READING GROUP ANNOUNCEMENT
L'ISOMORPHISME ENTRE LES TOURS DE LUBIN–TATE ET DE DRINFEL'D

ORGANIZER: ERIC PETERSON

When: Wednesdays, 11:00am – 1:00pm. Where: 891 Evans Hall.

1. Some references

- **Fargues, Genestier, and Lafforgue's *L'isomorphisme entre les tours de Lubin–Tate et de Drinfel'd*.** (Out of print, but available electronically to UC–Berkeley through SpringerLink. Printing a bound copy at Vick's on the corner costs about $35 and takes 15 minutes.)
- Harris's *The local Langlands correspondence*.
- Drinfel'd's Covers of p-adic symmetric regions.
- Drinfel'd's *Elliptic modules*.
- Weinstein's The geometry of Lubin–Tate spaces.
- Harris and Taylor's *The geometry and cohomology of some simple Shimura varieties*.
- Hopkins and Gross's *Equivariant vector bundles on the Lubin–Tate moduli space*.
- Messing's *The crystals associated to Barsotti–Tate groups*,
- Rapoport and Zink's *Period spaces for p-divisible groups*.
- Fargues's Groupes analytiques rigides p-divisible.
- Faltings's A relation between two moduli studied by Drinfel'd.
- Scholze's Moduli of p-divisible groups.
- Wang's Moduli spaces of p-divisible groups and period morphisms.
- Ando's Dieudonné crystals associated to Lubin–Tate formal groups.
- Hopkins and Gross's The rigid analytic period mapping, Lubin–Tate space, and stable homotopy theory.
- Kohlhaase's On the Iwasawa theory of the Lubin–Tate moduli space.
- Boutot and Carayol's Uniformisation p-adique des courbes de Shimura.
- Morava's Toward a fundamental groupoid for the stable homotopy category.
- ...If you find a helpful reference, please let me know and I'll add it to the list.

2. Motivation

My intention is to run this as a reading group, where we do what we can without a schedule or central speaker. This is dangerously unstructured, but I think it's better than having one person speak each week. This material is very difficult, and with that difficulty comes the temptation to let the designated speaker do all the hard work alone of understanding what's happening. I would rather the group stay unified, even if this requires that we're each more self-motivated and that we go more slowly.

I should give you an initial boost, though, by taking some time to motivate you all to join me in reading this. I'll begin the semester by talking about what I know about this subject and why I'm interested in understanding it. I should warn you immediately that I'm an algebraic topologist by training, and while that requires some chameleonic abilities you should take it to mean that I'm sort of circumeducated about number theory.[1]

[1] It also means that I'm strictly interested in the mixed characteristic case discussed in Fargues's section of the book. I would be happy to skip parts II and III, though I doubt we'll make it far enough to have to make such a decision.

2.1. **Abelian class field theory.** All of this sits right at the heart of some really intriguing number theory, which has the benefit of allowing me to drop some exciting names and sound more prepared than I actually am. The story begins with local class field theory, which was a program to understand a very easy part of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$: first, the theory localizes to a single prime $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, and second (more seriously) it reduces to the abelianization $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$. This group is relevant to the study of number fields with abelian Galois groups; we denote the maximal such extension by $\mathbb{Q}_p^{\mathrm{ab}}$, so that $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$. Generally, we might consider $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ for a nonarchimedean local field K.

The fundamental observation of abelian class field theory is a correspondence between subgroups of $K^\times$ with abelian extensions of K, realized by the Artin reciprocity map

$$\mathrm{rec}_K^{\mathrm{ab}} : K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

which after passing to the completion $\widehat{\mathrm{rec}}_K^{\mathrm{ab}} : \hat{K}^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ becomes an isomorphism. This assertion of an isomorphism leaves something to be desired, though: while it gives an identification of the Galois group in reasonable terms, it does not make clear any actual construction of these intermediate field extensions.

