

# Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations

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## Abstract

Let  $F \subset H^1(M^3, \mathbb{R})$  be a fibered face of the Thurston norm ball for a hyperbolic 3-manifold  $M$ .

Any  $\phi \in \mathbb{R}_+ \cdot F$  determines a measured foliation  $\mathcal{F}$  of  $M$ . Generalizing the case of Teichmüller geodesics and fibrations, we show  $\mathcal{F}$  carries a canonical *Riemann surface* structure on its leaves, and a transverse *Teichmüller flow* with pseudo-Anosov expansion factor  $K(\phi) > 1$ .

We introduce a polynomial invariant  $\Theta_F \in \mathbb{Z}[H_1(M, \mathbb{Z})/\text{torsion}]$  whose roots determine  $K(\phi)$ . The Newton polygon of  $\Theta_F$  allows one to compute fibered faces in practice, as we illustrate for closed braids in  $S^3$ . Using fibrations we also obtain a simple proof that the shortest geodesic on moduli space  $\mathcal{M}_g$  has length  $O(1/g)$ .

## Resumé

Soit  $M$  une variété hyperbolique de dimension 3, et  $F \subset H^1(M^3, \mathbb{R})$  une face fibrée de la boule unité dans la norme de Thurston.

Chaque  $\phi \in \mathbb{R}_+ \cdot F$  détermine un feuilletage mesuré  $\mathcal{F}$  de  $M$ . Généralisant le cas des géodésiques de Teichmüller et des fibrations, nous démontrons que  $\mathcal{F}$  porte une structure complexe canonique sur les feuilles, et admet un *flot transverse de Teichmüller*, avec facteur d'expansion pseudo-Anosov  $K(\phi) > 1$ .

Nous introduisons un invariant polynômial  $\Theta_F \in \mathbb{Z}[H_1(M, \mathbb{Z})/\text{torsion}]$ , dont les racines déterminent  $K(\phi)$ . Le polygone de Newton de  $\Theta_F$  permet le calcul pratique des faces fibrées, comme nous illustrons pour les tresses fermées dans  $S^3$ . Nous obtenons aussi, utilisant les fibrations, une preuve simple que la géodésique la plus courte sur l'espace de modules  $\mathcal{M}_g$  est de longueur  $O(1/g)$ .

# 1 Introduction

Every fibration of a 3-manifold  $M$  over the circle determines a closed loop in the moduli space of Riemann surfaces. In this paper we introduce a polynomial invariant for  $M$  that packages the Teichmüller lengths of these loops, and we extend the theory of Teichmüller geodesics from fibrations to measured foliations.

**Riemann surfaces and fibered 3-manifolds.** Let  $M$  be a compact oriented 3-manifold, possibly with boundary. Suppose  $M$  fibers over the circle  $S^1 = \mathbb{R}/\mathbb{Z}$ , with fiber  $S$  and pseudo-Anosov monodromy  $\psi : S \rightarrow S$ :

$$\begin{array}{ccc} \psi \circ S & \longrightarrow & M \\ & & \downarrow \pi \\ & & S^1. \end{array}$$

Then there is:

- a natural complex structure  $J_s$  along the fibers  $S_s = \pi^{-1}(s)$ , and
- a flow  $f : M \times \mathbb{R} \rightarrow M$ , circulating the fibers at unit speed,

such that the conformal distortion of  $f$  is minimized.

Indeed, the mapping-class  $\psi$  determines a loop in the moduli space of complex structures on  $S$ , represented by a unique Teichmüller geodesic

$$\gamma : S^1 \rightarrow \mathcal{M}_{g,n}.$$

The complex structure on the fibers is given by  $(S_s, J_s) = \gamma(s)$ . The time  $t$  map of the flow  $f$  is determined by the condition that on each fiber,  $f_t : (S_s, J_s) \rightarrow (S_{s+t}, J_{s+t})$  is a Teichmüller mapping. Outside a finite subset of  $S_s$ ,  $f_t$  is locally an affine stretch of the form

$$f_t(x + iy) = K^t x + iK^{-t} y, \tag{1.1}$$

where  $K > 1$  is the *expansion factor* of the monodromy  $\psi$ . The Teichmüller length of the loop  $\gamma$  in moduli space is  $\log K$ .

This well-known interplay between topology and complex analysis was developed by Teichmüller, Thurston and Bers (see [Bers]). The fibration  $\pi$ , the resulting geometric structure on  $M$  and the expansion factor  $K$  are all determined (up to isotopy) by the cohomology class  $\phi = [S] \in H^1(M, \mathbb{R})$ .

**Fibered faces.** In this paper we extend the theory of Teichmüller geodesics from fibrations to measured foliations.

The Thurston norm  $\|\phi\|_T$  on  $H^1(M, \mathbb{R})$  leads to a coherent picture of all the cohomology classes represented by fibrations and measured foliations of  $M$ . To describe this picture, we begin by defining the Thurston norm, which is a generalization of the genus of a knot; it measures the minimal complexity of an embedded surface in a given cohomology class. For an integral cohomology class  $\phi$ , the norm is given by:

$$\|\phi\|_T = \inf\{|\chi(S_0)| : (S, \partial S) \subset (M, \partial M) \text{ is dual to } \phi\},$$

where  $S_0 \subset S$  excludes any  $S^2$  or  $D^2$  components of  $S$ . The Thurston norm is extended to real classes by homogeneity and continuity. The unit ball of the Thurston norm is a polyhedron with rational vertices.

An embedded, oriented surface  $S \subset M$  is a *fiber* if it is the preimage of a point under a fibration  $M \rightarrow S^1$ . Any fiber minimizes  $|\chi(S)|$  in its cohomology class. Moreover,  $[S]$  belongs to the cone  $\mathbb{R}_+ \cdot F$  over an open *fibred face*  $F$  of the unit ball in the Thurston norm. Every integral class in  $\mathbb{R}_+ \cdot F$  is realized by a fibration  $M^3 \rightarrow S^1$ ; more generally, every real cohomology class  $\phi \in \mathbb{R}_+ \cdot F$  is represented by a *measured foliation*  $\mathcal{F}$  of  $M$ . Such a foliation is determined by a closed, nowhere-vanishing 1-form  $\omega$  on  $M$ , with  $T\mathcal{F} = \text{Ker } \omega$  and with measure

$$\mu(T) = \left| \int_T \omega \right|$$

for any connected transversal  $T$  to  $\mathcal{F}$ . For an integral class, the leaves of  $\mathcal{F}$  are closed and come from a fibration  $\pi : M \rightarrow S^1$  with  $\omega = \pi^*(dt)$ .

Generalizing the case of fibrations, we will show (§9):

**Theorem 1.1** *For any measured foliation  $\mathcal{F}$  of  $M$ , there is a complex structure  $J$  on the leaves of  $\mathcal{F}$ , a unit speed flow*

$$f : (M, \mathcal{F}) \times \mathbb{R} \rightarrow (M, \mathcal{F}),$$

*and a  $K > 1$ , such that  $f_t$  maps leaves to leaves by Teichmüller mappings with expansion factor  $K^{|t|}$ .*

*The foliation  $\mathcal{F}$ , the complex structure  $J$  along its leaves, the transverse flow  $f$  and the stretch factor  $K$  are all determined up to isotopy by the cohomology class  $[\mathcal{F}] \in H^1(M, \mathbb{R})$ .*

Here  $f$  has *unit speed* if it is generated by a vector field  $v$  with  $\omega(v) = 1$ , where  $\omega$  is the defining 1-form of  $\mathcal{F}$ . The complex structure  $J$  makes each leaf  $\mathcal{F}_\alpha$  of  $\mathcal{F}$  into a Riemann surface, and

$$f_t : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$$

is a *Teichmüller mapping* with *expansion factor*  $K$  if

$$\mu(f_t) = \frac{\bar{\partial} f_t}{\partial f_t} = \left( \frac{K^2 - 1}{K^2 + 1} \right) \frac{\bar{q}}{|q|}$$

for some holomorphic quadratic differential  $q(z) dz^2$  on  $\mathcal{F}_\alpha$ . Away from the zeros of  $q$ , such a mapping has the form of an affine stretch as in (1.1).

