Algebraic topology and algebraic number theory Graduate Student Topology & Geometry Conference

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http://math.berkeley.edu/~ericp/latex/talks/austin-2014.pdf

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Formal groups

In this talk, p is an odd prime and k is a finite field, char k = p.

Definition

A formal group law is a power series $x +_F y \in R[\![x,y]\!]$ satisfying

$$x +_F 0 = x$$
, $x +_F y = y +_F x$, $x +_F (y +_F z) = (x +_F y) +_F z$.

Idea

Complex geometry: these come from charts on Lie groups. Arithmetic geometry: take a more exotic R than \mathbb{C} , like \mathbb{F}_p .

Example: \mathbb{G}_m

The formal multiplicative group \mathbb{G}_m is presented by

$$x +_{\mathbb{G}_m} y = 1 - (1 - x)(1 - y) = x + y - xy.$$



Some local class field theory

Most number theoretic questions can be interpreted through \bar{K} and $\operatorname{Gal}(\bar{K}/K)$ for K some (local) number field.

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Theorem (Lubin–Tate)

For a local number field K, there is a maximal abelian extension:

$$\operatorname{\mathsf{Gal}}(\mathsf{K}^{\mathrm{ab}}/\mathsf{K}) = \operatorname{\mathsf{Gal}}(\bar{\mathsf{K}}/\mathsf{K})^{\mathrm{ab}}.$$

One can construct it explicitly by studying the torsion points of a certain formal group Γ_K over \mathcal{O}_K .

Example: $K = \mathbb{Q}_p$

 $\Gamma_{\mathbb{Q}_p}$ is given by \mathbb{G}_m . "Torsion points" of \mathbb{G}_m are unipotent elements, and in fact $\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p(\zeta_n : n > 0)$.



Fields in stable homotopy theory

A *field spectrum* is a ring spectrum with Künneth isomorphisms. E.g.:

$$\left. egin{aligned} H^{dR} &= H\mathbb{R}, \ H\mathbb{F}_p, \end{aligned}
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Theorem (Devinatz–Hopkins–Smith)

There is a bijection

$$\left\{\begin{array}{c} \text{formal groups} \\ \text{over } k \end{array}\right\} \stackrel{\mathcal{K}}{\longrightarrow} \left\{\begin{array}{c} \text{2-periodic} \\ \text{field spectra} \\ \text{with } \pi_0 = k \end{array}\right\}.$$

The spectrum $K(\Gamma)$ is called the Morava K-theory for Γ .

Example: $\Gamma = \mathbb{G}_m$

KU/p is a model for $K(\mathbb{G}_m)$.

Homology operations and Morava E-theories

Our three examples come with natural "deformations":

$$H\mathbb{R} \to H\mathbb{R}, \qquad H\mathbb{Z}_p \to H\mathbb{F}_p, \qquad KU_p^{\wedge} \to KU/p.$$

All of these are governed by Bockstein operations.

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Theorem (Morava et al.)

For $d = ht(\Gamma)$ finite, $K(\Gamma)$ has d Bocksteins, giving a spectrum

$$E(\Gamma) \rightarrow K(\Gamma)$$

called the Morava E-theory for Γ . It takes values in modules over $\mathrm{Def}(\Gamma) = \mathbb{W}(k)[\![u_1,\ldots,u_{d-1}]\!]$ with an $\mathrm{Aut}\,\Gamma$ action.

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Theorem (Harris–Taylor et al., "local Langlands correspondence")

There is a correspondence among certain representations of:

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Theorem (Salch(–Morava), special case of $\Gamma=\mathbb{G}_m$)

For nice enough finite cell complexes X, there is an L-function

$$L(E(\mathbb{G}_m)_*(X);s).$$

It is analytic to the right of a pole at $s = \dim X$, and its special values at $s > \dim X$ have denominators encoding the ranks of the $E(\mathbb{G}_m)$ -local homotopy groups of X away from 2.

Example: $L_{E(\mathbb{G}_m)}S^0$ (Adams–Hopkins–Ravenel)

The *L*-function associated to S^0 is the Riemann ζ -function.

n	1	2	3	4	5	6	7
$ \pi_{2n+1}L_{E(\mathbb{G}_m)}S^0 $	$2^{3}3^{1}$	1	$2^43^15^1$	2^1	$2^33^27^1$	1	$2^53^15^1$
$denom(\zeta(-n))$	2^23^1	1	$2^33^15^1$	1	$2^23^27^1$	1	$2^43^15^1$

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Idea

This set-up encourages us to work one prime at a time and invoke Euler factorizations and p-adic L-functions.

What might $n \in \mathbb{Z}_p$ mean on the homotopy groups side?

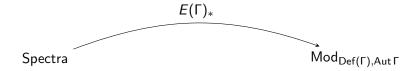


What's a sphere, anyway?

The \land -invertible spectra are exactly the stable spheres.

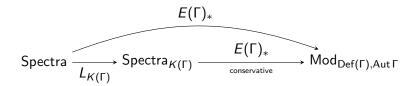
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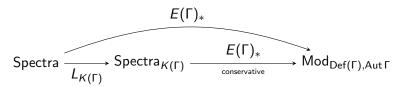
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Question

What are the \land -invertible objects in Spectra $_{K(\Gamma)}$?

Theorems (Hopkins–Mahowald–Sadofsky; Hopkins–Strickland)

For $\Gamma=\mathbb{G}_m$ and $p\geq 3$, $\mathrm{Pic}=\mathbb{Z}_p^\times\rtimes\mathbb{Z}/2$ (even spheres $\rtimes S^1$). The homotopy groups of $L_{K(\mathbb{G}_m)}S^{-1}$ indexed on the \mathbb{Z}_p^\times -factor in Pic agree with the p-adic interpolation of the ζ -function.

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Theorems (Goerss, Hopkins, Mahowald, Rezk, Sadofsky)

- $\Gamma = \mathbb{G}_m$, p = 2: $\mathbb{Z}/2 \times (\mathbb{Z}_2^{\times} \rtimes \mathbb{Z}/2)$.
- $\Gamma = C_{ss}$, $p \geq 5$: $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}/(p^2 1)) \rtimes \mathbb{Z}/2$.
- $\Gamma = C_{ss}$, p = 3: $\mathbb{Z}/3 \times \mathbb{Z}/3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}/(3^2 1)) \rtimes \mathbb{Z}/2)$.

All other values unknown. How can we compute them? What can these say about Salch's L-functions?



Bonus slide: Lines in the $K(\Gamma)$ -local category

Theorem (Hopkins–Mahowald–Sadofsky)

A spectrum X is $K(\Gamma)$ -locally invertible if and only if $K(\Gamma)_*X$ is a $K(\Gamma)_*$ -line (i.e., dim $K(\Gamma)_*X=1$).

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Theorem (P.)

If X is a space with $K(\Gamma)^*X$ a power series ring, there is a map

$$T_+L_{K(\Gamma)}\Sigma^{\infty}X \to L_{K(\Gamma)}\Sigma^{\infty}X$$

selecting its algebro-geometric tangent space on cohomology.

E.g.:

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$$T_+\mathbb{C}\mathrm{P}^\infty \simeq \mathbb{C}\mathrm{P}^1 \simeq S^2$$
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E.g.:

- $T_+\mathbb{C}\mathrm{P}^\infty \simeq \mathbb{C}\mathrm{P}^1 \simeq S^2$.
- $T_+K(\mathbb{Q}_p/\mathbb{Z}_p,d)\simeq S^0[\det]$ for $p\gg d$.

