The moduli space of Riemann surfaces is Kähler hyperbolic

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1 Introduction

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n punctures.

From a complex perspective, moduli space is hyperbolic. For example, $\mathcal{M}_{g,n}$ is abundantly populated by immersed holomorphic disks of constant curvature -1 in the Teichmüller (=Kobayashi) metric.

When $r = \dim_{\mathbb{C}} \mathcal{M}_{g,n}$ is greater than one, however, $\mathcal{M}_{g,n}$ carries no complete metric of bounded negative curvature. Instead, Dehn twists give chains of subgroups $\mathbb{Z}^r \subset \pi_1(\mathcal{M}_{g,n})$ reminiscent of flats in symmetric spaces of rank $r > 1$.

In this paper we introduce a new Kähler metric on moduli space that exhibits its hyperbolic tendencies in a form compatible with higher rank.

Definitions. Let (M, g) be a Kähler manifold. An n-form α is *d(bounded)* if $\alpha = d\beta$ for some bounded $(n-1)$-form β. The space (M, g) is *Kähler hyperbolic* if:

1. On the universal cover $\widetilde{M}$, the Kähler form ω of the pulled-back metric $\widetilde{g}$ is d(bounded);

2. (M, g) is complete and of finite volume;

3. The sectional curvature of (M, g) is bounded above and below; and

4. The injectivity radius of $(\widetilde{M}, \widetilde{g})$ is bounded below.

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Note that (2-4) are automatic if M is compact.

The notion of a Kähler hyperbolic manifold was introduced by Gromov. Examples include compact Kähler manifolds of negative curvature, products of such manifolds, and finite volume quotients of Hermitian symmetric spaces with no compact or Euclidean factors [Gr].

In this paper we show:

Theorem 1.1 (Kähler hyperbolic) *The Teichmüller metric on moduli space is comparable to a Kähler metric h such that $(\mathcal{M}_{g,n}, h)$ is Kähler hyperbolic.*

The bass note of Teichmüller space. The universal cover of $\mathcal{M}_{g,n}$ is the *Teichmüller space* $\mathcal{T}_{g,n}$. Recall that the Teichmüller metric gives *norms* $\|\cdot\|_T$ on the tangent and cotangent bundles to $\mathcal{T}_{g,n}$. The analogue of the lowest eigenvalue of the Laplacian for such a metric is:

$$\lambda_0(\mathcal{T}_{g,n}) \quad = \quad \inf_{f \in C_0^\infty(\mathcal{T}_{g,n})} \int \|df\|_T^2 \, dV \Big/ \int |f|^2 \, dV,$$

where dV is the volume element of unit norm.

Corollary 1.2 *We have $\lambda_0(\mathcal{T}_{g,n}) > 0$ in the Teichmüller metric.*

Proof. The Kähler metric h is comparable to the Teichmüller metric, so it suffices to bound $\lambda_0(\mathcal{T}_{g,n}, h)$. Since the Kähler form ω for h is d(bounded), say $\omega = d\theta$, the volume form $\omega^n = d\eta = d(\theta \wedge \omega^{n-1})$ is also d(bounded). Using the Cauchy-Schwarz inequality we then obtain

$$\begin{aligned} \langle f, f \rangle \quad &= \quad \int f^2 \, \omega^n = \int f^2 \, d\eta = -\int 2f \, df \wedge \eta \\ &\leq \quad C \langle f, f \rangle^{1/2} \langle df, df \rangle^{1/2}. \end{aligned}$$

The lower bound $\langle df, df \rangle / \langle f, f \rangle \ \geq \ 1/C^2 > 0$ follows, yielding $\lambda_0 > 0$. ■

Corollary 1.3 (Complex isoperimetric inequality) *For any compact complex submanifold $N^{2k} \subset \mathcal{T}_{g,n}$, we have*

$$\mathrm{vol}_{2k}(N) \quad \leq \quad C_{g,n} \cdot \mathrm{vol}_{2k-1}(\partial N)$$

in the Teichmüller metric.

Proof. Passing to the equivalent Kähler hyperbolic metric h, Stokes' theorem yields:

$$\mathrm{vol}_{2k}(N) = \int_N \omega^k = \int_{\partial N} \theta \wedge \omega^{k-1} = O(\mathrm{vol}_{2k-1}(\partial N)),$$

since $\theta \wedge \omega^{k-1}$ is a bounded $2k-1$ form. ■

(These two corollaries also hold in the Weil-Petersson metric, since its Kähler form is d(bounded) by Theorem 1.5 below.)

The Euler characteristic. Gromov shows the Laplacian on the universal cover $\widetilde{M}$ of a Kähler hyperbolic manifold M is positive on p-forms, so long as $p \neq n = \dim_{\mathbb{C}} M$. The L^2-cohomology of $\widetilde{M}$ is therefore concentrated in the middle dimension n. Atiyah's L^2-index formula for the Euler characteristic (generalized to complete manifolds of finite volume and bounded geometry by Cheeger and Gromov [CG]) then yields

$$\operatorname{sign} \chi(M^{2n}) = (-1)^n.$$

In particular Chern's conjecture on the sign of $\chi(M)$ for closed negatively curved manifolds holds in the Kähler setting. See [Gr, §2.5A].

For moduli space we obtain:

Corollary 1.4 *The orbifold Euler characteristic of moduli space satisfies* $\chi(\mathcal{M}_{g,n}) > 0$ *if* $\dim_{\mathbb{C}} \mathcal{M}_{g,n}$ *is even, and* $\chi(\mathcal{M}_{g,n}) < 0$ *if* $\dim_{\mathbb{C}} \mathcal{M}_{g,n}$ *is odd.*

This corollary was previously known by explicit computations. For example the Harer-Zagier formula gives

$$\chi(\mathcal{M}_{g,1}) = \zeta(1 - 2g)$$

for $g > 2$, and this formula alternates sign as g increases [HZ].

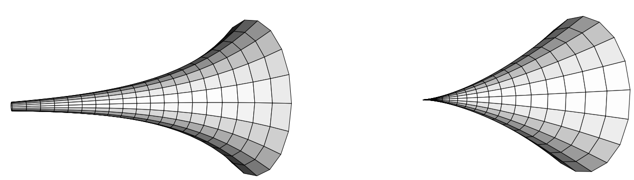

Figure 1. The cusp of moduli space in the Teichmüller and Weil-Petersson metrics.

Metrics on Teichmüller space. To discuss the Kähler hyperbolic metric $h = g_{1/\ell}$ used to prove Theorem 1.1, we begin with the Weil-Petersson and Teichmüller metrics.

Let S be a hyperbolic Riemann surface of genus g with n punctures, and let $\operatorname{Teich}(S) \cong \mathcal{T}_{g,n}$ be its Teichmüller space. The cotangent space $\mathrm{T}^*_X \operatorname{Teich}(S)$ is canonically identified with the space $Q(X)$ of holomorphic quadratic differentials $\phi(z)\, dz^2$ on $X \in \operatorname{Teich}(S)$. The Weil-Petersson and the Teichmüller metrics correspond to the norms

$$\begin{aligned} \|\phi\|^2_{\mathrm{WP}} &= \int_X \rho^{-2}(z) |\phi(z)|^2 \, |dz|^2 \qquad \text{and} \\ \|\phi\|_T &= \int_X |\phi(z)| \, |dz|^2 \end{aligned}$$

on $Q(X)$, where $\rho(z)|dz|$ is the hyperbolic metric on X. The Weil-Petersson metric is Kähler, but the Teichmüller metric is not even Riemannian when $\dim_{\mathbb{C}} \operatorname{Teich}(S) > 1$.

To compare these metrics, consider the case of punctured tori with $\mathcal{T}_{1,1} \cong \mathbb{H} \subset \mathbb{C}$. The Teichmüller metric on $\mathbb{H}$ is given by $|dz|/(2y)$, while the Weil-Petersson metric is asymptotic to $|dz|/y^{3/2}$ as $y \to \infty$. Indeed, the Weil-Petersson symplectic form is given in Fenchel-Nielsen length-twist coordinates by $\omega_{\mathrm{WP}} = d\ell \wedge d\tau$, and we have $\ell \sim 1/y$ while $\tau \sim x/y$. Compare [Mas].

The cusp of the moduli space $\mathcal{M}_{1,1} = \mathbb{H}/SL_2(\mathbb{Z})$ behaves like the surface of revolution for $y = e^x$, $x < 0$ in the Teichmüller metric; it is complete and of constant negative curvature. In Weil-Petersson geometry, on the other hand, the cusp behaves like the surface of revolution for $y = x^3$, $x > 0$. The Weil-Petersson metric on moduli space is convex but incomplete, and its curvature tends to $-\infty$ at the cusp. See Figure 1.

A quasifuchsian primitive for the Weil-Petersson form. Nevertheless the Weil-Petersson symplectic form ω_{WP} is d(bounded), and it serves as our point of departure for the construction of a Kähler hyperbolic metric. To describe a bounded primitive for ω_{WP}, recall that the *Bers embedding*

$$\beta_X : \operatorname{Teich}(\overline{S}) \to Q(X) \cong \mathrm{T}^*_X \operatorname{Teich}(S)$$

sends Teichmüller space to a bounded domain in the space of holomorphic quadratic differentials on X (§2).

Theorem 1.5 *For any fixed $Y \in \operatorname{Teich}(\overline{S})$, the 1-form*

$$\theta_{\mathrm{WP}}(X) = -\beta_X(Y)$$

is bounded in the Teichmüller and Weil-Petersson metrics, and satisfies $d(i\theta_{\mathrm{WP}}) = \omega_{\mathrm{WP}}$.

The complex projective structures on X are an affine space modeled on $Q(X)$, and we can also write

$$\theta_{\mathrm{WP}}(X) = \sigma_F(X) - \sigma_{QF}(X, Y),$$

where $\sigma_F(X)$ and $\sigma_{QF}(X, Y)$ are the Fuchsian and quasifuchsian projective structures on X (the latter coming from Bers' simultaneous uniformization of X and Y). The 1-form θ_{WP} is bounded by Nehari's estimate for the Schwarzian derivative of a univalent map (§7).

Theorem 1.5 is inspired by the formula

$$d(\sigma_F(X) - \sigma_S(X)) = -i\omega_{\mathrm{WP}} \tag{1.1}$$

discovered by Takhtajan and Zograf, where the projective structure $\sigma_S(X)$ comes from a Schottky uniformization of X [Tak, Thm. 3], [TZ]; see also [Iv1]. The proof of (1.1) by Takhtajan and Zograf leads to remarkable results on the classical problem of accessory parameters. It is based on an explicit Kähler potential for ω_{WP} coming from the *Liouville action* in string theory. Unfortunately Schottky uniformization makes the 1-form $\sigma_F(X) - \sigma_S(X)$ unbounded.

Our proof of Theorem 1.5 is quite different and invokes a new duality for Bers embeddings which we call *quasifuchsian reciprocity* (§6).

Theorem 1.6 *Given* $(X, Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$*, the derivatives of the Bers embeddings*

$$\begin{aligned} D\beta_X : \mathrm{T}_Y \mathrm{Teich}(\overline{S}) &\to \mathrm{T}^*_X \mathrm{Teich}(S) \quad \textit{and} \\ D\beta_Y : \mathrm{T}_X \mathrm{Teich}(S) &\to \mathrm{T}^*_Y \mathrm{Teich}(\overline{S}) \end{aligned}$$

are adjoint linear operators; that is, $D\beta_X^* = D\beta_Y$*.*

Using this duality, we find that $d\theta_{\mathrm{WP}}(X)$ is independent of the choice of Y. Theorem 1.5 then follows easily by setting $Y = \overline{X}$.