**Quantum geodesics.** Theorem 1.1 provides, for a general measured foliation  $\mathcal{F}$  with typical leaf  $S$ , a ‘quantum geodesic’

$$\gamma : \mathbb{R}/H_1(M, \mathbb{Z}) \rightarrow \text{Teich}(S)/H_1(M, \mathbb{Z}).$$

Here  $H_1(M, \mathbb{Z})$  acts on  $\mathbb{R}$  by translation by the periods  $\Pi$  of  $\omega$ , and on  $\text{Teich}(S)$  by monodromy around loops in  $M$ . Generically  $\Pi$  is a dense subgroup of  $\mathbb{R}$ , in which case  $\mathbb{R}/\Pi$  and  $\text{Teich}(S)/H_1(M, \mathbb{Z})$  are ‘quantum spaces’ in the sense of Connes [Con]. The map  $\gamma$  plays the role of a closed Teichmüller geodesic for the virtual mapping class determined by  $\mathcal{F}$ .

**The Teichmüller polynomial.** Next we introduce a polynomial invariant  $\Theta_F$  for a fibered face  $F \subset H^1(M, \mathbb{R})$ . This polynomial determines the Teichmüller expansion factors  $K(\phi)$  for all  $\phi = [\mathcal{F}] \in \mathbb{R}_+ \cdot F$ .

Like the Alexander polynomial,  $\Theta_F$  naturally resides in the group ring  $\mathbb{Z}[G]$ , where  $G = H_1(M, \mathbb{Z})/\text{torsion}$ . Observe that  $\mathbb{Z}[G]$  can be thought of as a ring of complex-valued functions on the character variety  $\widehat{G} = \text{Hom}(G, \mathbb{C}^*)$ , with

$$\left( \sum a_g \cdot g \right) (\rho) = \sum a_g \rho(g).$$

To define  $\Theta_F$ , we first show  $F$  determines a 2-dimensional lamination  $\mathcal{L} \subset M$ , transverse to every fiber  $[S] \in \mathbb{R}_+ \cdot F$  and with  $S \cap \mathcal{L}$  equal to the expanding lamination for the monodromy  $\psi : S \rightarrow S$ . Next we define, for every character  $\rho \in \widehat{G}$ , a group of twisted cycles  $Z_2(\mathcal{L}, \mathbb{C}_\rho)$ . Here a *cycle*  $\mu$  is simply an additive, holonomy-invariant function  $\mu(T)$  on compact, open transversals  $T$  to  $\mathcal{L}$ , with values in the complex line bundle specified by  $\rho$ .

The *Teichmüller polynomial*  $\Theta_F \in \mathbb{Z}[G]$  defines the largest hypersurface  $V \subset \widehat{G}$  such that

$$\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0 \quad \text{for all } \rho \in V. \tag{1.2}$$

More precisely, we associate to  $\mathcal{L}$  a module  $T(\widetilde{\mathcal{L}})$  over  $\mathbb{Z}[G]$ , and  $(\Theta_F)$  is the smallest principal ideal containing all the minor determinants in a presentation matrix for  $T(\widetilde{\mathcal{L}})$ . Thus  $\Theta_F$  is well-defined up to multiplication by a unit  $\pm g \in \mathbb{Z}[G]$ .

**Information packaged in  $\Theta_F$ .** Let  $\Theta_F = \sum a_g \cdot g$  be the Teichmüller polynomial of a fibered face  $F$  of the Thurston norm ball in  $H^1(M, \mathbb{R})$ . In §3 – §6 we will show:

1. *The Teichmüller polynomial is symmetric; that is,  $\Theta_F = \sum a_g \cdot g^{-1}$  up to a unit in  $\mathbb{Z}[G]$ .*
2. *For any fiber  $[S] = \phi \in \mathbb{R}_+ \cdot F$ , the expansion factor  $k = K(\phi)$  of its monodromy  $\psi$  is the largest root of the polynomial equation*

$$\Theta_F(k^\phi) = \sum a_g k^{\phi(g)} = 0. \quad (1.3)$$

3. *Equation (1.3) also determines the expansion factor for any measured foliation  $[\mathcal{F}] = \phi \in \mathbb{R}_+ \cdot F$ .*
4. *The function  $1/\log K(\phi)$  is real-analytic and strictly concave on  $\mathbb{R}_+ \cdot F$ .*
5. *The cone  $\mathbb{R}_+ \cdot F$  is dual to a vertex of the Newton polygon*

$$N(\Theta_F) = (\text{the convex hull of } \{g : a_g \neq 0\}) \subset H_1(M, \mathbb{R}).$$

To see the relation of  $\Theta_F$  to expansion factors, note that a fibration  $M \rightarrow S^1$  with fiber  $S$  determines a measured lamination  $(\lambda, \mu_0) \in \mathcal{ML}(S)$ , such that the transverse measure  $\mu_0$  on  $\lambda$  is expanded by a factor  $K > 1$  under monodromy. Thus the suspension of  $\mu_0$  gives a cycle  $\mu \in Z_2(\mathcal{L}, \mathbb{C}_\rho)$  with character

$$\rho(\gamma) = K^{[S] \cdot [\gamma]}$$

for loops  $\gamma \subset M$ . Therefore  $\Theta_F(\rho) = 0$  (as in (1.2) above), and thus  $K$  can be recovered from the zeros of  $\Theta_F$ .

The relation between  $F$  and the Newton polygon of  $\Theta_F$  ((5) above) comes from the fact that  $K(\phi) \rightarrow \infty$  as  $\phi \rightarrow \partial F$ .

**A formula for  $\Theta_F(\mathbf{t}, \mathbf{u})$ .** One can also approach the Teichmüller polynomial from a 2-dimensional perspective. Let  $\psi : S \rightarrow S$  be a pseudo-Anosov mapping, and let  $(t_1, \dots, t_b)$  be a multiplicative basis for

$$H = \text{Hom}(H^1(S, \mathbb{Z})^\psi, \mathbb{Z}) \cong \mathbb{Z}^b,$$

where  $H^1(S, \mathbb{Z})^\psi$  is the  $\psi$ -invariant cohomology of  $S$ . (When  $\psi$  acts trivially on cohomology, we can identify  $H$  with  $H_1(S, \mathbb{Z})$ .) By evaluating cohomology classes on loops, we obtain a natural map  $\pi_1(S) \rightarrow H$ . Choose a lift

$$\tilde{\psi} : \tilde{S} \rightarrow \tilde{S}$$

of  $\psi$  to the  $H$ -covering space of  $S$ .

