[h_{10}, h_{11}, h_{20}]$, corresponding to the surviving terms ξ_1, ξ_1^2, and ξ_2 in $\mathscr{A}(1)_*$. If you'll look up at the screen, I've had a computer program draw some pictures of the May spectral sequence for this filtration. The differentials in this spectral sequence encode pieces of the how $\mathscr{A}(1)_*$ is built out of simple pieces. For instance, there is a d_1-differential $d_1(h_{20}) = h_{10}h_{11}$, which records the relation between the symbol $Q_1 = \xi_2^\vee$ and the commutator $[\mathrm{Sq}^2, \mathrm{Sq}^1]$. After turning the page, there's also a d_2-differential $d_2(h_{20}^2) = h_{11}^3$.[1] This is recording the fact that there are three routes to the top of the $\mathscr{A}(1)^*$ picture: the two that pass through the $\mathrm{Sq}^2\,\mathrm{Sq}^1$ and $\mathrm{Sq}^1\,\mathrm{Sq}^2$ chains, and then also the middle path $\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2$. After applying this differential, you see a now familiar friend: the Adams E_2-page for $\pi_* kO$.

2. Multiplicative Thom spectra

Let's leave the May spectral sequence alone for a while and talk about things we're supposed to talk about: Thom complexes. These were introduced by Cary on the very first day, and since used by Brooke, Gabe, and Jean. The idea, as Cary told it, was to take a vector bundle, put a metric on it, think about the associated disk bundle, and quotient out its sub-spherical bundle. Brooke and Jean both mentioned in brief that vector bundles don't really have to enter this story: given a spherical bundle, you can build a Thom complex, and given a stable spherical bundle, you

[1] $d(\xi_2|\xi_2 + \xi_1^2\xi_2|\xi_1 + \xi_1^2|\xi_1\xi_2) = \xi_1^2|\xi_1^2|\xi_1^2$.