Let's look for clues. First, a choice of uniformizing element π gives a decomposition of $\hat{K}^\times$ into the product $\mathcal{O}_K^\times \times \pi^{\hat{\mathbb{Z}}}$, and so we can understand K^{ab} in terms of the fixed fields of these two factors of the Galois group. The fixed field of $\mathcal{O}_K^\times$ is simple enough: it is the maximal unramified extension K^{ur}. The other fixed field K_π is a priori more mysterious, but the case of $K = \mathbb{Q}_p$ gives a breadcrumb: in this case, we can make the explicit identifications $K^{\mathrm{ab}} = K^{\mathrm{cycl}} = K(\zeta_n : n \in \mathbb{N})$, $K^{\mathrm{ur}} = K(\zeta_n : p \nmid n)$, and hence $K_\pi = K(\zeta_{p^j} : j \in \mathbb{N})$. It turns out that it is not an accident that the elements ζ_{p^j} appear as the $\bar{\mathbb{Q}}_p$-points of $\mathbb{G}_m[p^j]$. It is also not an accident that a similar game can be played with torsion points of CM elliptic curves to give interesting extensions of quadratic imaginary number fields.

Theorem (Lubin–Tate). *Let $\widehat{\mathbb{G}}$ be any formal $\mathcal{O}_K$-module with a coordinate x satisfying $[\pi]_{\widehat{\mathbb{G}}}(x) = x^{|k|} \pmod{\pi}$ (i.e., the Frobenius map of $\widehat{\mathbb{G}}$ lifts that of $k = \mathcal{O}_K/\pi$), and let $T_\pi\widehat{\mathbb{G}} = \lim_j \widehat{\mathbb{G}}[\pi^j]$ be its π-adic Tate module. Then:*

1. *K_π is the minimal extension of K splitting the Weierstrass polynomial of $[\pi^j]_{\widehat{\mathbb{G}}}(x)$ for all j.*
2. *Let $\rho : \mathrm{Gal}(K_\pi/K) \to \mathrm{Aut}_{\mathcal{O}_K} T_\pi\widehat{\mathbb{G}}$ be the action map. The Artin reciprocity map $\mathrm{rec}_K^{K_\pi} : K^\times \to \mathrm{Gal}(K_\pi/K)$ is determined by $\mathrm{rec}_K^{K_\pi}(\pi) = 1$ and $\mathrm{rec}_K^{K_\pi}(\alpha \in \mathcal{O}_K^\times) = \rho^{-1}(1/\alpha)$.*

In this way, π-divisible formal $\mathcal{O}_K$-modules arrive on the scene of number theory: they're essential to describing the abelian arithmetic of local number fields.

2.2. **Algebraic topology.** Chromatic homotopy theory is a subfield of stable algebraic topology centered around a theorem, due to Quillen, that there is a functor

$$\mathsf{Spectra} \to D(\mathcal{M}_{\mathrm{fg}})$$

from the category of spectra (in the sense of an algebraic topologist — loosely, spaces with the suspension operation formally inverted) to a derived category of complexes of sheaves on the moduli of commutative one-dimensional formal Lie groups. Though it is not necessary for this statement, it rapidly becomes useful to transfer to a local setting by considering p-local spectra and formal $\mathbb{Z}_p$-modules. We know many results about this functor, all subject to the calibrating maxim:

This functor is close to being an equivalence,
but just far enough to prevent the trivialization of the study of topology.

This typically means that quantitative results do not pull through — stable homotopy theory is strictly more complicated than derived categories allow — but simple qualitative results often do. There is a tool which compares the gap between both sides: the Adams spectral sequence begins with the stack cohomology $H^*(\mathcal{M}_{\mathrm{fg}}; \mathcal{X})$ and converges to $\pi_* X$. To make use of this, it is helpful to know cohomological structural facts about $\mathcal{M}_{\mathrm{fg}}$:

- There is a unique closed codimension d substack of $\mathcal{M}_{\mathrm{fg}}$ for every $0 \leq d \leq \infty$, and they are all nested. The d^{th} substack corresponds to the submoduli of formal groups of height at least d.

- There is an exhaustive sequence of geometric points of $\mathcal{M}_{\mathbf{fg}}$. The d^{th} point lies on the d^{th} filtration layer.[2]
- The points on the layers $1 \le d < \infty$ are all smooth, and the d^{th} point has formal deformation space of dimension $(d-1)$.