Let  $M = S \times [0, 1] / \langle (x, 1) \sim (\psi(x), 0) \rangle$  be the mapping torus of  $\psi$ , let

$$G = H_1(M, \mathbb{Z}) / \text{torsion} \cong H \oplus \mathbb{Z},$$

and let  $F \subset H^1(M, \mathbb{R})$  be the fibered face with  $[S] \in \mathbb{R}_+ \cdot F$ . Then we can regard  $\Theta_F$  as a Laurent polynomial

$$\Theta_F(t, u) \in \mathbb{Z}[G] = \mathbb{Z}[H] \oplus \mathbb{Z}[u] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_b^{\pm 1}, u^{\pm 1}],$$

where  $u$  corresponds to  $[\tilde{\psi}]$ .

To give a concrete expression for  $\Theta_F$ , let  $E$  and  $V$  denote the edges and vertices of an invariant train track  $\tau \subset S$  carrying the expanding lamination of  $\psi$ . Then  $\tilde{\psi}$  acts by matrices  $P_E(t)$  and  $P_V(t)$  on the free  $\mathbb{Z}[H]$ -modules generated by the lifts of  $E$  and  $V$  to  $\tilde{S}$ . In terms of this action we show (§3):

6. *The Teichmüller polynomial is given by*

$$\Theta_F(t, u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))}.$$

Using this formula, many of the properties of  $\Theta_F$  follow from the theory of Perron-Frobenius matrices over a ring of Laurent polynomials, developed in the Appendix.

**Fixed-points on  $\mathbb{P}\mathcal{ML}_s(S)$ .** Let  $\mathcal{ML}_s(S)$  denote the space of measured laminations  $\Lambda = (\lambda, \mu)$  on  $S$  twisted by  $s \in H^1(S, \mathbb{R})$ , meaning  $\mu$  transforms by  $e^{s(\gamma)}$  under  $\gamma \in \pi_1(S)$ .

The mapping-class  $\psi$  acts on  $\mathcal{ML}_s(S)$  for all  $s \in H^1(S, \mathbb{R})^\psi$ , once we have chosen the lift  $\tilde{\psi}$ . As in the untwisted case,  $\psi$  has a unique pair of fixed-points  $[\Lambda_\pm]$  in  $\mathbb{P}\mathcal{ML}_s(S)$ , whose supports  $\lambda_\pm$  are independent of  $s$ . In §8 we show:

7. *The eigenvector  $\Lambda_+ \in \mathcal{ML}_s(S)$  satisfies*

$$\psi \cdot \Lambda_+ = k(s)\Lambda_+$$

where  $u = k(s) > 0$  is the largest root of the polynomial  $\Theta_F(e^s, u) = 0$ .  
The function  $\log k(s)$  is convex on  $H^1(S, \mathbb{R})^\psi$ .

**Short geodesics on moduli space.** It is known that the shortest geodesic loop on moduli space  $\mathcal{M}_g$  has Teichmüller length  $L(\mathcal{M}_g) \asymp 1/g$  (see [Pen2]).

In §10 we show mapping-classes with invariant cohomology provide a natural source of such short geodesics.

More precisely, let  $\psi : S \rightarrow S$  be a pseudo-Anosov mapping on a closed surface of genus  $g \geq 2$ , leaving invariant a primitive cohomology class

$$\xi_0 : \pi_1(S) \rightarrow \mathbb{Z}.$$

Let  $\tilde{S} \rightarrow S$  be the corresponding  $\mathbb{Z}$ -covering space, with deck group generated by  $h : \tilde{S} \rightarrow \tilde{S}$ , and fix a lift  $\tilde{\psi}$  of  $\psi$  to  $\tilde{S}$ . Then for all  $n \gg 0$ , the surface  $R_n = \tilde{S}/\langle h^n \tilde{\psi} \rangle$  has genus  $g_n \asymp n$ , and  $h : \tilde{S} \rightarrow \tilde{S}$  descends to a pseudo-Anosov mapping-class  $\psi_n : R_n \rightarrow R_n$ .

This renormalization construction gives mappings  $\psi_n$  with expansion factors satisfying

$$K(\psi_n) = K(\phi)^{1/n} + O(1/n^2),$$

and hence produces closed Teichmüller geodesics of length

$$L(\psi_n) = \frac{L(\psi)}{n} + O(n^{-2}) \asymp \frac{1}{g_n}.$$

This estimate is obtained by realizing the surfaces  $R_n$  as fibers in the mapping torus of  $\psi$ ; see §10.

**Closed braids.** The Teichmüller polynomial leads to a practical algorithm for computing a fibered face  $F \subset H^1(M, \mathbb{R})$  from the dynamics on a particular fiber  $[S] \in \mathbb{R}_+ \cdot F$ .

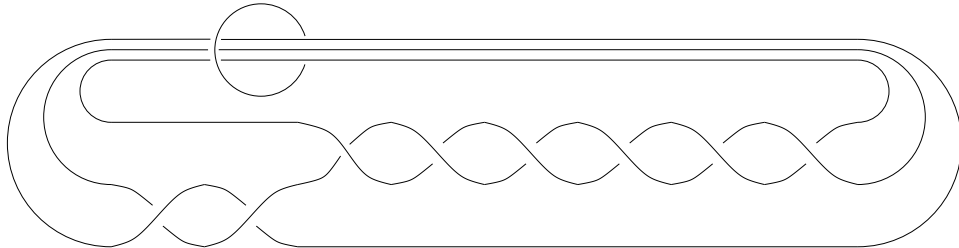


Figure 1. The 4 component fibered link  $L(\beta)$ , for the pure braid  $\beta = \sigma_1^2 \sigma_2^{-6}$ .

Closed braids in  $S^3$  provide a natural source of fibered 3-manifolds to which this algorithm can be applied, as we demonstrate in §11. For example, Figure 1 shows a 4-component link  $L(\beta)$  obtained by closing the braid  $\beta = \sigma_1^2 \sigma_2^{-6}$  after passing it through the unknot  $\alpha$ . The disk spanned by  $\alpha$  meets  $\beta$  in 3 points, providing a fiber  $S \subset M = S^3 - L(\beta)$  isomorphic to a 4-times punctured sphere.



The corresponding fibered face is a 3-dimensional polyhedron

$$F \subset H^1(M, \mathbb{R}) \cong \mathbb{R}^4;$$

its projection to  $H^1(S, \mathbb{R}) \cong \mathbb{R}^3$  is shown in Figure 2. Details of this example and others are presented in §11.

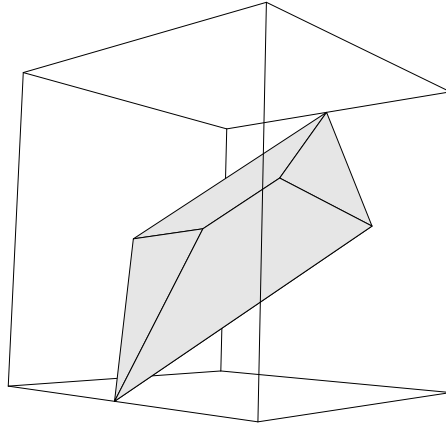


Figure 2. The fibered face of Thurston norm ball for  $M = S^3 - L(\beta)$ .

**Comparison with the Alexander polynomial.** In [Mc] we defined a norm  $\|\cdot\|_A$  on  $H^1(M, \mathbb{R})$  using the Alexander polynomial of  $M$ , and established the inequality

$$\|\phi\|_A \leq \|\phi\|_T$$

between the Alexander and Thurston norms (when  $b_1(M) > 1$ ). This inequality suggested that the Thurston norm should be refined to polynomial invariant, and  $\Theta_F$  provides such an invariant for the fibered faces of the Thurston norm ball.