In the Appendix we formulate a reciprocity law for general Kleinian groups, and sketch a new proof of the Takhtajan-Zograf formula (1.1).

The $1/\ell$ metric. For any closed geodesic γ on S, let $\ell_\gamma(X)$ denote the length of the corresponding hyperbolic geodesic on $X \in \mathrm{Teich}(S)$. A sequence $X_n \in \mathcal{M}(S)$ tends to infinity if and only if $\inf_\gamma \ell_\gamma(X_n) \to 0$ [Mum]. This behavior motivates our use of the reciprocal length functions $1/\ell_\gamma$ to define a complete Kähler metric $g_{1/\ell}$ on moduli space.

To begin the definition, let $\mathrm{Log} : \mathbb{R}_+ \to [0, \infty)$ be a C^∞ function such that

$$\mathrm{Log}(x) = \begin{cases} \log(x) & \text{if } x \geq 2, \\ 0 & \text{if } x \leq 1. \end{cases}$$

The $1/\ell$ *metric* $g_{1/\ell}$ is then defined, for suitable small ϵ and δ, by its Kähler form

$$\omega_{1/\ell} \;=\; w_{\mathrm{WP}} - i\delta \sum_{\ell_\gamma(X) < \epsilon} \partial\overline{\partial} \,\mathrm{Log}\, \frac{\epsilon}{\ell_\gamma} \cdot \tag{1.2}$$

The sum above is over primitive short geodesics γ on X; at most $3|\chi(S)|/2$ terms occur in the sum.

Since $g_{1/\ell}$ is obtained by modifying the Weil-Petersson metric, it is useful to have a comparison between $\|v\|_T$ and $\|v\|_{\mathrm{WP}}$ based on short geodesics.

Theorem 1.7 *For all* $\epsilon > 0$ *sufficiently small, we have:*

$$\|v\|_T^2 \;\asymp\; \|v\|_{\mathrm{WP}}^2 + \sum_{\ell_\gamma(X) < \epsilon} |(\partial \log \ell_\gamma)(v)|^2. \tag{1.3}$$

This estimate (§5) is based on a thick-thin decomposition for quadratic differentials (§4).

Proof of Theorem 1.1. We can now outline the proof that $h = g_{1/\ell}$ is Kähler hyperbolic and comparable to the Teichmüller metric.

We begin by showing that any geodesic length function is almost pluriharmonic (§3); more precisely,

$$\|\partial\overline{\partial}(1/\ell_\gamma)\|_T = O(1).$$

This means the term $\partial\overline{\partial}\,\mathrm{Log}(\epsilon/\ell_\gamma)$ in the definition (1.2) of $\omega_{1/\ell}$ can be replaced by $(\partial\,\mathrm{Log}\,\ell_\gamma)\wedge(\overline{\partial}\,\mathrm{Log}\,\ell_\gamma)$ with small error. Using the relation between the Weil-Petersson and Teichmüller metrics given by (1.3), we then obtain the comparability estimate $g_{1/\ell}(v,v) \asymp \|v\|_T^2$. This estimate implies moduli space is complete and of finite volume in the metric $g_{1/\ell}$, because the same statements hold for the Teichmüller metric.

To show $\omega_{1/\ell}$ is d(bounded), we note that $d(i\theta_{1/\ell}) = \omega_{1/\ell}$ where

$$\theta_{1/\ell} \;=\; \theta_{\mathrm{WP}} - \delta \sum_{\ell_\gamma(X)<\epsilon} \partial\,\mathrm{Log}\,\frac{\epsilon}{\ell_\gamma}\cdot$$

The first term θ_{WP} is bounded by Theorem 1.5, and the remaining terms are bounded by basic estimates for the gradient of geodesic length.

Finally we observe that ℓ_γ and θ_{WP} can be extended to holomorphic functions on the complexification of Teich(S). Local uniform bounds on these holomorphic functions control all their derivatives, and yield the desired bounds on the curvature and injectivity radius of $g_{1/\ell}$ (§8). ■

The $1/d$ metric and domains in the plane. To conclude we mention a parallel discussion of a Kähler metric $g_{1/d}$ comparable to the hyperbolic metric g_H on a bounded domain $\Omega\subset\mathbb{C}$ with smooth boundary.

The (incomplete) Euclidean metric g_E on Ω is defined by the Kähler form

$$\omega_E \;=\; \frac{i}{2}\,dz\wedge d\overline{z}.$$

A well-known argument (based on the Koebe 1/4-theorem) gives for $v\in \mathrm{T}_z\Omega$ the estimate

$$\|v\|_H^2 \;\asymp\; \frac{\|v\|_E^2}{d(z,\partial\Omega)^2}, \tag{1.4}$$

where $d(z,\partial\Omega)$ is the Euclidean distance to the boundary [BP].

Now consider the $1/d$ *metric* $g_{1/d}$, defined for small ϵ and δ by the Kähler form

$$\omega_{1/d}(z) = \omega_E(z) \;+\; i\,\delta\,\partial\overline{\partial}\,\mathrm{Log}\,\frac{\epsilon}{d(z,\partial\Omega)}\cdot$$

We claim that for suitable ϵ and δ, the metric $g_{1/d}$ is comparable to the hyperbolic metric g_H.

Sketch of the proof. Since $\partial\Omega$ is smooth, the function $d(z) = d(z,\partial\Omega)$ is also smooth near the boundary and satisfies $\|\partial\overline{\partial}d\|_H = O(d^2)$. Thus for $\epsilon>0$ sufficiently small, $\partial\overline{\partial}\,\mathrm{Log}(\epsilon/d)$ is dominated by the gradient term $(\partial d\wedge\overline{\partial}d)/d^2$. Since $|(\partial d)(v)|$ is comparable to the Euclidean length $\|v\|_E$, by (1.4) we find $g_H \asymp g_{1/d}$. ■

Like the function $1/d(z, \partial\Omega)$, the reciprocal length functions $1/\ell_\gamma(X)$ measure the distance from X to the boundary of moduli space, rendering the metric $g_{1/\ell}$ complete and comparable to the Teichmüller (=Kobayashi) metric on $\mathcal{M}(S)$.

References. The curvature and convexity of the Weil-Petersson metric and the behavior of geodesic length-functions are discussed in [Wol1] and [Wol2]. For more on $\pi_1(\mathcal{M}_{g,n})$, its subgroups and parallels with lattices in Lie groups, see [Iv2], [Iv3]. The hyperconvexity of Teichmüller space, which is related to Kähler hyperbolicity, is established by Krushkal in [Kru].

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Notation. We use the standard notation $A = O(B)$ to mean $A \le CB$, and $A \asymp B$ to mean $A/C < B < CA$, for some constant $C > 0$. Throughout the exposition, the constant C is allowed to depend on S but it is otherwise universal. In particular, all bounds will be uniform over the entire Teichmüller space of S unless otherwise stated.

2 Teichmüller space

This section reviews basic definitions and constructions in Teichmüller theory; for further background see [Gd], [IT], [Le], [Nag].

The hyperbolic metric. A Riemann surface X is *hyperbolic* if it is covered by the upper halfplane $\mathbb{H}$. In this case the metric

$$\rho = \frac{|dz|}{\operatorname{Im} z}$$

on $\mathbb{H}$ descends to the *hyperbolic metric* on X, a complete metric of constant curvature -1.

The Teichmüller metric. Let S be a hyperbolic Riemann surface. A Riemann surface X is *marked* by S if it is equipped with a quasiconformal homeomorphism $f : S \to X$. The *Teichmüller metric* on marked surfaces is defined by

$$d((f : S \to X), (g : S \to Y)) = \frac{1}{2} \inf \log K(h),$$

where $h : X \to Y$ ranges over all quasiconformal maps isotopic to $g \circ f^{-1}$ *rel* ideal boundary, and $K(h) \ge 1$ is the dilatation of h. Two marked surfaces are *equivalent* if their Teichmüller distance is zero; then there is a conformal map $h : X \to Y$ respecting the markings. The metric space of equivalence classes is the *Teichmüller space* of S, denoted $\operatorname{Teich}(S)$.

Teichmüller space is naturally a complex manifold. To describe its tangent and cotangent spaces, let $Q(X)$ denote the Banach space of holomorphic quadratic differentials $\phi = \phi(z)\,dz^2$ on X for which the L^1-norm

$$\|\phi\|_T \;=\; \int_X |\phi|$$

is finite; and let $M(X)$ be the space of L^∞ measurable Beltrami differentials $\mu(z)\,d\overline{z}/dz$ on X. There is a natural pairing between $Q(X)$ and $M(X)$ given by

$$\langle \phi, \mu \rangle = \int_X \phi(z)\mu(z)\,dz\,d\overline{z}.$$

A vector $v \in \mathrm{T}_X \operatorname{Teich}(S)$ is represented by a Beltrami differential $\mu \in M(X)$, and its *Teichmüller norm* is given by

$$\|\mu\|_T = \sup\{\operatorname{Re}\langle \phi, \mu\rangle \;:\; \|\phi\|_T = 1\}.$$

We have the isomorphism:

$$\mathrm{T}_X \operatorname{Teich}(S) \cong Q(X)^* \cong M(X)/Q(X)^\perp,$$

and $\|\mu\|_T$ gives the infinitesimal form of the Teichmüller metric.

Projective structures. A complex *projective structure* on X is a subatlas of charts whose transition functions are Möbius transformations. The space of projective surfaces marked by S is naturally a complex manifold $\operatorname{Proj}(S) \to \operatorname{Teich}(S)$ fibering over Teichmüller space. The Fuchsian uniformization, $X = \mathbb{H}/\Gamma(X)$, determines a canonical section

$$\sigma_F : \operatorname{Teich}(S) \to \operatorname{Proj}(S).$$

This section is real analytic but *not* holomorphic.

Let $P(X)$ be the Banach space of holomorphic quadratic differentials on X with finite L^∞-norm

$$\|\phi\|_\infty = \sup_X \rho^{-2}(z)|\phi(z)|.$$

The fiber $\operatorname{Proj}_X(S)$ of $\operatorname{Proj}(S)$ over $X \in \operatorname{Teich}(S)$ is an affine space modeled on $P(X)$. That is, given $X_0 \in \operatorname{Proj}_X(S)$ and $\phi \in P(X)$, there is a unique $X_1 \in \operatorname{Proj}_X(S)$ and a conformal map $f : X_0 \to X_1$ respecting markings, such that $Sf = \phi$. Here Sf is the *Schwarzian derivative*

$$Sf(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 dz^2.$$

Writing $X_1 = X_0 + \phi$, we have $\operatorname{Proj}_X(S) = \sigma_F(X) + P(X)$.

Nehari's bound. A *univalent function* is an injective, holomorphic map $f : \mathbb{H} \to \widehat{\mathbb{C}}$. The bounds of the next result [Gd, §5.4] play a key role in proving universal bounds on the geometry of $\operatorname{Teich}(S)$.