This analysis powers a Mayer–Vietoris decomposition of the stack cohomology, altogether assembling into a spectral sequence which produces the stack cohomology $H^*\mathcal{M}_{\mathbf{fg}}$ from the stack cohomologies of the deformation spaces of these geometric points, called Lubin–Tate spaces. So, the geometry of Lubin–Tate space has the potential to say a lot about the behavior of stable homotopy theory — something observed repeatedly over the past three decades.

2.3. **Nonabelian class field theory.** Let's investigate a fixed geometric point of the moduli for the rest of the discussion by selecting the associated formal group $\widehat{\mathbb{G}}_d$ of height d with deformation space $\mathcal{M}_{\mathbf{LT}}$ of dimension $(d-1)$. This space carries an action of three important groups:

- The automorphism group of $\widehat{\mathbb{G}}_d$, which can be identified with $\mathcal{O}_D^\times$ of a certain division algebra D.
- If we equip our formal groups with full level structures (i.e., we incorporate isomorphisms between $\mathcal{O}_K^d$ and their π-adic Tate modules), then the modified moduli $\mathcal{M}_{\mathbf{LT}}^\infty$ acquires an action of $\mathrm{GL}_d\mathcal{O}_K$.
- Finally, a Galois group also acts because $\widehat{\mathbb{G}}_d$ can be defined over k. If we pass to the rigid analytic fiber and restrict consideration to the "Weil subgroup" (i.e., the preimage of $\mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}}$ in $\mathrm{Gal}(\bar{K}/K)$), then this action commutes with the others.

Here is the fundamental conjecture about this situation, which forms the basis of the Fargues book:

"Conjecture" (Langlands, locally; now due to Harris–Taylor)**.** *Set $K = \mathbb{Q}_p$ for ease. The ℓ-adic cohomology of $\mathcal{M}_{\mathbf{LT}}^\infty$ carries a simultaneous action of these three groups.*[3] *Decomposing into a sum of simple representations gives a Langlands correspondence between nice $\mathcal{O}_D^\times$-representations, nice $\mathrm{GL}_d\mathbb{Q}_p$-representations, and nice d-dimensional Weil(–Deligne) representations.*[4] *Namely, if π is a nice irreducible $\mathrm{GL}_d\mathbb{Q}_p$-representation, then we have*

$$\mathrm{Hom}_{\mathrm{GL}_d\mathbb{Q}_p}(\pi, H_c^*(\mathcal{M}_{\mathbf{LT}}^\infty; \bar{\mathbb{Q}}_\ell)) \cong \mathrm{JL}(\pi) \otimes \widetilde{\mathrm{rec}}(\pi),$$

where $\mathrm{JL}(\pi)$ is the Jacquet–Langlands dual $\mathcal{O}_D^\times$-representation to π and $\widetilde{\mathrm{rec}}(\pi)$ specializes at $d = 1$ to pullback along the inverse to the Artin reciprocity map.

The conjecture is actually much stronger than this; for instance, the choice of K is not essential, and there is also a comparison of "L-series" and "ε-factors" for representations on all three sides of the correspondence. In any event, this is supposed to be viewed as a nonabelian class field theory: it describes some structural behavior of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ in representation theoretic terms generalizing Artin reciprocity.

2.4. **The Drinfel'd moduli.** This conjecture was actually originally made for a different space $\mathcal{M}_{\mathbf{Dr}}^\infty$. The motivation for this other object, as with many things in geometry, comes from elliptic curves: the moduli $\mathcal{M}_{\mathbf{ell},\mathbb{C}}$ of complex elliptic curves can be described as a stacky quotient of the upper half-plane by an action of $\mathrm{SL}_2(\mathbb{Z})$ by special Möbius transformations.[5] The key shift in perspective needed to get away from characteristic zero is to think of the upper half-plane as the complex projective line equipped with the Galois action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and with the fixed points removed: $\mathfrak{h} = \mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1_{\mathbb{R}}$. The following theorem generalizes this construction and matches it to the setting discussed previously:

Theorem (Drinfel'd)**.** *Define the rigid analytic space Ω_d by the complement:*

$$\Omega_d = \mathbb{P}^{d+1}_{\bar{K}} \setminus \mathbb{P}^{d+1}_K.$$

There exists a formal model $\widehat{\Omega}_d$ for Ω_d which classifies "special" formal $\mathcal{O}_K$-modules (height d, dimension d^2 formal $\mathcal{O}_D$-modules with a certain splitting into 1-dimensional submodules) up to quasi-isogeny.