The Alexander polynomial  $\Delta_M$  and the Teichmüller polynomial  $\Theta_F$  are compared in Table 3. Both polynomials are attached to modules over  $\mathbb{Z}[G]$ , namely  $A(M)$  and  $T(\tilde{\mathcal{L}})$ . These modules give rise to groups of (co)cycles with twisted coefficients, and  $\Delta$  and  $\Theta_F$  describe the locus of characters  $\rho \in \widehat{G}$  where  $\dim Z^1(M, \mathbb{C}_\rho) > 1$  and  $\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0$  respectively.

The polynomials  $\Delta$  and  $\Theta_F$  are related to the Alexander and Thurston norms on  $H^1(M, \mathbb{R})$ , and these norms agree on the cohomology classes of

Alexander	Teichmüller
3-manifold $M$	Fibered face $F$ for $M$
Alexander module $A(M)$	Teichmüller module $T(\tilde{\mathcal{L}})$
$\text{Hom}(A(M), B) = Z^1(M, B)$	$\text{Hom}(T(\tilde{\mathcal{L}}), B) = Z_2(\tilde{\mathcal{L}}, B)$
Alexander polynomial $\Delta_M$	Teichmüller polynomial $\Theta_F$
Alexander norm on $H^1(M, \mathbb{Z})$	Thurston norm on $H^1(M, \mathbb{Z})$
$\ \phi\ _A = b_1(\text{Ker } \phi) + p(M)$	$\ \phi\ _T = \inf\{ \chi(S)  : [S] = \phi\}$
$\ \phi\ _A = \ \phi\ _T$ for the cohomology class of a fibration $M \rightarrow S^1$	
Extended Torelli group of $S$ acts on $H^1(S)$ with twisted coefficients	Extended Torelli group acts on $\mathcal{ML}(S)$ with twisted coefficients

Table 3.

fibrations. Moreover, if the lamination  $\mathcal{L}$  for the fibered face  $F$  has transversally oriented leaves, then  $\Delta_M$  divides  $\Theta_F$  and  $F$  is also a face of the Alexander norm ball (§7).

From a 2-dimensional perspective, the polynomials attached to a fibered manifold  $M$  can be described in terms of a mapping-class  $\psi \in \text{Mod}(S)$ . The description is most uniform for  $\psi$  in the *Torelli group*  $\text{Tor}(S)$ , the subgroup of  $\text{Mod}(S)$  that acts trivially on  $H = H_1(S, \mathbb{Z})$ . By providing  $\psi$  with a lift  $\tilde{\psi}$  to the  $H$ -covering space of  $S$ , we obtain the *extended Torelli group*  $\widetilde{\text{Tor}}(S)$ , a central extension satisfying:

$$0 \rightarrow H_1(S, \mathbb{Z}) \rightarrow \widetilde{\text{Tor}}(S) \rightarrow \text{Tor}(S) \rightarrow 0.$$

The lifted mappings  $\tilde{\psi} \in \widetilde{\text{Tor}}(S)$  preserve twisted coefficients for any  $s \in H^1(S, \mathbb{R})$ , so we obtain a *linear* representation of  $\widetilde{\text{Tor}}(S)$  on  $H^1(S, \mathbb{C}_s)$  and a *piecewise-linear* action on  $\mathcal{ML}_s(S)$ . For example, when  $S$  is a sphere with  $n + 1$  boundary components, the pure braid group  $P_n$  is a subgroup of  $\widetilde{\text{Tor}}(S)$ , and its action on  $H^1(S, \mathbb{C}_s)$  is the *Gassner representation* of  $P_n$  [Bi].

Characteristic polynomials for these actions then give the Alexander and Teichmüller invariants  $\Delta_M$  and  $\Theta_F$ .

**Other foliations.** Gabai has shown that every norm-minimizing surface

$S \subset M$  is the leaf of a taut foliation  $\mathcal{F}$  (see [Ga1]), and the construction of pseudo-Anosov flows transverse to taut foliations is a topic of current research. It would be interesting to obtain polynomial invariants for these more general foliations, and in particular for the non-fibered faces of the Thurston norm ball.

**Notes and references.** Contributions related to this paper have been made by many authors.

For a pseudo-Anosov mapping with transversally orientable foliations, Fried investigated a twisted Lefschetz zeta-function  $\zeta(t, u)$  similar to  $\Theta_F(t, u)$ . For example, the homology directions of these special pseudo-Anosov mappings can be recovered from the support of  $\zeta(t, u)$ , just as  $\mathbb{R}_+ \cdot F$  can be recovered from  $\Theta_F$ ; and the concavity of  $1/\log(K(\phi))$  holds in a general setting. See [Fr2], [Fr4].

Laminations, foliations and branched surfaces with affine invariant measures have been studied in [HO], [CLR], [LO], [CLR], [Bon2], [Oe2] and elsewhere. The Thurston norm can also be studied using taut foliations [Ga2], branched surfaces [Oe1], [Mos] and Seiberg-Witten theory [KM]. Another version of Theorem 1.1 is presented by Thurston in [Th4, Thm. 5.8].

Background on pseudo-Anosov mappings, laminations and train tracks can be found, for example, in [FLP], [Th1, §8.9], [Th3], [Bers], [HP], [BH] and the references therein. Additional notes and references are collected at the end of each section.

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## 2 The module of a lamination

**Laminations.** Let  $\lambda$  be a Hausdorff topological space. We say  $\lambda$  is an  $n$ -dimensional *lamination* if there exists a collection of compact, totally disconnected spaces  $K_\alpha$  such that  $\lambda$  is covered by open sets  $U_\alpha$  homeomorphic to  $K_\alpha \times \mathbb{R}^n$ .

The *leaves* of  $\lambda$  are its connected components.

A compact, totally disconnected set  $T \subset \lambda$  is a *transversal* for  $\lambda$  if there is an open neighborhood  $U$  of  $T$  and a homeomorphism

$$(U, T) \cong (T \times \mathbb{R}^n, T \times \{0\}). \quad (2.1)$$

Any compact open subset of a transversal is again a transversal.

**Modules and cycles.** We define the *module of a lamination*,  $T(\lambda)$ , to be the  $\mathbb{Z}$ -module generated by all transversals  $[T]$ , modulo the relations:

- (i)  $[T] = [T'] + [T'']$  if  $T$  is the disjoint union of  $T'$  and  $T''$ ; and
- (ii)  $[T] = [T']$  if there is a neighborhood  $U$  of  $T \cup T'$  such (2.1) holds for both  $T$  and  $T'$ .

Equivalently, (ii) identifies transversals that are equivalent under holonomy (sliding along the leaves of the lamination).

For any  $\mathbb{Z}$ -module  $B$ , we define the space of  $n$ -cycles on an  $n$ -dimensional lamination  $\lambda$  with values in  $B$  by:

$$Z_n(\lambda, B) = \text{Hom}(T(\lambda), B).$$

For example, cycles  $\mu \in Z_n(\lambda, \mathbb{R})$  correspond to finitely-additive transverse signed measures; the measure of a transversal  $\mu(T)$  is holonomy invariant by relation (ii), and it satisfies

$$\mu(T \sqcup T') = \mu(T) + \mu(T')$$

by relation (i).