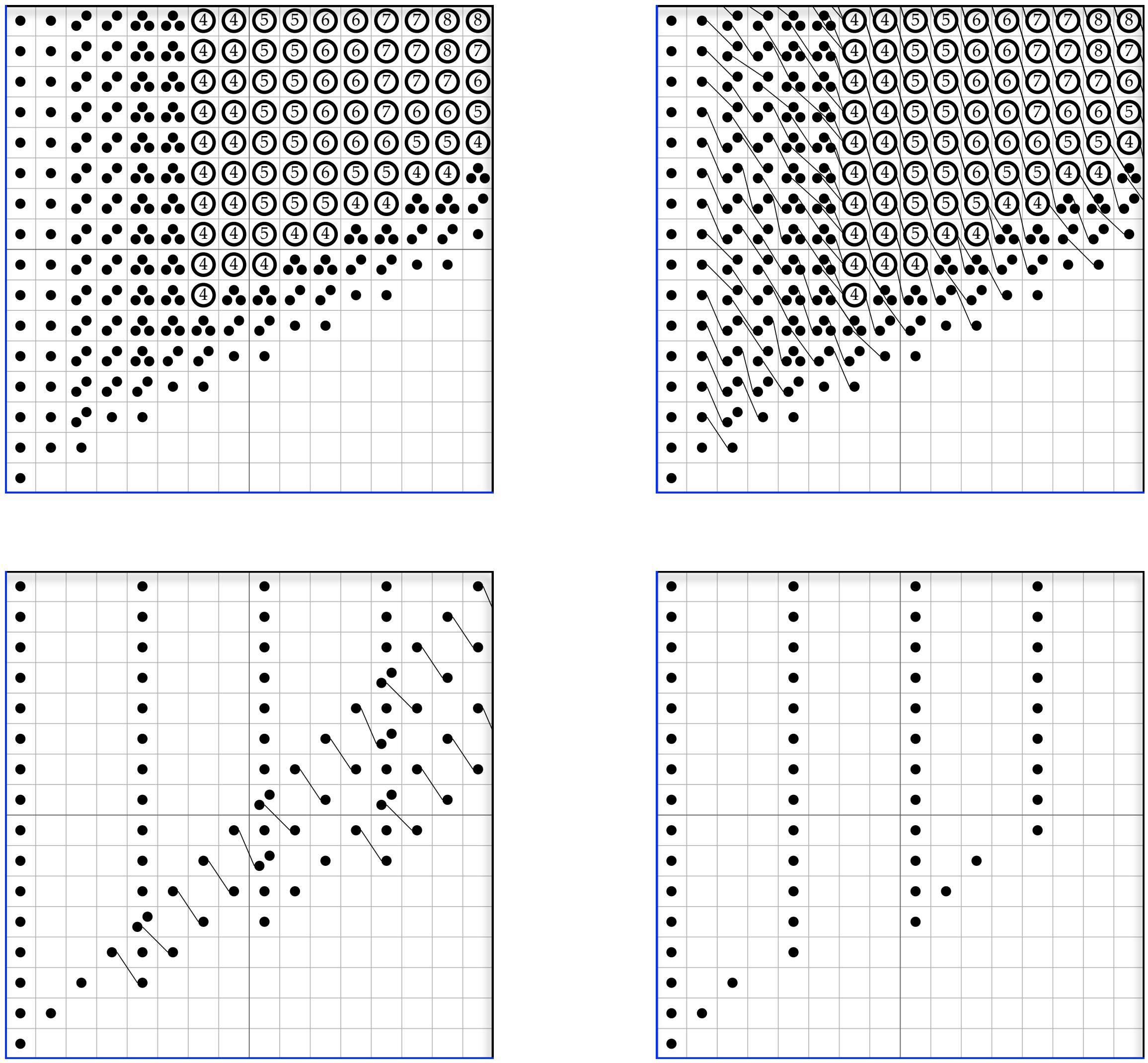


FIGURE 1. The E_1 page without differentials, pages E_1 and E_2 with differentials, and the $E_3 = E_\infty$ page.

can build a Thom *spectrum*. They even mentioned that there's a classifying space for such stable spherical bundles, and while they called it BF, and I'm going to write $BGL_1\mathbb{S}$. These are exactly the same space, but if you attend algebraic topology talks in the future, this is probably the name by which you'll encounter this object.

So, given a (homotopy class of a) map $\varphi : X \to BGL_1\mathbb{S}$, I can build a Thom spectrum $T\varphi$. This construction is actually extremely well-behaved. For starters, it's functorial: is I have a pair of spherical fibrations which are related by a homotopy-commuting triangle

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "\varphi'"'] & & X \arrow[dl, "\varphi"] \\ & BGL_1\mathbb{S}, & \end{tikzcd}$$

then I get an induced map $Tf : T(\varphi') \to T\varphi$. These is also a method for sticking two spherical bundles together: given a pair of spherical bundles $\nu : X \to BGL_1\mathbb{S}$ and $\tau : Y \to BGL_1\mathbb{S}$, I can build a pair of spherical bundles over $X \times Y$ by pulling back along the projections. Then, given by two bundles over this common base, I can smash them together fiberwise — and since the smash of two spheres is another sphere, this gives me yet another stable spherical fibration. This all compiles into a group operation $BGL_1\mathbb{S} \times BGL_1\mathbb{S} \to BGL_1\mathbb{S}$. What's remarkable is that this operation interacts well with Thomification; there is an identity[2]

$$T(\nu \times \tau) = T\nu \wedge T\tau.$$

Mahowald's first big idea is to request that X and φ are themselves compatible with this structure: let X now be a group-like H-space, and let φ be a homomorphism $\varphi : X \to BGL_1\mathbb{S}$. The fact that it's a homomorphism can be encoded in the commuting diagram

$$\begin{tikzcd} X \times X \arrow[r, "\mu"] \arrow[d, "\varphi \times \varphi"] \arrow[dr, "\varphi \times \varphi"] & X \arrow[d, "\varphi"] \\ BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}. \end{tikzcd}$$

Thomifying the maps φ and $\varphi \times \varphi$, we get a map $T\varphi \wedge T\varphi \to T\varphi$ by functoriality — and you can check that this is the multiplication data for a ring spectrum.