Theorem 2.1 (Nehari) *Let Sf be the Schwarzian derivative of a holomorphic map $f : \mathbb{H} \to \widehat{\mathbb{C}}$. Then we have the implications:*

$$\|Sf\|_\infty < 1/2 \implies (f \text{ is univalent}) \implies \|Sf\|_\infty < 3/2.$$

Quasifuchsian groups. The space $QF(S)$ of marked quasifuchsian groups provides a complexification of Teich(S) that plays a crucial role in the sequel.

Let $\widehat{\mathbb{C}} = \mathbb{H} \cup \mathbb{L} \cup \mathbb{R}_\infty$ denote the partition of the Riemann sphere into the upper and lower halfplanes and the circle $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Let $S = \mathbb{H}/\Gamma(S)$ be a presentation of S as the quotient $\mathbb{H}$ by the action of a Fuchsian group $\Gamma(S) \subset PSL_2(\mathbb{R})$.

Let $\overline{S} = \mathbb{L}/\Gamma$ denote the complex conjugate of S. Any Riemann surface $X \in$ Teich(S) also has a complex conjugate $\overline{X} \in$ Teich$(\overline{S})$, admitting an anti-conformal map $\overline{X} \to X$ compatible with marking.

The *quasifuchsian space* of S is defined by

$$QF(S) = \text{Teich}(S) \times \text{Teich}(\overline{S}).$$

The map $X \mapsto (X, \overline{X})$ sends Teichmüller space to the totally real *Fuchsian subspace* $F(S) \subset QF(S)$, and thus $QF(S)$ is a complexification of Teich(S).

The space $QF(S)$ parametrizes marked quasifuchsian groups equivalent to $\Gamma(S)$, as follows. Given

$$(f : S \to X, g : \overline{S} \to Y) \in QF(S),$$

we can pull back the complex structure from $X \cup Y$ to $\mathbb{H} \cup \mathbb{L}$, solve the Beltrami equation, and obtain a quasiconformal map $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

- ϕ transports the action of $\Gamma(S)$ to the action of a Kleinian group $\Gamma(X, Y) \subset PSL_2(\mathbb{C})$;
- ϕ maps $(\mathbb{H} \cup \mathbb{L}, \mathbb{R}_\infty)$ to $(\Omega(X, Y), \Lambda(X, Y))$, where $\Lambda(X, Y)$ is a quasicircle; and
- there is an isomorphism $\Omega(X, Y)/\Gamma(X, Y) \cong X \cup Y$ such that

  $$\phi : (\mathbb{H} \cup \mathbb{L}) \to \Omega(X, Y)$$

  is a lift of $(f \cup g) : (S \cup \overline{S}) \to (X \cup Y)$.

Then $\Gamma(X, Y)$ is a *quasifuchsian group* equipped with a conjugacy ϕ to $\Gamma(S)$. Here (X, Y) determines $\Gamma(X, Y)$ up to conjugacy in $PSL_2(\mathbb{C})$, and ϕ up to isotopy *rel* $(\mathbb{R}_\infty, \Lambda(X, Y))$.[1]

There is a natural holomorphic map

$$\sigma : \text{Teich}(S) \times \text{Teich}(\overline{S}) \to \text{Proj}(S) \times \text{Proj}(\overline{S}),$$

which records the projective structures on X and Y inherited from $\Omega(X, Y) \subset \widehat{\mathbb{C}}$. We denote the two coordinates of this map by

$$\sigma(X, Y) = (\sigma_{QF}(X, Y), \overline{\sigma}_{QF}(X, Y)).$$

[1] When S has finite area, the limit set of $\Gamma(X, Y)$ coincides with $\Lambda(X, Y)$; in general it may be smaller.

The *Bers embedding* $\beta_Y : \mathrm{Teich}(S) \to P(Y)$ is given by

$$\beta_Y(X) \;=\; \overline{\sigma}_{QF}(X, Y) \,-\, \sigma_F(Y).$$

Writing $Y = \mathbb{H}/\Gamma(Y)$, we have $\beta_Y(X) = Sf$, where $f : \mathbb{H} \to \Omega(X, Y)$ is a Riemann mapping conjugating $\Gamma(Y)$ to $\Gamma(X, Y)$. Amplifying Theorem 2.1 we have:

Theorem 2.2 *The Bers embedding maps Teichmüller space to a bounded domain in $P(Y)$, with*

$$B(0, 1/2) \subset \beta_Y(\mathrm{Teich}(S)) \subset B(0, 3/2),$$

where $B(0, r)$ is the norm ball of radius r in $P(Y)$. The Teichmüller metric agrees with the Kobayashi metric on the image of β_Y.

See [Gd, §5.4, §7.5]. (This reference has different constants, because there the hyperbolic metric ρ is normalized to have curvature -4 instead of -1.)

Real and complex length. Given a hyperbolic geodesic γ on S, let $\ell_\gamma(X)$ denote the hyperbolic length of the corresponding geodesic on $X \in \mathrm{Teich}(S)$. For $(X, Y) \in QF(S)$, we can normalize coordinates on $\widehat{\mathbb{C}}$ so that the element $g \in \Gamma(X, Y)$ corresponding to γ is given by $g(z) = \lambda z$, $|\lambda| > 1$, and so that 1 and λ belong to $\Lambda(X, Y)$. By analytically continuing the logarithm from 1 to λ along $\Lambda(X, Y)$, starting with $\log(1) = 0$, we obtain the *complex length*

$$\mathcal{L}_\gamma(X, Y) = \log \lambda = L + i\theta.$$

In the hyperbolic 3-manifold $\mathbb{H}^3/\Gamma(X, Y)$, γ corresponds to a closed geodesic of length L and torsion θ.

The group $\Gamma(X, Y)$ varies holomorphically as a function of $(X, Y) \in QF(S)$, so we have:

Proposition 2.3 *The complex length $\mathcal{L}_\gamma : QF(S) \to \mathbb{C}$ is holomorphic, and satisfies $\ell_\gamma(X) = \mathcal{L}_\gamma(X, \overline{X})$.*

The Weil-Petersson metric. Now suppose S has finite hyperbolic area. The *Weil-Petersson metric* is defined on the cotangent space $Q(X) \cong T^*_X \,\mathrm{Teich}(S)$ by the L^2-norm

$$\|\phi\|^2_{\mathrm{WP}} = \int_X \rho^{-2}(z)\, |\phi|^2\, |dz|^2.$$

By duality we obtain a Riemannian metric g_{WP} on the tangent space to $\mathrm{Teich}(S)$, and in fact g_{WP} is a Kähler metric.

Proposition 2.4 *For any tangent vector v to* $\mathrm{Teich}(S)$ *we have*

$$\|v\|_{\mathrm{WP}} \le |2\pi\chi(S)|^{1/2} \cdot \|v\|_T.$$

Proof. By Cauchy-Schwarz, if $\phi \in Q(X)$ represents a cotangent vector then we have

$$\|\phi\|_T = \int_X \frac{|\phi|}{\rho^2}\rho^2 \leq \left(\int_X 1 \cdot \rho^2\right)^{1/2} \left(\int_X \frac{|\phi|^2}{\rho^4}\rho^2\right)^{1/2} = |2\pi\chi(S)|^{1/2} \cdot \|\phi\|_{\mathrm{WP}},$$

where Gauss-Bonnet determines the hyperbolic area of S. By duality the reverse inequality holds on the tangent space. ■

3 $1/\ell$ is almost pluriharmonic

In this section we begin a more detailed study of geodesic length functions and prove a universal bound on $\partial\overline{\partial}(1/\ell_\gamma)$.

The Teichmüller metric $\|v\|_T$ on tangent vectors determines a norm $\|\theta\|_T$ for n-forms on $\mathrm{Teich}(S)$ by

$$\|\theta\|_T = \sup\{|\theta(v_1, \ldots, v_n)| \ : \ \|v_i\|_T = 1\},$$

where the sup is over all $X \in \mathrm{Teich}(S)$ and all n-tuples (v_i) of unit tangent vectors at X.

Theorem 3.1 (Almost pluriharmonic) *Let $\ell_\gamma : \mathrm{Teich}(S) \to \mathbb{R}_+$ be the length function of a closed geodesic on S. Then*

$$\|\partial\overline{\partial}(1/\ell_\gamma)\|_T = O(1).$$

The bound is independent of γ and S.

We begin by discussing the case where S is an annulus and γ is its core geodesic. To simplify notation, set $\ell = \ell_\gamma$ and $\mathcal{L} = \mathcal{L}_\gamma$. Each annulus $X \in \mathrm{Teich}(S)$ can be presented as a quotient:

$$X \;\; = \;\; \mathbb{H}/\langle z \mapsto e^{\ell(X)} z\rangle.$$

The metric $|dz|/|z|$ makes X into a right cylinder of area $A = \pi\ell$ and circumference $C = \ell(X)$; the *modulus* of X is the ratio

$$\mathrm{mod}(X) = \frac{A}{C^2} = \frac{\pi}{\ell(X)} \cdot$$

Given a pair of Riemann surfaces $(X, Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$ we can glue X to Y along their ideal boundaries (which are canonically identified using the markings by S) to obtain a complex torus

$$T(X,Y) \;\; = \;\; X \cup (\partial X = \partial Y) \cup Y \cong \mathbb{C}^*/\langle e^{\mathcal{L}(X,Y)}\rangle,$$

where $\mathcal{L}(X,Y)$ is the complex length introduced in §2. This torus is simply the quotient Riemann surface for the Kleinian group

$$\Gamma(X,Y) \cong \langle z \mapsto e^{\mathcal{L}(X,Y)} z \rangle.$$

The metric $|dz|/|z|$ makes $T(X,Y)$ into a flat torus with area $A = 2\pi \operatorname{Re} \mathcal{L}$ in which ∂X is represented by a geodesic loop of length $C = |\mathcal{L}|$. We define the *modulus* of the torus by

$$\operatorname{mod}(T(X,Y)) = \frac{A}{C^2} = \operatorname{Re} \frac{2\pi}{\mathcal{L}(X,Y)}.$$

Note that $T(X,\overline{X})$ is obtained by doubling the annulus X, and $\operatorname{mod}(T(X,\overline{X})) = 2\operatorname{mod}(X)$.

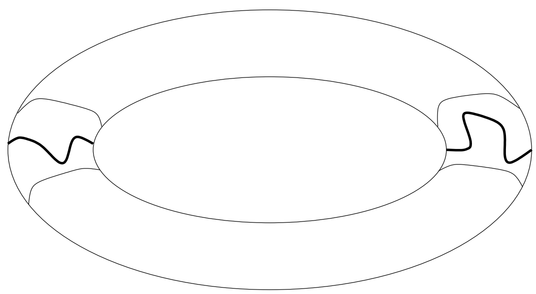

Figure 2. Two annuli joined to form the torus $T(X,Y)$.

Lemma 3.2 *If the Teichmüller distance from X to $\overline{Y}$ is bounded by* 1*, then*

$$\operatorname{mod}(T(X,Y)) = \operatorname{mod}(X) + \operatorname{mod}(Y) + O(1).$$

Proof. Since $d_T(X,\overline{Y}) \le 1$, there is a K-quasiconformal map from $T(X,\overline{X})$ to $T(X,Y)$ with $K = O(1)$. The annuli $X, Y \subset T(X,Y)$ are thus separated by a pair of K-quasicircles. A quasicircle has bounded turning [LV, §8.7], with a bound controlled by K, so we can find a pair of geodesic cylinders (with respect to the flat metric on $T(X,Y)$) such that $\partial X = \partial Y \subset A \cup B$ and $\operatorname{mod}(A) = \operatorname{mod}(B) = O(1)$. See Figure 2. (The cylinders A and B will be embedded if $\operatorname{mod}(X)$ and $\operatorname{mod}(Y)$ are large; otherwise they may be just immersed.)