**Action of homeomorphisms.** Let  $\psi : \lambda_1 \rightarrow \lambda_2$  be a homeomorphism between laminations. Then  $\psi$  determines an isomorphism

$$\psi^* : T(\lambda_2) \rightarrow T(\lambda_1),$$

defined by pulling back transversals:

$$\psi^*([T]) = [\psi^{-1}(T)].$$

Applying  $\text{Hom}(\cdot, B)$ , we obtain a pushforward map on cycles,

$$\psi_* : Z_n(\lambda_1, B) \rightarrow Z_n(\lambda_2, B),$$

satisfying  $(\psi_*(\mu))(T) = \mu(\psi^{-1}(T))$  and thus generalizing the pushforward of measures.

**The mapping-torus.** Now let  $\psi : \lambda \rightarrow \lambda$  be a homeomorphism of an  $n$ -dimensional lamination to itself. The *mapping torus*  $\mathcal{L}$  of  $\psi$  is the  $(n+1)$ -dimensional lamination defined by

$$\mathcal{L} = \lambda \times [0, 1] / \langle (x, 1) \sim (\psi(x), 0) \rangle.$$

The lamination  $\mathcal{L}$  fibers over  $S^1$  with fiber  $\lambda$  and monodromy  $\psi$ . Since cycles on  $\mathcal{L}$  correspond to  $\psi$ -invariant cycles on  $\lambda$ , we have:

**Proposition 2.1** *The module of the mapping torus of  $\psi : \lambda \rightarrow \lambda$  is given by*

$$T(\mathcal{L}) = \text{Coker}(\psi^* - I) = T(\lambda)/(\psi^* - I)(T(\lambda)).$$

**Example:**  $(\mathbb{Z}_p, x + 1)$ . Let  $\lambda = \mathbb{Z}_p$  be the  $p$ -adic integers, considered as a 0-dimensional lamination, and let  $\psi : \lambda \rightarrow \lambda$  be the map  $\psi(x) = x + 1$ . Then the mapping torus  $\mathcal{L}$  of  $\psi$  is a 1-dimensional solenoid, satisfying

$$T(\mathcal{L}) \cong \mathbb{Z}[1/p],$$

where  $\mathbb{Z}[1/p] \subset \mathbb{Q}$  is the subring generated by  $1/p$ . Indeed, the transversals  $T_n = p^n \mathbb{Z}_p$  and their translates generate  $T(\lambda)$ , so their images  $[T_n]$  generate  $T(\mathcal{L})$ . Since  $T_n$  is the union of  $p$  translates of  $T_{n+1}$ , we have  $[T_n] = p[T_{n+1}]$ , and therefore  $T(\mathcal{L}) \cong \mathbb{Z}[1/p]$  by the map sending  $[T_n]$  to  $p^{-n}$ .

Observe that

$$Z_1(\mathcal{L}, \mathbb{R}) = \text{Hom}(\mathbb{Z}[1/p], \mathbb{R}) = \mathbb{R},$$

showing there is a unique finitely-additive probability measure on  $\mathbb{Z}_p$  invariant under  $x \mapsto x + 1$ .

**Twisted cycles.** Next we describe cycles with twisted coefficients.

Let  $\tilde{\lambda} \rightarrow \lambda$  be a Galois covering space with abelian deck group  $G$ . Then  $G$  acts on  $T(\tilde{\lambda})$ , making the latter into a module over the *group ring*  $\mathbb{Z}[G]$ . Any  $G$ -module  $B$  determines a bundle of twisted local coefficients over  $\lambda$ , and we define

$$Z_n(\lambda, B) = \text{Hom}_G(T(\tilde{\lambda}), B).$$

For example, any homomorphism

$$\rho : G \rightarrow \mathbb{R}_+$$

makes  $\mathbb{R}$  into a module  $\mathbb{R}_\rho$  over  $\mathbb{Z}[G]$ . The cycles  $\mu \in Z_n(\lambda, \mathbb{R}_\rho)$  can then be interpreted as either:

- (i) cycles on  $\tilde{\lambda}$  satisfying  $g_*\mu = \rho(g)\mu(T)$  for all  $g \in G$ ; or
- (ii) cycles on  $\lambda$  with values (locally) in the real line bundle over  $\lambda$  determined by  $\rho \in H^1(\lambda, \mathbb{R}_+)$ .

**Geodesic laminations on surfaces.** Now let  $S$  be a compact orientable surface with  $\chi(S) < 0$ . Fix a complete hyperbolic metric of finite volume on  $\text{int}(S)$ .

A *geodesic lamination*  $\lambda \subset S$  is a compact lamination whose leaves are hyperbolic geodesics.

A *train track*  $\tau \subset S$  is a finite 1-complex such that

- (i) every  $x \in \tau$  lies in the interior of a smooth arc embedded in  $\tau$ ,
- (ii) any two such arcs are tangent at  $x$ , and
- (iii) for each component  $U$  of  $S - \tau$ , the double of  $U$  along the smooth part of  $\partial U$  has negative Euler characteristic.

A geodesic lamination  $\lambda$  is *carried* by a train track  $\tau$  if there is a continuous *collapsing map*  $f : \lambda \rightarrow \tau$  such that for each leaf  $\lambda_0 \subset \lambda$ ,

- (i)  $f|_{\lambda_0}$  is an immersion, and
- (ii)  $\lambda_0$  is the geodesic representative of the path or loop  $f : \lambda_0 \rightarrow S$ .

Collapsing maps between train tracks are defined similarly. Every geodesic lamination is carried by some train track [HP, 1.6.5].

The *vertices* (or *switches*) of a train track,  $V \subset \tau$ , are the points where 3 or more smooth arcs come together. The *edges*  $E$  of  $\tau$  are the components of  $\tau - V$ ; some ‘edges’ may be closed loops.

A train track is *trivalent* if only 3 edges come together at each vertex. A trivalent train track has *minimal complexity* for  $\lambda$  if it has the minimal number of edges among all trivalent  $\tau$  carrying  $\lambda$ .

**The module of a train track.** Let  $T(\tau)$  denote the  $\mathbb{Z}$ -module generated by the edges  $E$  of  $\tau$ , modulo the relations

$$[e_1] + \cdots + [e_r] = [e'_1] + \cdots + [e'_s]$$

for each vertex  $v \in V$  with incoming edges  $(e_i)$  and outgoing edges  $(e'_j)$ . (The distinction between incoming and outgoing edges depends on the choice a direction along  $\tau$  at  $v$ .) Since there is one relation for each vertex, we obtain a presentation for  $T(\tau)$  of the form:

$$\mathbb{Z}^V \xrightarrow{D} \mathbb{Z}^E \rightarrow T(\tau) \rightarrow 0. \quad (2.2)$$

As for a geodesic lamination, we define the 1-cycles on  $\tau$  with values in  $B$  by

$$Z_1(\tau, B) = \text{Hom}(T(\tau), B).$$

**Theorem 2.2** *Let  $\lambda \subset S$  be a geodesic lamination, and let  $\tau$  be a train track carrying  $\lambda$  with minimal complexity. Then there is a natural isomorphism*

$$T(\lambda) \cong T(\tau).$$

**Corollary 2.3** *For any geodesic lamination  $\lambda$ , the module  $T(\lambda)$  is finitely-generated.*

**Corollary 2.4** *If  $\lambda$  is connected and carried by a train track  $\tau$  of minimal complexity, then we have*

$$T(\lambda) \cong \mathbb{Z}^{|\chi(\tau)|} \oplus \begin{cases} \mathbb{Z} & \text{if } \tau \text{ is orientable,} \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

(Here  $\chi(\tau)$  is the Euler characteristic of  $\tau$ .)