[2] That is: height is a perfect invariant for formal groups over a separably closed field.

[3] Note that even defining the ℓ-adic cohomology of $\mathcal{M}_{\mathbf{LT}}^\infty$ takes Berkovitch's theory of vanishing cycles, so this is all highly nontrivial.

[4] "Nice" expands into a lengthy bunch of words in each case. In particular, "nice" forces the $\mathrm{GL}_d\mathbb{Q}_p$ representations to be ∞-dimensional.

[5] This is also the preferred place to talk about modular / automorphic forms, which certainly formed part of the motivation for Drinfel'd and Langlands, but I am largely still ignorant of this story.

The action of $\pi_1^{\text{ét}}\Omega_d$ on the universal formal $\mathscr{O}_D$-module gives rise to a representation

$$\pi_1^{\text{ét}}\Omega_d \to \operatorname{Aut}_{\mathscr{O}_D} T_\pi(\widehat{\mathbb{G}}_{/\widehat{\Omega}_d}) \cong \mathrm{GL}_1 \mathscr{O}_D = \mathscr{O}_D^\times.$$

Drinfel'd defines a sequence of covers $\mathscr{M}_{\mathbf{Dr}}^U \to \mathscr{M}_{\mathbf{Dr}}$ for $U \subseteq \mathscr{O}_D^\times$, and taking the limit of these rigid analytic spaces determines a moduli object $\mathscr{M}_{\mathbf{Dr}}^\infty$. Drinfel'd's original conjecture concerns the ℓ-adic cohomology of this space, and Fargues's climactic theorem is the following comparison:

Theorem (Théorème I.IV.13.2)**.** *Similarly, in the Lubin–Tate setting, one can define a tower of rigid analytic spaces $\mathscr{M}_{\mathbf{LT}}^V$ indexed by open subgroups $V \subseteq \mathrm{GL}_n\mathbb{Z}_p$, so that the map $\mathscr{M}_{\mathbf{LT}}^\infty \to \mathscr{M}_{\mathbf{LT}}^V$ is a cover with fiber V.*[6] *While the limiting objects $\mathscr{M}_{\mathbf{LT}}^\infty$ and $\mathscr{M}_{\mathbf{Dr}}^\infty$ do not make sense as rigid analytic spaces, there is an equivariant isomorphism of their étale pro-topoi, and this isomorphism induces an equivariant isomorphism on ℓ-adic cohomology.*

There are lots of reasons to find this statement interesting — I can tell you at least a few from the perspective of an algebraic topologist. Sufficiently deep understanding of this could point toward Langlands-type phenomena in stable homotopy theory, for one. (Already there has been fruitful work stemming from a comparison of special values of the p-adic ζ-function and the $K(1)$-local stable homotopy groups of spheres. More recently, a result was announced giving a perfect characterization of finite $E(1)$-local complexes in terms of p-adic L-functions.) The moduli $\widehat{\Omega}_d$ is constructed extremely explicitly and is quite different from what is usually considered in homotopy theory, and so perhaps it can indicate things not previously considered about the cohomology of $\mathscr{M}_{\mathbf{LT}}$. More generally, many objects associated to the geometry of Lubin–Tate space have been instantiated in homotopy theory (e.g., Gross–Hopkins duality), and the Fargues book is a treasure-trove of information about Lubin–Tate space[7], so it will be healthy to read the book regardless of the end goal.

[6] That is, these intermediate covers of $\mathscr{M}_{\mathbf{LT}}$ track partial level structures.

[7] In particular it has loads to say about period maps, which I've omitted from this discussion.