The geodesic cylinders $X \cup A \cup B$ and $Y \cup A \cup B$ cover $T(X,Y)$ with bounded overlap, so their moduli sum to $\operatorname{mod}(T(X,Y)) + O(1)$. Combining this fact with monotonicity of the modulus [LV, §4.6], we have

$$\begin{aligned}\operatorname{mod}(X) + \operatorname{mod}(Y) &\le \operatorname{mod}(X \cup A \cup B) + \operatorname{mod}(Y \cup A \cup B) \\ &= \operatorname{mod}(T) + O(1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{mod}(T(X,Y)) &= \mathrm{mod}(X-A-B)+\mathrm{mod}(Y-A-B)+O(1)\\ &\leq \mathrm{mod}(X)+\mathrm{mod}(Y)+O(1),\end{aligned}$$

establishing the Theorem. ■

Proof of Theorem 3.1 (Almost pluriharmonic). We continue with the case of an annulus and its core geodesic as above. Consider $X_0 \in \mathrm{Teich}(S)$ and $v \in \mathrm{T}_{X_0}\,\mathrm{Teich}(S)$ with $\|v\|_T = 1$. Let Δ be the unit disk in $\mathbb{C}$. Using the Bers embedding of $\mathrm{Teich}(S)$ into $P(\overline{X}_0)$ and Theorem 2.2, we can find a holomorphic disk

$$\iota : (\Delta, 0) \to (\mathrm{Teich}(S), X_0),$$

tangent to v at the origin, such that the Teichmüller and Euclidean metrics are comparable on Δ, and $\mathrm{diam}_T(\iota(\Delta)) \leq 1$. (For example, we can take $\iota(s) = sv/10$ using the linear structure on $P(\overline{X}_0)$.)

Let $X_s = \iota(s)$ and $Y_t = \overline{X}_{\overline{t}} \in \mathrm{Teich}(\overline{S})$; then $(X_s, Y_t) \in QF(S)$ is a holomorphic function of $(s,t) \in \Delta^2$. Set

$$M(X,Y) = \mathrm{mod}(T(X,Y)) = \mathrm{Re}\,\frac{2\pi}{\mathcal{L}(X,Y)},$$

and define $f : \Delta^2 \to \mathbb{R}$ by

$$f(s,t) = M(X_s,Y_t) - M(X_s,Y_0) - M(X_0,Y_t) + M(X_0,Y_0).$$

By Lemma 3.2 above, we have $f(s,t) = O(1)$. On the other hand, $\mathcal{L}(X,Y)$ is holomorphic, so $f(s,t)$ is pluriharmonic. Thus the bound $f(s,t) = O(1)$ controls the full 2-jet of $f(s,t)$ at $(0,0)$; in particular we have

$$\left.\frac{\partial^2 f(s,t)}{\partial s\,\partial t}\right|_{0,0} = O(1).$$

Letting $g(s) = f(s,\overline{s})$, it follows that $(\partial\overline{\partial} g)(0) = O(1)$ in the Euclidean metric on Δ. On the other hand,

$$\partial\overline{\partial} g(s) = \partial\overline{\partial} M(X_s, \overline{X}_s) = \partial\overline{\partial}(\pi/\ell(X_s)),$$

since the remaining terms in the expression for $f(s,\overline{s})$ are pluriharmonic in s. Thus $\|\partial\overline{\partial}(1/\ell)\|_T = O(1)$, and the proof is complete for annuli.

To treat the case of general (S,γ), let $\widetilde{S} \to S$ be the annular covering space determined by $\langle\gamma\rangle \subset \pi_1(S)$, and let $\pi : \mathrm{Teich}(S) \to \mathrm{Teich}(\widetilde{S})$ be the holomorphic map obtained by lifting complex structures. Then we have:

$$\|\partial\overline{\partial}(1/\ell_\gamma)\|_T = \|\pi^*(\partial\overline{\partial}(1/\ell))\|_T \leq \|\partial\overline{\partial}(1/\ell)\|_T = O(1),$$

since holomorphic maps do not expand the Teichmüller (=Kobayashi) metric.
■

Remark. It is known that on finite-dimensional Teichmüller spaces, ℓ_γ is strictly plurisubharmonic [Wol2].

4 Thick-thin decomposition of quadratic differentials

Let S be a hyperbolic surface of finite area, and let $\phi \in Q(X)$ be a quadratic differential on $X \in \operatorname{Teich}(S)$. In this section we will present a canonical decomposition of ϕ adapted to the short geodesics γ on X.

To each γ we will associate a residue $\operatorname{Res}_\gamma : Q(X) \to \mathbb{C}$ and a differential $\phi_\gamma \in Q(X)$ proportional to $\partial \log \ell_\gamma$ with $\operatorname{Res}_\gamma(\phi_\gamma) \approx 1$. We will then show:

Theorem 4.1 (Thick-thin) *For $\epsilon > 0$ sufficiently small, any $\phi \in Q(X)$ can be uniquely expressed in the form*

$$\phi = \phi_0 + \sum_{\ell_\gamma(X)<\epsilon} a_\gamma \phi_\gamma \tag{4.1}$$

with $\operatorname{Res}_\gamma(\phi_0) = 0$ *for all γ in the sum above. Each term ϕ_0 and $a_\gamma\phi_\gamma$ has Teichmüller norm $O(\|\phi\|_T)$.*

We will also show that $\|\phi_0\|_{\mathrm{WP}} \asymp \|\phi_0\|_T$ (Theorem 4.4). Thus the thick-thin decomposition accounts for the discrepancy between the Teichmüller and Weil-Petersson norms on $Q(X)$ in terms of short geodesics on X.

The quadratic differential $\boldsymbol{\partial \log \ell_\gamma}$. Let γ be a closed hyperbolic geodesic on S. Given $X \in \operatorname{Teich}(S)$, let $\pi : X_\gamma \to X$ be the covering space corresponding to $\langle\gamma\rangle \subset \pi_1(S)$. We may identify X_γ with a round annulus

$$X_\gamma \cong A(R) = \{z \,:\, R^{-1} < |z| < R\}.$$

By requiring that $\widetilde{\gamma} \subset X_\gamma$ and S^1 agree as oriented loops, we can make this identification unique up to rotations.

Consider the natural 1-form $\theta_\gamma = dz/z$ on X_γ. In the $|\theta|$-metric, X_γ is a right cylinder of circumference $C = 2\pi$ and area $A = 4\pi \log R$. Thus we have

$$\begin{aligned} \operatorname{mod}(X_\gamma) &= \frac{A}{C^2} = \frac{\log R}{\pi} = \frac{\pi}{\ell_\gamma(X)}, \quad \text{and} \\ \|\theta^2\|_T &= A = \frac{4\pi^3}{\ell_\gamma(X)}. \end{aligned}$$

Define $\phi_\gamma \in Q(X)$ by

$$\phi_\gamma = \pi_*(\theta_\gamma^2) = \pi_*\left(\frac{dz^2}{z^2}\right).$$

The importance of ϕ_γ comes from its well-known connection to geodesic length:

$$(\partial \log \ell_\gamma)(X) = -\frac{\ell_\gamma(X)}{2\pi^3}\,\phi_\gamma \tag{4.2}$$

in $T^*_X \operatorname{Teich}(S) \cong Q(X)$ (cf. [Wol2, Thm 3.1]).

Theorem 4.2 *The differential $(\partial \log \ell_\gamma)(X)$ is proportional to ϕ_γ. We have $\|\partial \log \ell_\gamma\|_T \le 2$, and $\|\partial \log \ell_\gamma\|_T \to 2$ as $\ell_\gamma \to 0$.*

Proof. Equation (4.2) gives the proportionality and implies the bound

$$\|\partial \log \ell_\gamma\|_T = \frac{\ell_\gamma(X)}{2\pi^3}\|\phi_\gamma\|_T \le \frac{\ell_\gamma(X)}{2\pi^3}\|\theta^2\|_T = 2.$$

To analyze the behavior of $\partial \log \ell_\gamma$ when $\ell_\gamma(X)$ is small, note that the collar lemma [Bus] provides a universal $\epsilon_0 > 0$ such that for

$$T = \epsilon_0 R,$$

the map π sends $A(T) \subset A(R)$ injectively into a collar neighborhood of γ on X. Since $\int_{A(R)-A(T)} |\theta_\gamma|^2 = O(1)$, we obtain

$$\|\phi_\gamma\|_T = \int_{\pi(A(T))} |\phi_\gamma| + O(1) = \|\theta^2\|_T + O(1),$$

which implies $\|\partial \log \ell_\gamma\|_T = 2 + O(\ell_\gamma)$. ∎

The residue of a quadratic differential. Let us define the *residue* of $\phi \in Q(X)$ around γ by

$$\mathrm{Res}_\gamma(\phi) = \frac{1}{2\pi i}\int_{S^1} \frac{\pi^*(\phi)}{\theta_\gamma}\cdot$$

In terms of the Laurent expansion

$$\pi^*(\phi) = \left(\sum_{-\infty}^{\infty} a_n z^n\right)\frac{dz^2}{z^2}$$

on $A(R)$, we have $\mathrm{Res}_\gamma(\phi) = a_0$.

Proof of Theorem 4.1 (Thick-thin). To begin we will show that for any γ with $\ell_\gamma(X) < \epsilon$, we have

$$\mathrm{Res}_\gamma(\phi) = O\left(\frac{\|\phi\|_T}{\|\phi_\gamma\|_T}\right). \tag{4.3}$$

To see this, identify X_γ with $A(R)$, set $T = \epsilon_0 R$ as in the proof of Theorem 4.2, and consider the Beltrami coefficient on $A(T)$ given by

$$\mu = \frac{\overline{\theta_\gamma}^2}{|\theta_\gamma|} = \frac{z}{\overline{z}}\frac{d\overline{z}}{dz}\cdot$$

Then we have:

$$\mathrm{Res}_\gamma(\phi) = \frac{1}{4\pi \log T}\int_{A(T)} \pi^*(\phi)\,\mu = O\left((\log T)^{-1}\int_{A(T)} |\pi^*\phi|\right). \tag{4.4}$$

Since $\pi|A(T)$ is injective, we have $\int_{A(T)} |\pi^*\phi| = O(\|\phi\|_T)$ and $\log T \asymp \|\phi_\gamma\|_T$, yielding (4.3).

By similar reasoning, all γ and δ shorter than ϵ satisfy:

$$\mathrm{Res}_\gamma(\phi_\delta) = \left\{\begin{matrix} 1 & \text{if } \gamma=\delta, \\ 0 & \text{otherwise} \end{matrix}\right\} \; + \; O\left(\frac{1}{\|\phi_\gamma\|}\right). \tag{4.5}$$

Indeed, if $\delta \neq \gamma$ then most of the mass of $|\phi_\delta|$ resides in the thin part associated to δ, which is disjoint from $\pi(A(T))$. More precisely, we have $\int_{A(T)} |\pi^*\phi_\delta| = O(1)$, and the desired bound on $\mathrm{Res}_\gamma(\phi_\delta)$ follows from (4.4). The estimate when $\delta = \gamma$ is similar, using the fact that $\pi^*\phi_\gamma = \pi^*\pi_*(\theta_\gamma^2) \approx \theta_\gamma^2$ on $A(T)$.