**Proof.** Use the fact that the transpose  $D^* : \mathbb{Z}^E \rightarrow \mathbb{Z}^V$  of the presentation matrix (2.2) for  $T(\tau)$  behaves like a boundary map, and  $\sum n_i v_i$  is in the image of  $D^*$  iff  $\sum n_i = 0$  (in the orientable case) or  $\sum n_i = 0 \pmod{2}$  (in the non-orientable case). ■

**Proof of Theorem 2.2.** Let  $\tau_0 = \tau$ . The collapsing map  $f_0 : \lambda \rightarrow \tau_0$  determines a map of modules

$$f_0^* : T(\tau_0) \rightarrow T(\lambda)$$

sending each edge  $e \in E$  to the transversal defined by

$$T = f_0^*(e) = f_0^{-1}(x)$$

for any  $x \in e$ . We will show  $f_0^*$  is an isomorphism.

We begin by using  $\lambda$  to guide a sequence of splittings of  $\tau_0$  into finer and finer train tracks  $\tau_n$ , converging to  $\lambda$  itself, in the sense that there are collapsing maps  $f_n : \lambda \rightarrow \tau_n$  converging to the inclusion  $\lambda \subset S$ . We will also have collapsing maps  $g_n : \tau_{n+1} \rightarrow \tau_n$  such that  $f_n = g_n \circ f_{n+1}$ . Each  $\tau_n$  will be of minimal complexity.

The train track  $\tau_{n+1}$  is constructed from  $\tau_n$  as follows. First, observe that each edge of  $\tau_n$  carries at least one leaf of  $\lambda$  (since  $\tau_n$  has minimal complexity). Thus each cusp of a component  $U$  of  $S - \tau$  (where tangent edges  $a, b$  in  $\tau$  come together) corresponds to pair of adjacent leaves  $\lambda_a, \lambda_b$  of  $\lambda$ . Choose a particular cusp, and split  $\tau_n$  between  $a$  and  $b$  so that the train track continues to follow  $\lambda_a$  and  $\lambda_b$ . When we split past a vertex, we obtain a new trivalent train track  $\tau_{n+1}$ . There are 3 possible results of splitting, recorded in Figure 4.

In the middle case, the leaves  $\lambda_1$  and  $\lambda_2$  diverge, and we obtain a train track  $\tau_{n+1}$  carrying  $\lambda$  but with fewer edges than  $\tau_n$ ; this is impossible, since  $\tau_n$  has minimal complexity.

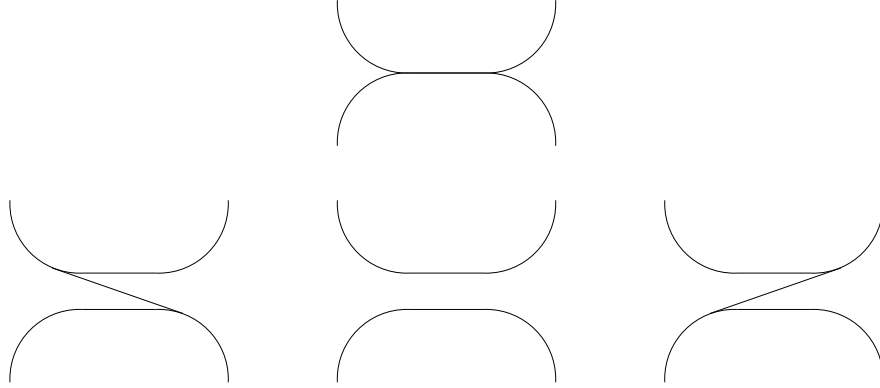


Figure 4. Three possible splittings

In the right and left cases, we obtain a train track  $\tau_{n+1}$  of the same complexity as  $\tau_n$ , with a natural collapsing map  $g_{n+1} : \tau_{n+1} \rightarrow \tau_n$ . Since the removed and added edges  $e$  and  $f$  are both in the span of  $\langle a, b, c, d \rangle$ , the module map

$$g_n^* : T(\tau_n) \rightarrow T(\tau_{n+1}) \quad (2.3)$$

is an isomorphism.

By repeatedly splitting every cusp of  $S - \tau$ , we obtain train tracks with longer and longer edges, following the leaves of  $\lambda$  more and more closely; thus the collapsing maps can be chosen such that  $f_n : \lambda \rightarrow \tau_n$  converges to the identity. Compare [Th1, Prop. 8.9.2], [HP, §2].

To prove  $T(\lambda) \cong T(\tau_0)$ , we will define a map

$$\phi : T(\lambda) \rightarrow T_\infty = \varinjlim T(\tau_n)$$

(where the direct limit is taken with respect to the collapsing maps  $g_n^*$ ). Given any transversal  $T$  to  $\lambda$ , there is a neighborhood  $U$  of  $T$  in  $\lambda$  homeomorphic to  $T \times \mathbb{R}$ . Then for all  $n \gg 0$ , we have

$$\sup_{x \in \lambda} d(f_n(x), x) < d(T, \partial U),$$

and thus all the leaves of  $\lambda$  carried by  $\tau \cap U$  are accounted for by  $T$ . Therefore  $T$  is equivalent to a finite sum of edges in  $T(\tau_n)$ :

$$f_n^*([e_1] + \cdots + [e_i]) = [T],$$

and we define  $\phi(T) = [e_1] + \cdots + [e_i]$ .



It is now straightforward to verify that  $\phi$  is a map of modules, inverting the map  $T_\infty \rightarrow T(\lambda)$  obtained as the inverse limit of the collapsings  $f_n^* : T(\tau_n) \rightarrow T(\lambda)$ . But the maps  $g_n^*$  of (2.3) are isomorphisms, so we have  $T(\lambda) \cong T_\infty \cong T(\tau_0)$ . ■

**Twisted train tracks.** Train tracks also provide a convenient description of twisted cycles on a geodesic lamination.

Let  $\lambda \subset S$  be a geodesic lamination carried by a train track  $\tau$ . Let

$$\pi : \tilde{S} \rightarrow S$$

be a Galois covering space with abelian deck group  $G$ . We can then construct modules  $T(\tilde{\lambda})$  and  $T(\tilde{\tau})$  attached to the induced covering spaces of  $\lambda$  and  $\tau$ . The deck group acts naturally on  $\tilde{\lambda}$  and  $\tilde{\tau}$ , so we obtain modules over the group ring  $\mathbb{Z}[G]$ . The arguments of Theorem 2.2 can then be applied to the lift of a collapsing map  $f : \lambda \rightarrow \tau$ , to establish:

**Theorem 2.5** *The  $\mathbb{Z}[G]$ -modules  $T(\tilde{\lambda})$  and  $T(\tilde{\tau})$  are naturally isomorphic. A choice of lifts for the edges and vertices  $(E, V)$  of  $\tau$  to  $\tilde{\tau}$  determines a finite presentation*

$$\mathbb{Z}[G]^V \xrightarrow{D} \mathbb{Z}[G]^E \rightarrow T(\tilde{\tau}) \rightarrow 0$$

for  $T(\tilde{\tau})$  as a  $\mathbb{Z}[G]$ -module.