That much is pretty standard, but here's something a bit nuttier. One thing you might have seen is the notion of a G-torsor, which is a set X equipped with a free and transitive G-action. This means that X is of the same size and shape as G, but that it's missing a choice of basepoint — you can still move around inside of X using G, but you can only move relative to where you are.[3] A different way to phrase this condition is that the "shearing map"

$$\sigma : G \times X \xrightarrow{(x,y)\mapsto(x,x^{-1}y)} X \times X$$

is a bijection — injectivity is faithfulness, and having a section is transitivity.

A prime example of a G-torsor is G itself, acted on by multiplication. Let's go back to our X and φ, and let's extend this diagram by the shearing map:

$$\begin{tikzcd} X \times X \arrow[r, "\sigma"] \arrow[rr, bend left, "0 \times \mathrm{id}"] \arrow[dr, dashed, "0 \times \varphi"] & X \times X \arrow[r, "\mu"] \arrow[d, "\varphi \times \varphi"] & X \arrow[d, "\varphi"] \\ & BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}. \end{tikzcd}$$

We can identify its long composite to $BGL_1\mathbb{S}$ too: $\mu \circ \sigma$ is the same as $0 \times \mathrm{id}$, and hence the long composite is $0 \times \varphi$. This clearly lifts to the square of $BGL_1\mathbb{S}$, and so we can identify the Thom spectrum: it's $\Sigma^\infty_+ X \times T\varphi$. What's more is that because σ is an isomorphism, it begets an equivalence

$$\Sigma^\infty_+ X \wedge T\varphi \xrightarrow{T\sigma} T\varphi \wedge T\varphi.$$

Now, that's *really* cool! Let's interpret this: in the case that φ is the trivial spherical bundle, the associated Thom spectrum is given by $\Sigma^\infty_+ X$. On the other hand, a nontrivial spherical bundle will beget some weird Thom spectrum $T\varphi$, about which little can be said. A ring spectrum E is said to be oriented for φ when smashing through with E gives an (E-module) equivalence

$$E \wedge \Sigma^\infty_+ X \simeq E \wedge T\varphi.$$

This equivalence is supposed to be a spectrum level incarnation of the Thom isomorphism — taking homotopy of both sides, as Mark told us on Wednesday, gives the E-homology of B and of $T\varphi$, and this equivalence is the Thom isomorphism between the homology of the base space and the homology of the (oriented) Thom spectrum. So, Mahowald's theorem says that certain Thom spectra are automatically oriented for *themselves*. Some quick thinking extends this further: not only is $T\varphi$ oriented for $T\varphi$, but it's also oriented for the Thom spectrum of any bundle that factors through φ. Also, if $T\varphi \to E$ is map of ring spectra, then any bundle for which $T\varphi$ is oriented, E is oriented as well. So, this example is sort of "doubly universal."

[2]Equivalently, T can be said to be a monoidal functor.

[3]The canonical physical example of a torsor is temperature, on which $\mathbb{R}$ acts, but there's no natural "basepoint" temperature.

3. A COUPLE EXAMPLES

Before we go any further, I should produce some examples for you. Where do we get such H-spaces and homomorphisms? For one, as you may have already guessed, the maps $BO \to BGL_1\mathbb{S}$ and $BU \to BGL_1\mathbb{S}$ fit the bill — they're in fact "infinite loop maps," which is even better. The second trick up Mahowald's sleeve is that he knows how to use these to produce a bunch of examples. Brooke told us the homotopy groups of BO:

n	1	2	3	4	5	6	7	8
$\pi_n BO$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$.

These groups aren't just abstract things — there's a map $\eta : S^1 \to BO$, for instance, which selects the nontrivial element of $\mathbb{Z}/2$.