By (4.5), the matrix $\mathrm{Res}_\gamma(\phi_\delta)$ is close to the identity when ϵ is small, since $\|\phi_\gamma\|_T^{-1} = O(\epsilon)$. Therefore we have unique coefficients a_γ satisfying equation (4.1) in the statement of the theorem.

To estimate $|a_\gamma|$, we first use the matrix equation

$$[a_\gamma] = [\mathrm{Res}_\gamma\, \phi_\delta]^{-1} [\mathrm{Res}_\delta(\phi)]$$

to obtain the bound

$$|a_\gamma| = O(\|\phi\|_T) \tag{4.6}$$

from (4.3). (Note that size of the matrix $\mathrm{Res}_\gamma\, \phi_\delta$ is controlled by the genus of X.) Then we make the more precise estimate

$$|a_\gamma| \asymp |\mathrm{Res}_\gamma(a_\gamma\phi_\gamma)| = \left| \mathrm{Res}_\gamma(\phi) - \sum_{\delta\neq\gamma} a_\delta\, \mathrm{Res}_\gamma(\phi_\delta) \right| = O(\|\phi\|_T / \|\phi_\gamma\|_T)$$

by (4.3), (4.5) and (4.6). The bound $\|a_\gamma\phi_\gamma\|_T = O(\|\phi\|_T)$ follows. ■

The bound on the terms in (4.1) above can be improved when ϕ is also associated to a short geodesic.

Theorem 4.3 *If $\phi = \phi_\delta$ with $2\epsilon > \ell_\delta(X) > \epsilon$, then we have*

$$\|a_\gamma\phi_\gamma\|_T \;\; = \;\; O(\ell_\delta(X)\, \|\phi\|_T)$$

in equation (4.1).

Proof. For any γ with $\ell_\gamma(X) < \epsilon$, the short geodesics δ and γ correspond to disjoint components $X(\delta)$ and $X(\gamma)$ of the thin part of X. The total mass of $|\phi_\delta|$ in $X(\gamma)$ is $O(1)$. Now $a_\gamma\phi_\gamma$ is chosen to cancel the residue of ϕ_δ in $X(\gamma)$, so we also have $\|a_\gamma\phi_\gamma\|_T = O(1)$. Since $\|\phi_\delta\|_T \asymp \ell_\delta(X)^{-1}$, we obtain the bound above. ■

Theorem 4.4 *If* $\mathrm{Res}_\gamma(\phi) = 0$ *for all geodesics with* $\ell_\gamma(X) < \epsilon$*, then we have*

$$\|\phi\|_{\mathrm{WP}} \le C(\epsilon)\,\|\phi\|_T.$$

Proof. Let X_r, $r = \epsilon/2$, denote the subset of X with hyperbolic injectivity radius less than r. Since the area of X is $2\pi|\chi(S)|$, the thick part $X - X_r$ can be covered by $N(r)$ balls of radius $r/2$. The L^1-norm of ϕ on a ball $B(x, r)$ controls its L^2-norm on $B(x, r/2)$, so we have:

$$\int_{X-X_r} \rho^{-2}|\phi|^2 \;=\; O(\|\phi\|_T^2).$$

It remains to control the L^2-norm of ϕ over the thin part X_r. For ϵ sufficiently small, every component of X_r is either a horoball neighborhood of a cusp or a collar neighborhood $X_r(\gamma)$ of a geodesic γ with $\ell_\gamma(X) < \epsilon$.

To bound the integral of $\rho^{-2}|\phi|^2$ over a collar $X_r(\gamma)$, identify the covering space $X_\gamma \to X$ with $A(R)$ as before, and note that (for small ϵ) we have $X_r(\gamma) \subset \pi(A(T))$ with $T = \epsilon_0 R$. Since $\pi|A(T)$ is injective we have

$$\int_{X_r(\gamma)} \rho^{-2}|\phi|^2 \le \int_{A(T)} \rho^{-2}|\pi^*\phi|^2.$$

Now because $\mathrm{Res}_\gamma(\phi) = 0$, we can use the Laurent expansion on $A(R)$ to write

$$\pi^*\phi \;=\; z f(z)\frac{dz^2}{z^2} + \frac{1}{z}\, g\left(\frac{1}{z}\right)\frac{dz^2}{z^2} \;=\; F + G,$$

where $f(z)$ and $g(z)$ are holomorphic on $\Delta(R) = \{z : |z| < R\}$. Then F and G are orthogonal in $L^2(A(R))$, so we have

$$\int_{A(T)} \rho^{-2}|\pi^*\phi|^2 = \int_{A(T)} \rho^{-2}(|F|^2 + |G|^2).$$

The inclusion $A(R) \subset \Delta^*(R)$ contracts the hyperbolic metric, so to obtain an upper bound on the integral above we can replace $\rho(z)$ with $\rho_{\Delta(R)^*}(z) = 1/|z\log(R/|z|)|$. Moreover $|f(z)|^2$ is subharmonic, so its mean over the circle of radius t is an increasing function of t. Combining these facts, we have

$$\begin{aligned}
\int_{A(T)} \rho^{-2}|F|^2 \;&=\; \int_{A(T)} \frac{|zf(z)|^2}{|z|^4\rho^2(z)}\,|dz|^2 \le \int_{\Delta(T)} |f(z)|^2\,|\log(R/|z|)|^2\,|dz|^2 \\
&=\; \int_0^T t(\log(R/t))^2 \int_0^{2\pi} |f(te^{i\theta})|^2\,d\theta\,dt \\
&\le\; 2\pi T^2 |\log\epsilon_0|^2 \sup_{S^1(T)} |f(z)|^2 = O\left(\sup_{S^1(T)} |zf(z)|^2\right).
\end{aligned}$$

Applying a similar argument to $|G|^2$, we obtain

$$\int_{A(T)} \rho^{-2}|\pi^*\phi|^2 = O\left(\sup_{S^1(T)} |zg(z)|^2 + |zf(z)|^2\right).$$

Without loss of generality we may assume $\sup_{S^1(T)} |f(z)| \geq \sup_{S^1(T)} |g(z)|$. Since $\rho \asymp |dz|/|z|$ on $S^1(T)$, we then have

$$\sup_{S^1(T)} \frac{|\pi^*\phi|}{\rho^2} \asymp \sup_{S^1(T)} |zf(z) + g(1/z)/z| \asymp \sup_{S^1(T)} |zf(z)|.$$

Now $\pi(S^1(T))$ is contained in the thick part $X - X_r$, so we may conclude that

$$\int_{X_r(\gamma)} \rho^{-2}|\phi|^2 = O\left(\sup_{X-X_r} \rho^{-4}|\phi|^2\right).$$

But the sup-norm of ϕ in the thick part is controlled by its L^1-norm, so finally we obtain

$$\int_{X_r(\gamma)} \rho^{-2}|\phi|^2 = O(\|\phi\|_T^2).$$

The bound on the L^2-norm of ϕ over the cuspidal components of the thin part X_r is similar, using the fact that ϕ has at worst simple poles at the cusps. Since the number of components of X_r is bounded in terms of $|\chi(S)|$, we obtain $\int_{X_r} \rho^{-2}|\phi|^2 = O(\|\phi\|_T^2)$, completing the proof. ∎

Remark. Masur has shown the Weil-Petersson metric extends to $\overline{\mathcal{M}}_{g,n}$, using a construction similar to the thick-thin decomposition to trivialize the cotangent bundle of $\mathcal{M}_{g,n}$ near a curve with nodes [Mas].

5 The $1/\ell$ metric

In this section we turn to the Kähler metric $g_{1/\ell}$ on Teichmüller space, and show it is comparable to the Teichmüller metric.

Recall that a positive $(1,1)$-form ω on $\mathrm{Teich}(S)$ determines a Hermitian metric $g(v,w) = \omega(v, iw)$, and g is Kähler if ω is closed. We say g is *comparable* to the Teichmüller metric if we have $\|v\|_T^2 \asymp g(v,v)$ for all v in the tangent space to $\mathrm{Teich}(S)$.

Theorem 5.1 (Kähler $\asymp$ Teichmüller) *Let S be a hyperbolic surface of finite volume. Then for all $\epsilon > 0$ sufficiently small, there is a $\delta > 0$ such that the $(1,1)$-form*

$$\omega_{1/\ell} = \omega_{\mathrm{WP}} - i\delta \sum_{\ell_\gamma(X)<\epsilon} \partial\overline{\partial} \operatorname{Log} \frac{\epsilon}{\ell_\gamma} \tag{5.1}$$

defines a Kähler metric $g_{1/\ell}$ on $\mathrm{Teich}(S)$ that is comparable to the Teichmüller metric.

Since the Teichmüller metric is complete we have:

Corollary 5.2 (Completeness) *The metric $g_{1/\ell}$ is complete.*

Notation. To present the proof of Theorem 5.1, let $N = 3|\chi(S)|/2 + 1$ be a bound on the number of terms in the expression for $\omega_{1/\ell}$, and let

$$\psi_\gamma = \partial \log \ell_\gamma = \frac{\partial \ell_\gamma}{\ell_\gamma};$$

then we have

$$|\psi_\gamma(v)|^2 = \frac{i}{2} \frac{\partial \ell_\gamma \wedge \overline{\partial} \ell_\gamma}{\ell_\gamma^2}(v, iv). \tag{5.2}$$

Lemma 5.3 *There is a Hermitian metric g of the form*

$$g(v,v) = A(\epsilon)\, \|v\|^2_{\mathrm{WP}} \; + \; B \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|^2 \tag{5.3}$$

such that $\|v\|^2_T \le g(v,v) \le O(\|v\|^2_T)$ for all $\epsilon > 0$ sufficiently small.

Proof. By Propositions 2.4 and 4.2, we have $\|v\|_{\mathrm{WP}} = O(\|v\|_T)$ and $\|\psi_\gamma(v)\| \le 2\|v\|_T$, and there are at most N terms in the sum (5.3), so $g(v,v) \le O(\|v\|^2_T)$.

To make the reverse comparison for a given $v \in \mathrm{T}_X \operatorname{Teich}(S)$, pick $\phi \in Q(X)$ with $\|\phi\|_T = 1$ and $\phi(v) = \|v\|_T$. So long as $\epsilon > 0$ is sufficiently small, we can apply the thick-thin decomposition for quadratic differentials (Theorem 4.1) to obtain

$$\phi = \phi_0 + \sum_{\ell_\gamma(X)<\epsilon} a_\gamma \psi_\gamma \tag{5.4}$$

with $\mathrm{Res}_\gamma(\phi_0) = 0$ and with $\|\psi_\gamma\|_T \ge 1$. (Recall from Theorem 4.2 that ϕ_γ and ψ_γ are proportional, and that $\|\psi_\gamma\|_T \to 2$ as $\epsilon \to 0$.)