**Example.** Let  $S$  be a sphere with 4 disks removed. Let  $\tilde{S} \rightarrow S$  be the maximal abelian covering of  $S$ , with deck group

$$G = H_1(S, \mathbb{Z}) = \langle A, B, C \rangle \cong \mathbb{Z}^3$$

generated by counterclockwise loops around 3 boundary components of  $S$ .

Let  $\tau \subset S$  be the train track shown in Figure 5. Then for suitable lifts of the edges of  $\tau$ , the module  $T(\tilde{\tau})$  is generated over  $\mathbb{Z}[G]$  by  $\langle a, b, c, d, e, f \rangle$ , with the relations:

$$\begin{aligned} b &= a + d \\ A^{-1}d &= a + e \\ b &= c + f \\ c &= B^{-1}e + Cf, \end{aligned}$$

coming from the 4 vertices of  $\tau$ . Simplifying, we find  $T(\tilde{\tau})$  is generated by  $\langle a, b, c \rangle$  with the single relation

$$(1 + A)a + AB(1 + C)c = (1 + ABC)b.$$

This relation shows, for example, that

$$\dim Z_1(\tau, \mathbb{C}_\rho) = \begin{cases} 3 & \text{if } \rho(A) = \rho(B) = \rho(C) = -1, \\ 2 & \text{otherwise,} \end{cases}$$

for any 1-dimensional representation  $\rho : G \rightarrow \mathbb{C}^*$ .

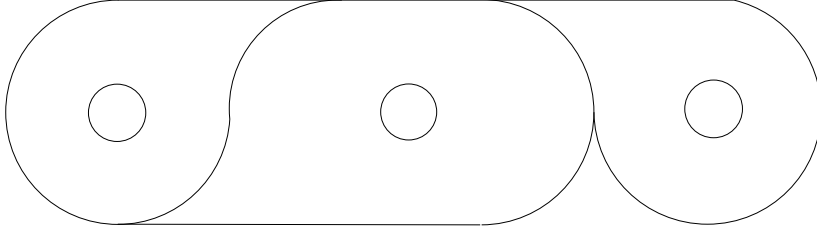


Figure 5. Presenting a track track.

### Notes.

1. The usual (positive, countably-additive) transverse measures on a geodesic lamination  $\lambda$  generally span a *proper* subspace  $M(\lambda)$  of the space of cycles  $Z_1(\lambda, \mathbb{R})$ . Indeed, a generic measured lamination  $\lambda$  on a closed surface cuts  $S$  into ideal triangles, so any train track  $\tau$  carrying  $\lambda$  is the 1-skeleton of a triangulation of  $S$ . At the same time  $\lambda$  is typically uniquely ergodic, and therefore

$$\dim M(\lambda) = 1 < \dim Z_1(\lambda, \mathbb{R}) = \dim Z_1(\tau, \mathbb{R}) = 6g(S) - 6.$$

2. Bonahon has shown that cycles  $\mu \in Z_1(\lambda, \mathbb{R})$  correspond to transverse invariant *Hölder distributions*; that is, the pairing

$$\langle f, \mu \rangle = \int_T f(x) d\mu(x)$$

can be defined for any transversal  $T$  and Hölder continuous function  $f : T \rightarrow \mathbb{R}$  [Bon2, Thm. 17]. See also [Bon2, Thm. 11] for a variant of Theorem 2.2, and [Bon1] for additional results.

3. One can also describe  $Z_1(\lambda, \mathbb{R})$  as a space of closed *currents* carried by  $\lambda$ , since these cycles are distributional in nature and they need not be compactly supported (when  $\lambda$  is noncompact).

### 3 The Teichmüller polynomial

In this section we define the Teichmüller polynomial  $\Theta_F$  of a fibered face  $F$ , and establish the *determinant formula*

$$\Theta_F(t, u) = \det(uI - P_E(t)) / \det(uI - P_V(t)).$$

We begin by introducing some notation that will be used throughout the sequel.

Let  $M^3$  be a compact, connected, orientable, irreducible, atoroidal 3-manifold. Let  $\pi : M \rightarrow S^1$  be a fibration with fiber  $S \subset M$  and monodromy  $\psi$ . Then:

- $S$  is a compact, orientable surface with  $\chi(S) < 0$ , and
- $\psi : S \rightarrow S$  is a pseudo-Anosov map, with an expanding invariant lamination
- $\lambda \subset S$ , unique up to isotopy.

Adjusting  $\psi$  by isotopy, we can assume  $\psi(\lambda) = \lambda$ .

By the general theory of pseudo-Anosov mappings, there is a *positive* transverse measure  $\mu \in Z_1(\lambda, \mathbb{R})$ , unique up to scale, and  $\psi_*(\mu) = k\mu$  for some  $k > 1$ . Then  $[\Lambda] = [(\lambda, \mu)]$  is a fixed-point of  $\psi$  in the space of projective measured laminations  $\mathbb{P}\mathcal{ML}(S)$ . Moreover  $[\psi^n(\gamma)] \rightarrow [\Lambda]$  for every simple closed curve  $[\gamma] \in \mathbb{P}\mathcal{ML}(S)$ .

Associated to  $(M, S)$  we also have:

- $\mathcal{L} \subset M$ , the mapping torus of  $\psi : \lambda \rightarrow \lambda$ , and
- $F \subset H^1(M, \mathbb{R})$ , the open face of unit ball in the Thurston norm with  $[S] \in \mathbb{R}_+ \cdot F$ .

We say  $F$  is a *fibered face* of the Thurston norm ball, since every point in  $H^1(M, \mathbb{Z}) \cap \mathbb{R}_+ \cdot F$  is represented by a fibration of  $M$  over the circle [Th2, Thm 5].

**The flow lines of  $\psi$ .** Using  $\psi$  we can present  $M$  in the form

$$M = (S \times \mathbb{R}) / \langle (s, t) \sim (\psi(s), t - 1) \rangle,$$

and the lines  $\{s\} \times \mathbb{R}$  descend to the leaves of an oriented 1-dimensional foliation  $\Psi$  of  $M$ , the *flow lines* of  $\psi$ . The 2-dimensional lamination  $\mathcal{L} \subset M$  is swept out by the leaves of  $\Psi$  passing through  $\lambda$ .

**Invariance of  $\mathcal{L}$ .** We now show  $\mathcal{L}$  depends only on  $F$ .

**Theorem 3.1 (Fried)** *Let  $[S'] \in \mathbb{R}_+ \cdot F$  be a fiber of  $M$ . Then after an isotopy,*

- $S'$  is transverse to the flow lines  $\Psi$  of  $\psi$ , and
- the first return map of the flow coincides with the pseudo-Anosov monodromy  $\psi' : S' \rightarrow S'$ .

For this result, see [Fr1, Thm. 7 and Lemma] and [Fr3].

**Corollary 3.2** *Any two fibers  $[S], [S'] \in \mathbb{R}_+ \cdot F$  determine the same lamination  $\mathcal{L} \subset M$  (up to isotopy).*

**Proof.** Consider two fibers  $S$  and  $S'$  for the same face  $F$ . Let  $\psi, \psi'$  denote their respective monodromy transformations,  $\lambda, \lambda'$  their expanding laminations, and  $\mathcal{L}, \mathcal{L}' \subset M$  the mapping tori of  $\lambda, \lambda'$ .