Unfortunately, there's no reason to expect such maps to be multiplicative, but the trick is that we can trade them in for multiplicative ones. Aaron told us how to construct classifying spaces with his B functor, and one thing we know about $B(BO)$ is that its homotopy groups are the same as those of BO, but shifted up by one dimension. So, there is also a map $\eta : S^2 \to B(BO)$ encoding the same element of homotopy. The turn to the trick, then, is that we can apply the loopspace functor to move back down, producing a map $\sigma(\eta) : \Omega S^2 \to BO$ which, by construction, is multiplicative, and which captures η in the sense that the natural composite $S^1 \to \Omega S^2 \to BO$ is η. Mahowald calls the resulting Thom spectrum X_2, and the same trick played on η^2, ν, and λ produces spectra he calls X_3, X_5, and X_9 respectively.

He *loves* these spectra. He goes on to prove a whole bunch of things using them as auxiliary tools: they appear in a paper on bo-resolutions, a paper on his η-family in unstable homotopy, and in a paper where he gives a partially completed program for resolving a famous open problem called the telescope conjecture. They have all kinds of interesting properties: X_3 is abelian, for instance, and there are relations like $X_3 \wedge M^0(2) \simeq X_2$. For us, though, we're only going to love one of them: you can play this same game but with double loopspaces instead, yielding a map $\sigma^2(\eta) : \Omega^2 S^3 \to BO$, and the resulting Thom spectrum can be identified as $T(\sigma^2\eta) \simeq H\mathbb{Z}/2$.

This is really nuts — at least to me. It's hard for me to properly convey how nuts this is. This is computed to be true, rather than shown by any conceptual method, making it super mysterious. That won't stop us from using it, though.

4. THOM SPECTRA AND THE ADAMS SPECTRAL SEQUENCE

We'd like to apply these facts somewhere, so the question is: where? Where have we seen smash powers $H\mathbb{Z}/2^{\wedge q}$ before, so that we can use our shearing isomorphism? Of course! — we saw them earlier today, when Mark constructed the Adams spectral sequence. You'll have to forgive me for going through some of it again. The idea is to start by considering the triangle associated to the unit map $\mathbb{S} \to H\mathbb{Z}/2$:

$$\begin{tikzcd} \mathbb{S} \arrow[dr] & I \arrow[l, "{[-1]}"'] \\ & H\mathbb{Z}/2 \arrow[u] \end{tikzcd}$$

Then, by smashing this triangle through with I, $I^{\wedge 2}$, and so on, we can translate it around and join up the resulting triangles:

$$\begin{tikzcd} \mathbb{S} \arrow[dr] & I \arrow[l, "{[-1]}"'] \arrow[dr] & I^{\wedge 2} \arrow[l, "{[-1]}"'] \arrow[dr] & I^{\wedge 3} \arrow[l, "{[-1]}"'] \arrow[dr] & \cdots \arrow[l, "{[-1]}"'] \\ & H\mathbb{Z}/2 \arrow[u] \arrow[r, "d_0"] & H\mathbb{Z}/2 \wedge I \arrow[u] \arrow[r, "d_1"] & H\mathbb{Z}/2 \wedge I^{\wedge 2} \arrow[u] \arrow[r, "d_2"] & \cdots . \end{tikzcd}$$

These bottom maps come from filling in the commuting triangle, and Aaron told us that they're exactly the differentials on the E_1-page of the Adams spectral sequence. We're close to being able to apply Mahowald's theorems, but we have to translate from I to $H\mathbb{Z}/2$. We can perform the same translation trick: take this trapezoidal shape and iteratively smash it through with $H\mathbb{Z}/2$ to build the following infinite triangle:

$$
\begin{tikzcd}
\mathbb{S} & I \arrow[l, "{[-1]}"'] \arrow[dr] & I^{\wedge 2} \arrow[l, "{[-1]}"'] \arrow[dr] & I^{\wedge 3} \arrow[l, "{[-1]}"'] \arrow[dr] & \cdots \arrow[l, "{[-1]}"'] \\
& H\mathbb{Z}/2 \arrow[u] \arrow[r, "d_0"] \arrow[dr, "\delta_0"'] \arrow[from=ul, crossing over] & H\mathbb{Z}/2 \wedge I \arrow[u] \arrow[r, "d_1"] \arrow[dr] & H\mathbb{Z}/2 \wedge I^{\wedge 2} \arrow[u] \arrow[r, "d_2"] \arrow[dr] & \cdots \\
& & H\mathbb{Z}/2^{\wedge 2} \arrow[u] \arrow[r] \arrow[dr, "\delta_1"'] & H\mathbb{Z}/2^{\wedge 2} \wedge I \arrow[u] \arrow[r] \arrow[dr] & \cdots \\
& & & H\mathbb{Z}/2^{\wedge 3} \arrow[u] \arrow[r] \arrow[dr, "\delta_2"'] & \cdots \\
& & & & \cdots .
\end{tikzcd}
$$

Now, I've been flippant about labeling things, but if you're careful (like Mahowald was), you can identify not just the objects here using his theorem but also the maps between them:

$$
\begin{tikzcd}
H\mathbb{Z}/2 \arrow[r, "\delta_0"] \arrow[d, equal] & H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \arrow[r, "\delta_1"] & H\mathbb{Z}/2 \wedge H\mathbb{Z}/2^{\wedge 2} \arrow[r, "\delta_2"] & \cdots \\
H\mathbb{Z}/2 \arrow[r, "\partial_0"] & H\mathbb{Z}/2 \wedge \Omega^2 S^3_+ \arrow[r, "\partial_1"] \arrow[u, "\simeq"] & H\mathbb{Z}/2 \wedge (\Omega^2 S^3_+)^{\wedge 2} \arrow[r, "\partial_2"] \arrow[u, "\simeq"] & \cdots ,
\end{tikzcd}
$$

$$\partial_n = \left(\sum_{i=1}^{n} 1 \wedge \cdots \wedge T(\Delta) \wedge \cdots \wedge 1 \right) + (-1)^{n+1} 1^{\wedge n} \wedge \eta.$$

5. Back to the May spectral sequence

Mahowald comes across his final observation when he goes to put this program into place by studying $H_*(\Omega^2 S^3; \mathbb{Z}/2)$. Vitaly told us yesterday about models for detecting when something is a loopspace, and though he didn't quite tell us this fact, his work begets a filtration on $\Omega^2\Sigma^2 X$ based on the number of little squares in his little squares operad. Victor Snaith showed that this filtration trivializes upon passing to suspension spectra:

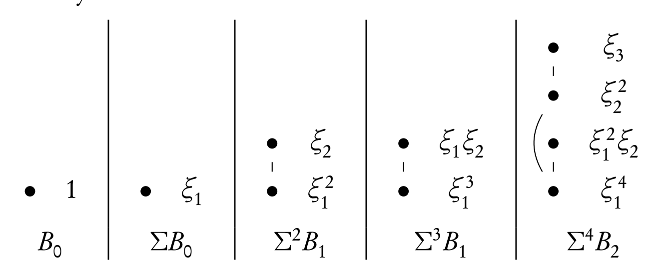
$$\Sigma^\infty_+ \Omega^2 S^3 = \Sigma^\infty_+ \Omega^2 \Sigma^2 S^1 \simeq \bigvee_{j=0}^{\infty} B_{\lfloor j/2 \rfloor},$$

where B_n is something called the nth Brown–Gitler spectrum. It has the property

$$H^*(B_n; \mathbb{Z}/2) \cong \mathscr{A} / \mathscr{A}\{\chi \operatorname{Sq}^i \mid i > n\}$$

and that there are cofiber sequences $B_{n-1} \to B_n \to B_{\lfloor n/2 \rfloor}$, though these don't quite characterize it.[4]

Let me draw a few of these for you:



Now, we know that these collectively form the homology groups $H_*(\Omega^2 S^3; \mathbb{F}_2)$, which we also know to be the dual Steenrod algebra $\pi_* H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$, so I've taken the liberty of naming some of these elements.