By Theorem 4.1 each term on the right in (5.4) has Teichmüller norm $O(\|\phi\|_T) = O(1)$. Since the residues of ϕ_0 along the short geodesics vanish, the Teichmüller and Weil-Petersson norms of ϕ_0 are comparable, with a bound depending on ϵ (Theorem 4.4). Therefore we have $|\phi_0(v)| \le D(\epsilon)\|v\|_{\mathrm{WP}}$. Since we have $\|\psi_\gamma\|_T \ge 1$ and $\|a_\gamma \psi_\gamma\|_T = O(1)$, we also have $|a_\gamma| \le E$, where E is independent of ϵ. So from (5.4) we obtain

$$\phi(v) = \|v\|_T = \phi(v) \le D(\epsilon)\, \|v\|_{\mathrm{WP}} + E \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|.$$

There are at most N terms in the sum above, so we have

$$\|v\|^2_T \;\; \le \;\; N D(\epsilon)^2 \|v\|^2_{\mathrm{WP}} + N E^2 \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|^2.$$

Setting $A(\epsilon) = N D(\epsilon)^2$ and $B = N E^2$, from (5.3) we obtain $\|v\|^2_T \le g(v,v)$. ∎

Corollary 5.4 *For $\epsilon > 0$ sufficiently small, we have*

$$\|v\|_T^2 \asymp \|v\|_{\mathrm{WP}}^2 + \sum_{\ell_\gamma(X)<\epsilon} |(\partial \log \ell_\gamma)(v)|^2$$

for all vectors v in the tangent space to $\mathrm{Teich}(S)$*.*

Next we control the terms in (5.1) coming from geodesics of length near ϵ.

Lemma 5.5 *For $\epsilon < \ell_\delta(X) < 2\epsilon$ we have*

$$|\psi_\delta(v)|^2 \leq D(\epsilon)\|v\|_{\mathrm{WP}}^2 + O\left(\epsilon \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|^2\right)$$

for any tangent vector $v \in \mathrm{T}_X \,\mathrm{Teich}(S)$.

Proof. By Theorem 4.3 we have $\psi_\delta = \psi_0 + \sum_{\ell_\gamma(X)<\epsilon} a_\gamma \psi_\gamma$ with $a_\gamma = O(\ell_\delta(X)) = O(\epsilon)$, and with $\|\psi_0\|_T \leq C(\epsilon)\|\psi_\delta\|_{\mathrm{WP}}$ by Theorem 4.4. Evaluating this sum on v, we obtain the Lemma. ∎

Proof of Theorem 5.1 (Kähler $\asymp$ Teichmüller). Consider the $(1,1)$-form

$$\omega = (F(\epsilon) + A(\epsilon))\omega_{\mathrm{WP}} - B \sum_{\ell_\delta(X)<2\epsilon} \frac{i}{2}\, \partial\overline{\partial} \operatorname{Log} \frac{2\epsilon}{\ell_\delta}, \tag{5.5}$$

where

$$F(\epsilon) = 16NBD(\epsilon) \sup_{[1,2]} |\operatorname{Log}''(x)|,$$

and where $A(\epsilon)$, B and $D(\epsilon)$ come from the Lemmas above. Then ω and $\omega_{1/\ell}$ are of the same form (up to scaling and replacing ϵ with $\epsilon/2$), so to prove the Theorem it suffices to show $g(v,v) = \omega(v, iv)$ is comparable to the Teichmüller metric.

Let $v \in \mathrm{T}_X \,\mathrm{Teich}(S)$ be a vector with $\|v\|_T = 1$. To begin the evaluation of $g(v,v)$, we compute

$$\partial\overline{\partial} \operatorname{Log} \frac{2\epsilon}{\ell_\delta} = 2\epsilon \left(\operatorname{Log}' \frac{2\epsilon}{\ell_\delta}\right) \partial\overline{\partial}\left(\frac{1}{\ell_\delta}\right) + \frac{4\epsilon^2}{\ell_\delta^2}\left(\operatorname{Log}'' \frac{2\epsilon}{\ell_\delta}\right) \frac{\partial \ell_\delta \wedge \overline{\partial}\ell_\delta}{\ell_\delta^2} \cdot \tag{5.6}$$

By Theorem 3.1, the function $1/\ell_\delta$ is almost pluriharmonic; more precisely,

$$\left|\partial\overline{\partial}\left(\frac{1}{\ell_\delta}\right)(v, iv)\right| = O(1).$$

Since $\operatorname{Log}'(x)$ is bounded, the term in (5.6) involving $\partial\overline{\partial}(1/\ell_\delta)$ is $O(\epsilon)$. Using (5.2) we then obtain

$$\frac{i}{2}\left(\partial\overline{\partial} \operatorname{Log} \frac{2\epsilon}{\ell_\delta}\right)(v, iv) \;=\; \frac{4\epsilon^2}{\ell_\delta^2}\left(\operatorname{Log}'' \frac{2\epsilon}{\ell_\delta}\right) |\psi_\delta(v)|^2 + O(\epsilon). \tag{5.7}$$

Using expression (5.5) to compute $\omega(v, iv)$, we obtain a sum of terms like that above, with $\ell_\delta(X) < 2\epsilon$. If $\ell_\delta(X) < \epsilon$, then $\operatorname{Log}''(2\epsilon/\ell_\gamma) = \log''(2\epsilon/\ell_\gamma) = -\ell_\gamma^2/(4\epsilon^2)$, and hence

$$-\frac{i}{2}\left(\partial\overline{\partial}\operatorname{Log}\frac{2\epsilon}{\ell_\delta}\right)(v, iv) = |\psi_\delta(v)|^2 + O(\epsilon).$$

On the other hand, if $\epsilon \le \ell_\delta(X) < 2\epsilon$, then from (5.7) and Lemma 5.5 we obtain:

$$\begin{aligned}\left|\frac{i}{2}\left(\partial\overline{\partial}\operatorname{Log}\frac{2\epsilon}{\ell_\delta}\right)(v, iv)\right| &\le 16|\psi_\delta(v)|^2 \sup_{[1,2]} |\operatorname{Log}''(x)| + O(\epsilon)\\ &\le \frac{F(\epsilon)}{NB}\|v\|_{\mathrm{WP}}^2 + O\left(\epsilon \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|^2\right) + O(\epsilon).\end{aligned}$$

Applying these two bounds to $g(v, v) = \omega(v, iv)$, we obtain:

$$\begin{aligned}g(v, v) &\ge A(\epsilon)\|v\|_{\mathrm{WP}}^2 + B \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|^2\\ &\quad + B \sum_{\epsilon\le\ell_\delta(X)<2\epsilon} \left(\frac{F(\epsilon)}{NB}\|v\|_{\mathrm{WP}}^2 - \left|\frac{i}{2}\left(\partial\overline{\partial}\operatorname{Log}\frac{2\epsilon}{\ell_\delta}\right)(v, iv)\right|\right) + O(\epsilon)\\ &\ge A(\epsilon)\|v\|_{\mathrm{WP}}^2 + B(1 + O(\epsilon)) \sum_{\ell_\gamma(X)<\epsilon} |\psi_\gamma(v)|^2 + O(\epsilon).\end{aligned}$$

By Lemma 5.3 we then have:

$$g(v, v) \ge \|v\|_T^2 + O(\epsilon) = 1 + O(\epsilon).$$

Thus $\|v\|_T^2 = O(g(v, v))$ when ϵ is small enough. The reverse comparison, $g(v, v) = O(\|v\|_T^2)$, follows the same lines as Lemma 5.3. ■

6 Quasifuchsian reciprocity

Let S be a hyperbolic surface of finite area with quasifuchsian space $QF(S) = \operatorname{Teich}(S) \times \operatorname{Teich}(\overline{S})$. In this section we define a map

$$q : \mathrm{T}QF(S) \to \mathbb{C},$$

providing a natural bilinear pairing $q(\mu, \nu)$ on $M(X) \times M(Y)$ for each $(X, Y) \in QF(S)$. The symmetry of this pairing (explained below) will play a key role in the discussion of a bounded primitive for the Weil-Petersson metric in the next section.

To define q, recall that a small change in the conformal structure on X determines a change in the projective structure on Y, by the derivative of the Bers embedding

$$D\beta_Y : \mathrm{T}_X \operatorname{Teich}(S) \to P(Y).$$

Since S has finite area, we also have

$$P(Y) \cong Q(Y) \cong \mathrm{T}_Y^* \operatorname{Teich}(\overline{S}).$$

Thus for $(\mu, \nu) \in M(X) \times M(Y)$, we can define the *quasifuchsian pairing*

$$q(\mu, \nu) = \langle D\beta_Y(\mu), \nu \rangle$$

by evaluating the cotangent vector $D\beta_Y(\mu)$ on the tangent vector $[\nu] \in \mathrm{T}_Y \operatorname{Teich}(S)$. This pairing only depends on the equivalence class represented by (μ, ν) in $\mathrm{T}_{(X,Y)} QF(S)$.

Interchanging the roles of X and Y, we obtain a similar pairing from the Bers embedding

$$\beta_X : \operatorname{Teich}(\overline{S}) \to P(X).$$

The main result of this section is that these pairings are equal.

Theorem 6.1 (Quasifuchsian reciprocity) *For any $(\mu, \nu) \in \mathrm{T}_{(X,Y)} QF(S)$, we have*

$$q(\mu, \nu) \;=\; \int_Y (D\beta_Y(\mu)) \cdot \nu \;=\; \int_X (D\beta_X(\nu)) \cdot \mu.$$

This Theorem says q is *symmetric*, meaning $q \circ D\rho = q$, where

$$D\rho : \mathrm{T}QF(S) \to \mathrm{T}QF(S)$$

is the derivative of the involution $\rho(X, Y) = (\overline{Y}, \overline{X})$.

Proof. Recall that $1/(\pi z)$ is the fundamental solution to the $\overline{\partial}$ equation on $\widehat{\mathbb{C}}$. Thus a solution to the infinitesimal Beltrami equation $\overline{\partial} v = \mu$ is given by

$$v(z) \frac{\partial}{\partial z} \;=\; \left(\frac{1}{\pi} \int_{\widehat{\mathbb{C}}} \frac{\mu(w)}{(z - w)} \, |dw|^2 \right) \frac{\partial}{\partial z}.$$

Since $1/(\pi z)''' = -6/(\pi z^4)$, outside the support of μ the vector field v is holomorphic with infinitesimal Schwarzian derivative given by

$$\phi = v'''(z) \, dz^2 = K * \mu$$

where K is the kernel

$$K = -\frac{6}{\pi (z - w)^4} \, dz^2 \, dw^2.$$

This kernel on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ is natural and symmetric, in the sense that:

- $(\gamma \times \gamma)^* K = K$ for any Möbius transformation $\gamma \in \operatorname{Aut}(\widehat{\mathbb{C}})$; and
- $\iota^* K = K$ where $\iota(w, z) = (z, w)$.

The symmetry of q will come from the symmetry of K.

To compute the derivative of Bers' embedding β_Y, let us regard $\mu \in M(X)$ and $\phi = D\beta_Y(\mu) \in Q(Y)$ as $\Gamma(X,Y)$-invariant forms on $\Omega(X,Y)$. Then we have

$$D\beta_Y(\mu) = \phi(z)\,dz^2 = \left(-\frac{6}{\pi}\int_{\widehat{\mathbb{C}}} \frac{\mu(w)}{(z-w)^4}\,|dw|^2\right) dz^2. \tag{6.1}$$

(See [Bers], [Gd, §5.7].)

Now consider the kernel on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ given by

$$K_0 = \sum_{\gamma\in\Gamma(X,Y)} (\gamma, \mathrm{id})^* K.$$

Since K was already invariant under the diagonal action of $\mathrm{Aut}(\widehat{\mathbb{C}})$, the kernel K_0 is invariant under $\Gamma(X,Y)\times\Gamma(X,Y)$, and so it descends to a form on $X\times Y$. We then have

$$q_0(\mu,\nu) = \langle D\beta_Y(\mu), \nu\rangle = \int_{X\times Y} K_0(w,z)\mu(w)\nu(z)\,|dw|^2\,|dz|^2.$$

The reverse pairing is given similarly by

$$q_1(\mu,\nu) = \langle D\beta_X(\nu), \mu\rangle = \int_{Y\times X} K_1(w,z)\nu(w)\mu(z)\,|dw|^2\,|dz|^2,$$

where the form K_1 on $Y\times X$ is given upstairs by

$$K_1 = \sum_{\gamma\in\Gamma(X,Y)} (\mathrm{id}, \gamma)^* K.$$

By symmetry of K, $K_0 = \pi^*(K_1)$ for the natural map $\pi : X\times Y \to Y \times X$; therefore $q_0(\mu,\nu) = q_1(\mu,\nu)$, establishing reciprocity. ■

7 The Weil-Petersson form is d(bounded)

In this section we construct an explicit 1-form θ_{WP} on $\mathrm{Teich}(S)$ such that $d(i\theta_{\mathrm{WP}}) = \omega_{\mathrm{WP}}$. Using this 1-form and Nehari's bound for univalent functions, we then show the Kähler metrics corresponding to ω_{WP} and $\omega_{1/\ell}$ are both d(bounded).

Recall $\sigma_{QF}(X,Y) \in \mathrm{Proj}_X(S)$ is the projective structure on X coming from the quasifuchsian uniformization of $X \cup Y$.

Theorem 7.1 (Quasifuchsian primitive) *Fix $Y \in \mathrm{Teich}(\overline{S})$, and let θ_{WP} be the $(1,0)$-form on $\mathrm{Teich}(S)$ given by*

$$\begin{aligned}\theta_{\mathrm{WP}}(X) &= \sigma_F(X) - \sigma_{QF}(X,Y)\\ &= -\beta_X(Y).\end{aligned}$$

Then $i\theta_{\mathrm{WP}}$ is a primitive for the Weil-Petersson Kähler form; that is, $d(i\theta_{\mathrm{WP}}) = \omega_{\mathrm{WP}}$.

A key ingredient in the proof is:

Theorem 7.2 *For any* $Y_0, Y_1 \in \mathrm{Teich}(\overline{S})$*, we have*

$$d(\sigma_{QF}(X, Y_1) - \sigma_{QF}(X, Y_0)) = 0$$

as a 2-form on $\mathrm{Teich}(S)$*.*

Proof. Let Y_t be a smooth path in $\mathrm{Teich}(\overline{S})$ joining Y_0 to Y_1, and let

$$\theta_t(X) = \sigma_{QF}(X, Y_t) - \sigma_{QF}(X, Y_0).$$

We will show $\theta_1 = dF$ for an explicit function $F : \mathrm{Teich}(S) \to \mathbb{C}$.

Let $\nu_t = dY_t/dt \in \mathrm{T}_{Y_t} \mathrm{Teich}(\overline{S})$. The quasifuchsian uniformization of (X, Y_t) determines a Schwarzian quadratic differential $\beta_{Y_t}(X)$ on Y_t. Let

$$f_t(X) = \langle \sigma_{QF}(X, Y_t) - \sigma_F(Y_t), \nu_t \rangle = \langle \beta_{Y_t}(X), \nu_t \rangle.$$

We claim $df_t = \partial\theta_t/\partial t$ as 1-forms on $\mathrm{Teich}(S)$. Indeed, by quasifuchsian reciprocity, for any $\mu \in M(X)$ we have

$$df_t(\mu) = \langle D\beta_{Y_t}(\mu), \nu_t \rangle = \langle D\beta_X(\nu_t), \mu \rangle = \left\langle \frac{\partial \sigma_{QF}(X, Y_t)}{\partial t}, \mu \right\rangle = \frac{\partial \theta_t}{\partial t}(\mu)\cdot$$

Set $F(X) = \int_0^1 f_t(X)\, dt$. Then, since $\theta_0 = 0$, we have

$$\theta_1 = \int_0^1 \frac{\partial \theta_t}{\partial t}\, dt = d\int_0^1 f_t\, dt = dF,$$

and thus $d\theta_1 = d^2 F = 0$. ■

Proof of Theorem 7.1 (Quasifuchsian primitive). Let us compute the 2-form $d\theta_{\mathrm{WP}}$ at $X_0 \in \mathrm{Teich}(S)$. By the preceding result, we may freely modify the choice of Y without changing $d\theta_{\mathrm{WP}}$. Setting $Y = \overline{X}_0$ we obtain

$$\theta_{\mathrm{WP}} = \sigma_{QF}(X, \overline{X}) - \sigma_{QF}(X, \overline{X}_0).$$

Now $\sigma_{QF}(X, \overline{Y})$ is holomorphic in X and antiholomorphic in Y, so to compute the $(2,0)$ part of $d\theta_{\mathrm{WP}}$ we can replace $\overline{X}$ by $\overline{X}_0$ in $\sigma_{QF}(X, \overline{X})$; then we obtain:

$$\partial\theta_{\mathrm{WP}} = \partial(\sigma_{QF}(X, \overline{X}_0) - \sigma_{QF}(X, \overline{X}_0)) = 0.$$

As for the $(1,1)$-part, we can similarly replace X by X_0 in $-\sigma_{QF}(X, \overline{X}_0)$ to obtain:

$$\overline{\partial}\theta_{\mathrm{WP}} = \overline{\partial}(\sigma_{QF}(X_0, \overline{X}) - \sigma_{QF}(X_0, \overline{X}_0)) = \overline{\partial}\beta_{X_0}(\overline{X}).$$

To complete the proof, it suffices to show

$$(\overline{\partial}\theta_{\mathrm{WP}})(\mu, i\mu) = -i\|\mu\|^2_{\mathrm{WP}}$$

for all $[\mu] \in \mathrm{T}_{X_0} \operatorname{Teich}(S)$.

Since S has finite hyperbolic area, any tangent direction to $\operatorname{Teich}(S)$ at X_0 is represented by a harmonic Beltrami differential

$$\mu = \rho^{-2}\overline{\phi},$$

where $\phi \in Q(X_0)$ and $\|\mu\|_{\mathrm{WP}} = \|\phi\|_{\mathrm{WP}}$. Let $\overline{\mu} \in M(\overline{X}_0)$ be the corresponding conjugate vector tangent to $\operatorname{Teich}(\overline{S})$ at $\overline{X}_0$. Since $\theta_{\mathrm{WP}}(X_0) = 0$, we have

$$\begin{aligned}(\overline{\partial}\theta_{\mathrm{WP}})(\mu, i\mu) &= \langle D\beta_{X_0}(\overline{\mu}), i\mu\rangle - \langle D\beta_{X_0}(-i\overline{\mu}), \mu\rangle \\ &= 2i\,\langle D\beta_{X_0}(\overline{\mu}), \mu\rangle.\end{aligned}$$

The evaluation of $D\beta_{X_0}(\overline{\mu})$ is a standard calculation of the derivative of the Bers embedding at the origin. Namely writing $X_0 = \mathbb{H}/\Gamma_0$, we can interpret ϕ and $\overline{\mu}$ as Γ_0-invariant forms on $\mathbb{H}$ and $\mathbb{L} = -\mathbb{H}$ respectively. Then $\overline{\mu}(w) = \phi(\overline{w})$, and by (6.1) we have

$$D\beta_{X_0}(\overline{\mu}) = \psi(z)\,dz^2 = \left(-\frac{6}{\pi}\int_{\mathbb{L}} \frac{\rho^2(\overline{w})\phi(\overline{w})}{(z-w)^4}\,|dw|^2\right) dz^2.$$

A well-known reproducing formula (see [Gd, §5.7], [Bers, (5.2)]) gives $\psi = (-1/2)\phi$. Therefore we have

$$(\overline{\partial}\theta_{\mathrm{WP}})(\mu, i\mu) = 2i\,\langle(-1/2)\phi, \mu\rangle = -i\int_X \phi\rho^{-2}\overline{\phi} = -i\,\|\mu\|^2_{\mathrm{WP}},$$

completing the proof. ∎

Corollary 7.3 (d(bounded)) *The Kähler form of the $1/\ell$ metric on Teichmüller space is d(bounded), with primitive*

$$\theta_{1/\ell} \quad = \quad \theta_{\mathrm{WP}} - \delta \sum_{\ell_\gamma(X)<\epsilon} \partial \operatorname{Log} \frac{\epsilon}{\ell_\gamma} \tag{7.1}$$

satisfying $d(i\theta_{1/\ell}) = \omega_{1/\ell}$.

Proof. By (5.1) we have $\omega_{1/\ell} = d(i\theta_{1/\ell})$. To bound the first term, we note that $-\theta_{\mathrm{WP}}(X) = \sigma_{QF}(X,Y) - \sigma_F(X)$ is the Schwarzian derivative of the univalent map

$$f_{X,Y} : \mathbb{H} \to \widetilde{X} \subset \Omega(X,Y) \subset \widehat{\mathbb{C}},$$

so by Nehari's bound (Theorem 2.1) we have $\|\theta_{\mathrm{WP}}(X)\|_\infty < 3/2$. Therefore

$$\|\theta_{\mathrm{WP}}\|_T = \int_X \frac{|\phi|}{\rho^2}\rho^2 \le 3\pi|\chi(S)|, \tag{7.2}$$

since $\int \rho^2 = 2\pi|\chi(S)|$ by Gauss-Bonnet.

For the remaining terms in (7.1) we appeal to Theorem 4.2, which gives the bound $\|\partial \ell_\gamma\|_T \le 2\ell_\gamma$. The latter implies:

$$\left\| \partial \operatorname{Log} \frac{\epsilon}{\ell_\gamma} \right\|_T = \left| \frac{\epsilon}{\ell_\gamma^2} \operatorname{Log}' \frac{\epsilon}{\ell_\gamma} \right| \cdot \|\partial \ell_\gamma\|_T \le \left| \frac{\epsilon}{\ell_\gamma} \operatorname{Log}' \frac{\epsilon}{\ell_\gamma} \right| = O(1).$$

Putting these bounds together, we have $\|\theta_{1/\ell}\|_T = O(1)$. Since the $1/\ell$ metric is comparable to the Teichmüller metric (Theorem 5.1), we have shown that $\omega_{1/\ell}$ is d(bounded). ■

The L^2-version of (7.2) gives $\|\theta_{\mathrm{WP}}\|^2_{\mathrm{WP}} \le (9\pi/2)|\chi(S)|$, so we also have:

Corollary 7.4 *The Kähler form ω_{WP} is d(bounded) for the Weil-Petersson metric.*

8 Volume and curvature of moduli space

In this section we prove that $(\mathcal{M}(S), g_{1/\ell})$ has bounded geometry and finite volume, completing the proof of the Kähler hyperbolicity of moduli space.