Here's where things start to get trippy: Mahowald puts all these observations together to produce some black magic that computes d_1 differentials in the May spectral sequence. Let me explain by example: for instance, look

[4]Another useful (if obvious) fact is that $H\mathbb{Z}/2 \simeq \operatorname{colim}_n B_n$.

at the second quotient $\Sigma^2 B_1$. The top cell here is ξ_2, called h_{20} in the May spectral sequence, which is attached by a Sq^1, called h_{10}, to the bottom cell ξ_1^2, called h_{11}. Mahowald asserts that it's no accident that $d_1(h_{20}) = h_{10}h_{11}$. The column after that is even fancier; the longest attaching map in it reads off the differential

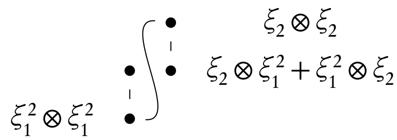
$$d_1(h_{30}) = h_{10}h_{21} + h_{20}h_{13}.$$

You can also see a smaller differential if you start at the second-topmost cell: $d_1(h_{21}) = h_{11}h_{12}$.[5]

That's really cool. And it gets trippier still: you can use these same methods to produce higher order differentials as well. The next term in our Adams resolution looks like

$$H\mathbb{Z}/2 \wedge (\Omega^2 S^3_+)^{\wedge 2} \simeq \bigvee_{j,k=0}^{\infty} H\mathbb{Z}/2 \wedge \Sigma^{j+k} B_{\lfloor j/2 \rfloor} \wedge B_{\lfloor k/2 \rfloor}.$$

Let's try the case $j = k = 2$, so we're studying $\Sigma^4 B_1 \wedge B_1$, which is the smash-square of the mod-2 Moore spectrum $M(2)$. Its cell structure looks like this:



Here, I've labeled the relevant cells using tensors of elements from the old picture. You can read off some kind of formula from this picture: b_{20} is attached to $h_{10}(“\xi_1^2 \otimes \xi_2 + \xi_2 \otimes \xi_1^2”) + h_{11}(h_{11}h_{11})$, . This observation doesn't record the d_2-differential on its own, since some things are still labeled by ξ's, but we have a differential from $\Sigma^4 B_2$ on the previous page which helps us out: $d_1(\xi_1^2\xi_2) = \xi_1^2 \otimes \xi_2 + \xi_2 \otimes \xi_1^2 + h_{10}h_{12}$. Since this sum is sent to zero by the quotient, it begets the relation $\xi_1^2 \otimes \xi_2 + \xi_2 \otimes \xi_1^2 = h_{10}h_{12}$ on the E_2-page, and so this gives:[6]

$$d_2(b_{20}) = h_{10}(h_{10}h_{12}) + h_{11}(h_{11}h_{11}).$$

(Apologies that I'm not able to outline the full method for computing these higher differentials. Not only is the talk a mere 45 minutes, but there are some wrinkles which I myself don't understand, so I'm not comfortable writing something that pretends to be complete.)

[5] You could have gotten this using Nakamura's squaring operations, since this is $d_1(\mathrm{Sq}^0 h_{20}) = (\mathrm{Sq}^0 h_{10})(\mathrm{Sq}^0 h_{11})$.

[6] This, too, is accessible by Nakamura's theorem: $d_2(\mathrm{Sq}^1 h_{20}) = \mathrm{Sq}^1 d_1 h_{20} = \mathrm{Sq}^1(h_{10}h_{11}) = \mathrm{Sq}^1 h_{10}\,\mathrm{Sq}^0 h_{11} + \mathrm{Sq}^0 h_{10}\,\mathrm{Sq}^1 h_{11} = h_{10}^2 h_{12} + h_{11}h_{11}^2$.