Theorem 8.1 (Finite volume) *The metric $g_{1/\ell}$ descends to the moduli space $\mathcal{M}(S)$, and $(\mathcal{M}(S), g_{1/\ell})$ has finite volume.*

Proof. By its definition (5.1), the metric $g_{1/\ell}$ is invariant under the action of the mapping class group $\operatorname{Mod}(S)$ on $\mathcal{M}(S)$, so it descends to a metric on moduli space.

Let $n = \dim_{\mathbb{C}} \mathcal{M}(S)$, and let $\overline{\mathcal{M}(S)}$ be the Deligne-Mumford compactification of $\mathcal{M}(S)$. Consider a stable curve $Z \in \overline{\mathcal{M}(S)}$ with k nodes. Then Z has a neighborhood U in $\overline{\mathcal{M}(S)}$ satisfying

$$\begin{aligned} (U, Z) &\cong (\Delta^n, 0)/G \quad \text{and} \\ U \cap \mathcal{M}(S) &\cong ((\Delta^*)^k \times \Delta^{n-k})/G, \end{aligned}$$

where G is a finite group. (Compare [Wol3, §3].)

A small neighborhood of $(0,0)$ has finite volume in the Kobayashi metric on $(\Delta^*)^k \times \Delta^{n-k}$, and inclusions contract the Kobayashi metric, so there is a neighborhood V of Z in $\overline{\mathcal{M}(S)}$ with $\mathrm{vol}(V \cap \mathcal{M}(S)) < \infty$ in the Kobayashi = Teichmüller metric on $\mathcal{M}(S)$. By compactness of $\overline{\mathcal{M}(S)}$, the Teichmüller volume of $\mathcal{M}(S)$ is finite.

Since the Teichmüller metric is comparable to $g_{1/\ell}$, $\mathcal{M}(S)$ also has finite volume in the $1/\ell$ metric. ■

Theorem 8.2 (Bounded geometry) *The sectional curvatures of the metric $g_{1/\ell}$ are bounded above and below over* $\operatorname{Teich}(S)$*, and the injectivity radius of $g_{1/\ell}$ is uniformly bounded below.*

Proof. We use the method of the proof of Theorem 3.1: namely we realize $\mathrm{Teich}(S)$ as the locus $(X, \overline{X})$ in $QF(S)$, and extend the functions σ_{QF} and ℓ_γ used in the definition of $g_{1/\ell}$ to holomorphic functions on $QF(S)$. Uniform bounds on these extensions then give C^∞ bounds on $g_{1/\ell}$.

Pick $X_0 \in \mathrm{Teich}(S)$, and let $n = \dim_{\mathbb{C}} \mathrm{Teich}(S)$. By an elementary result in convex geometry, there is an isomorphism $A : \mathbb{C}^n \to P(\overline{X}_0)$ such that $\|Az\|_\infty \asymp |z|$ with constants depending only on n. Using the Bers embedding of $\mathrm{Teich}(S)$ into $P(\overline{X}_0)$ and Theorem 2.2, we can find an embedded polydisk

$$f : (\Delta^n, 0) \to (\mathrm{Teich}(S), X_0),$$

such that the Teichmüller and Euclidean metrics are comparable on Δ^n. (Indeed we can take $f(z) = A(\alpha z)$ for a suitable $\alpha > 0$.)

Let $X_s = f(s) \in \mathrm{Teich}(S)$ and $Y_t = \overline{X}_{\overline{t}} \in \mathrm{Teich}(\overline{S})$; then $(X_s, Y_t) \in QF(S)$ is a holomorphic function of $(s, t) \in \Delta^n \times \Delta^n$. For any closed geodesic γ on S, the complex length satisfies $\operatorname{Re} \mathcal{L}_\gamma(X_s, Y_t) > 0$, so the holomorphic function

$$L_\gamma(s, t) = \log \mathcal{L}_\gamma(X_s, Y_t)$$

maps $\Delta^n \times \Delta^n$ into the strip $\{z \,:\, |\operatorname{Im} z| < \pi\}$. By the Schwarz lemma, all the derivatives of $L_\gamma(s, t)$ at $(0, 0)$ of order $\leq k$ are bounded by a constant C_k in the Euclidean metric. Therefore the derivatives of

$$\log \ell_\gamma(X_s) = \log \mathcal{L}_\gamma(X_s, \overline{X}_s)$$

are also controlled at $s = 0$.

Fixing $Z \in \mathrm{Teich}(\overline{S})$, the holomorphic 1-form

$$\tau(X, Y) = \sigma_{QF}(X, Y) - \sigma_{QF}(X, Z)$$

is bounded in the Teichmüller metric on $QF(S)$ (by Nehari's bound, Theorem 2.1). Hence $f^*\tau$ is bounded in the Euclidean metric, and thus the derivatives of

$$(f^*\tau)(s) = f^*(\sigma_F(X_s) - \sigma_{QF}(X_s, Z))$$

at $s = 0$ are also uniformly controlled.

Now by Corollary 7.3, if we set

$$\theta \;=\; f^*(\sigma_F(X_s) - \sigma_{QF}(X_s, Z)) \;-\; \delta \sum_{\ell_\gamma(X) < \epsilon} \partial \operatorname{Log} \frac{\epsilon}{\ell_\gamma(X_s)},$$

then the 1-form $i\theta$ is a primitive for the Kähler form of $g = f^* g_{1/\ell}$ on Δ^n. Since the derivatives of each term are controlled, and g is comparable to the Euclidean metric, we find that the sectional curvatures of g are bounded above and below at $s = 0$. (Indeed g ranges in a compact family of metrics in the C^∞ topology on Δ^n.) Thus the curvatures of $g_{1/\ell}$ are $O(1)$ at X_0, and hence uniformly bounded over $\mathrm{Teich}(S)$.

We then have curvature bounds on g throughout Δ^n, from which it follows that the injectivity radius of g at 0 is bounded below. Since f is one-to-one, the injectivity radius of $g_{1/\ell}$ on $\mathrm{Teich}(S)$ is also bounded below. ∎

Proof of Theorem 1.1 (Kähler hyperbolic). The metric $h = g_{1/\ell}$ is comparable to the Teichmüller metric by Theorem 5.1, and therefore $g_{1/\ell}$ is complete (Corollary 5.2). Its symplectic form $\omega_{1/\ell}$ is d(bounded) by Corollary 7.3. In this section we have shown that $(\mathcal{M}(S), g_{1/\ell})$ has finite volume and bounded geometry, and therefore moduli space is Kähler hyperbolic in the $g_{1/\ell}$ metric. ■

9 Appendix: Reciprocity for Kleinian groups

This appendix formulates a version of quasifuchsian reciprocity for general Kleinian groups. As an application, we sketch a proof of the Takhtajan-Zograf formula

$$d(\sigma_F(X) - \sigma_S(X)) = -i\omega_{\mathrm{WP}}.$$

Kleinian groups. Let $\Gamma \subset \mathrm{Aut}(\widehat{\mathbb{C}})$ be a finitely generated Kleinian group, with domain of discontinuity Ω and (possibly disconnected) quotient Riemann surface $X = \Omega/\Gamma$. By Ahlfors' finiteness theorem, X has finite hyperbolic area.

Let $\mu \in M(X)$ be a Beltrami differential, regarded as a Γ-invariant form on $\widehat{\mathbb{C}}$, vanishing outside Ω. Let v be a quasiconformal vector field on the sphere with $\overline{\partial} v = \mu$. Then the infinitesimal Schwarzian derivative

$$\phi_\mu = \frac{\partial^3 v}{\partial z^3}\, dz^2$$

is a Γ-invariant quadratic differential, holomorphic *outside* the support of μ.

On the support of μ, the third derivative of v exists only as a distribution. If, however, μ is sufficiently smooth — for example, if μ is a harmonic Beltrami differential ($\mu = \rho^{-2}\overline{\phi}$, $\phi \in Q(X)$) — then ϕ_μ is smooth on Ω and descends to a quadratic differential

$$K(\mu) = \phi_\mu \in L^1(X, dz^2).$$

We refer to ϕ_μ as the *projective distortion* of μ.

Theorem 9.1 (Kleinian reciprocity) *Let $X = \Omega/\Gamma$ be the quotient Riemann surface for a finitely generated Kleinian group Γ, and let $\mu, \nu \in M(X)$ be a pair of sufficiently smooth Beltrami differentials. Then we have:*

$$\int_X \phi_\mu\, \nu = \int_X \phi_\nu\, \mu,$$

where $\phi_\mu, \phi_\nu \in L^1(X, dz^2)$ give the projective distortions of μ and ν.

Note that if Γ is quasifuchsian, with $X = X_1 \sqcup X_2$, then by taking $\mu \in M(X_1)$ and $\nu \in M(X_2)$ we recover Theorem 6.1 (Quasifuchsian reciprocity). The proof of the general version is essentially the same; it turns on the symmetry of the kernel $K = -6\, dz^2\, dw^2/(\pi(z-w)^4)$.

Schottky uniformization. As an application, let S be a closed surface of genus $g \geq 2$, and fix a maximal collection $(a_1, \ldots, a_g)$ of disjoint simple closed curves on S, linearly independent in $H_1(S, \mathbb{R})$. Then to each $X \in \mathrm{Teich}(S)$ one can associate a *Schottky group* Γ_S, such that $X = \Omega_S/\Gamma_S$ and the curves a_i lift to Ω_S. Let $\sigma_S(X)$ be the projective structure on X inherited from Ω_S.

Theorem 9.2 (Takhtajan-Zograf) *The difference between the Fuchsian and Schottky projective structures gives a 1-form on* $\mathrm{Teich}(S)$ *satisfying:*

$$d(\sigma_F(X) - \sigma_S(X)) = -i\omega_{\mathrm{WP}}.$$

Sketch of the proof. Since σ_S and σ_{QF} are holomorphic, we have $\overline{\partial}(\sigma_F - \sigma_S) = \overline{\partial}(\sigma_F - \sigma_{QF}) = -i\omega_{\mathrm{WP}}$ (Theorem 7.1). Thus we just need to check that the (2,0)-form $\partial(\sigma_F - \sigma_S)$ on $\mathrm{Teich}(S)$ vanishes. To this end, let us represent the tangent space to $\mathrm{Teich}(S)$ at X by the space $\mathcal{H}(X)$ of harmonic Beltrami differentials. Using the Fuchsian and Schottky representations of X as a quotient Riemann surface for Kleinian groups Γ_F and Γ_S, we obtain operators

$$K_F, K_S : \mathcal{H}(X) \to L^1(X, dz^2)$$

sending $\mu \in \mathcal{H}(X)$ to its projective distortion ϕ_μ. (In the Fuchsian case, $\Omega_F/\Gamma_F = X \sqcup \overline{X}$; we set $\mu = 0$ on $\overline{X}$.)

We then compute:

$$\partial(\sigma_F - \sigma_S)(\mu, \nu) = \langle K_F(\mu) - K_S(\mu), \nu\rangle - \langle K_F(\nu) - K_S(\nu), \mu\rangle,$$

where $\langle \phi, \mu\rangle = \int_X \phi\, \mu$. The key to this computation is to observe that the usual chain rule for the Schwarzian derivative, $S(g \circ f) = Sf + f^* Sg$, holds even when just one of f and g is conformal (assuming the other is sufficiently smooth).

By reciprocity, the bracketed expressions above agree, so $\partial(\sigma_F - \sigma_S) = 0$.
■

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