By the Theorem above, we can assume  $S'$  is transverse to  $\Psi$  and hence transverse to  $\mathcal{L}$ .

Let  $\mu' = \mathcal{L} \cap S'$ . Then  $\mu' \subset S'$  is a  $\psi'$ -invariant lamination with no isolated leaves. By invariance,  $\mu'$  must contain the expanding or contracting lamination of  $\psi'$ . Since flowing along  $\Psi$  expands the leaves of  $\mathcal{L}$ , we find  $\mu' \supset \lambda'$ .

By irreducibility of  $\psi'$ , the complementary regions  $S' - \lambda'$  are  $n$ -gons or punctured  $n$ -gons. In such regions, the only geodesic laminations are isolated leaves running between cusps. Since  $\mu'$  has no isolated leaves, we conclude that  $\mu' = \lambda'$  and thus  $\mathcal{L} = \mathcal{L}'$  (up to isotopy). ■

**Modules and the Teichmüller polynomial.** By the preceding Corollary, the lamination  $\mathcal{L} \subset M$  depends only on  $F$ . Associated to the pair  $(M, F)$  we now have:

- $G = H_1(M, \mathbb{Z})/\text{torsion}$ , a free abelian group;
- $\widetilde{M} \rightarrow M$ , the Galois covering space corresponding to  $\pi_1(M) \rightarrow G$ ;
- $\widetilde{\mathcal{L}} \subset \widetilde{M}$ , the preimage of the lamination  $\mathcal{L}$  determined by  $F$ ; and

- $T(\tilde{\mathcal{L}})$ , the  $\mathbb{Z}[G]$ -module of transversals to  $\tilde{\mathcal{L}}$ .

Since  $\mathcal{L}$  is compact,  $T(\mathcal{L})$  is finitely-generated and  $T(\tilde{\mathcal{L}})$  is finitely-presented over the ring  $\mathbb{Z}[G]$ .

Choose a presentation

$$\mathbb{Z}[G]^r \xrightarrow{D} \mathbb{Z}[G]^s \rightarrow T(\tilde{\mathcal{L}}) \rightarrow 0,$$

and let  $I \subset \mathbb{Z}[G]$  be the ideal generated by the  $s \times s$  minors of  $D$ . The ideal  $I$  is the *Fitting ideal* of the module  $T(\tilde{\mathcal{L}})$ , and it is independent of the choice of presentation; see [La, Ch. XIII, §10], [Nor].

Using the fact that  $\mathbb{Z}[G]$  is a unique factorization domain, we define the *Teichmüller polynomial* of  $(M, F)$  by

$$\Theta_F = \gcd(f : f \in I) \in \mathbb{Z}[G]. \quad (3.1)$$

The polynomial  $\Theta_F$  is well-defined up to multiplication by a unit  $\pm g \in \mathbb{Z}[G]$ , and it depends only on  $(M, F)$ .

Note that  $\mathbb{Z}[G]$  can be identified with a ring of complex algebraic functions on the character variety

$$\widehat{G} = \text{Hom}(G, \mathbb{C}^*)$$

by setting  $(\sum a_g \cdot g)(\rho) = \sum a_g \rho(g)$ .

**Theorem 3.3** *The locus  $\Theta_F(\rho) = 0$  is the largest hypersurface  $V \subset \widehat{G}$  such that  $\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0$  for all  $\rho \in V$ .*

**Proof.** A character  $\rho$  belongs to the zero locus of the ideal  $I \iff$  the presentation matrix  $\rho(M)$  has rank  $r < s \iff$  we have

$$\dim_{\mathbb{C}} Z_2(\mathcal{L}, \mathbb{C}_\rho) = \dim \text{Hom}(T(\tilde{\mathcal{L}}), \mathbb{C}_\rho) = s - r > 0;$$

and the greatest common divisor of the elements of  $I$  defines the largest hypersurface contained in  $V(I)$ . ■

**Computing the Teichmüller polynomial.** We now describe a procedure for computing  $\Theta_F$  as an explicit Laurent polynomial.

Consider again a fiber  $S \subset M$  with monodromy  $\psi$  and expanding lamination  $\lambda$ . Associated to this data we have:

- $H = \text{Hom}(H^1(S, \mathbb{Z})^\psi, \mathbb{Z}) \cong \mathbb{Z}^b$ , the dual of the  $\psi$ -invariant cohomology of  $S$ ;

- $\tilde{S} \rightarrow S$ , the Galois covering space corresponding to the natural map

$$\pi_1(S) \rightarrow H_1(S, \mathbb{Z}) \rightarrow H;$$

- $\tau \subset S$ , a  $\psi$ -invariant train track carrying  $\lambda$ ; and
- $\tilde{\lambda}, \tilde{\tau} \subset \tilde{S}$ , the preimages of  $\lambda, \tau \subset S$ .

Note that pullback by  $S \subset M$  determines a surjection  $H^1(M, \mathbb{Z}) \rightarrow H^1(S, \mathbb{Z})^\psi$ , and hence a natural inclusion

$$H \subset G = H_1(M, \mathbb{Z})/\text{torsion} = \text{Hom}(H^1(M, \mathbb{Z}), \mathbb{Z}).$$

Alternatively, we can regard  $\tilde{S}$  as a component of the preimage of  $S$  in the covering  $\tilde{M} \rightarrow M$  with deck group  $G$ ; then  $H \subset G$  is the stabilizer of  $\tilde{S} \subset \tilde{M}$ .

Now *choose* a lift

$$\tilde{\psi} : \tilde{S} \rightarrow \tilde{S}$$

of the pseudo-Anosov mapping  $\psi$ . Then we obtain a splitting

$$G = H \oplus \mathbb{Z}\tilde{\Psi},$$

where  $\tilde{\Psi} \in G$  acts on  $\tilde{M} = \tilde{S} \times \mathbb{R}$  by

$$\tilde{\Psi}(s, t) = (\tilde{\psi}(s), t - 1). \quad (3.2)$$

If we further choose a basis  $(t_1, \dots, t_b)$  for  $H$ , written multiplicatively, and set  $u = [\tilde{\Psi}]$ , then we obtain an isomorphism

$$\mathbb{Z}[G] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_b^{\pm 1}, u^{\pm 1}]$$

between the group ring of  $G$  and the ring of integral Laurent polynomials in the variables  $t_i$  and  $u$ .

**Remark.** Under the fibration  $M \rightarrow S^1$ , the element  $u \in H_1(M, \mathbb{Z})/\text{torsion}$  maps to  $-1$  in  $H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}$ , as can be seen from (3.2).

**A presentation for  $T(\tilde{\mathcal{L}})$ .** The next step in the computation of  $\Theta_F$  is to obtain a concrete description of the module  $T(\tilde{\mathcal{L}})$ .

We begin by using the train track  $\tau$  to give a presentation of  $T(\tilde{\lambda})$  over  $\mathbb{Z}[H]$ . Let  $E$  and  $V$  denote the sets of edges and vertices of the train track  $\tau \subset S$ . By choosing a lift of each edge and vertex to the covering space  $\tilde{S} \rightarrow S$  with deck group  $H$ , we can identify the edges and vertices of  $\tilde{\tau}$  with the products  $H \times E$  and  $H \times V$ . These lifts yield a presentation

$$\mathbb{Z}[H]^V \xrightarrow{D} \mathbb{Z}[H]^E \rightarrow T(\tilde{\tau}) \rightarrow 0 \quad (3.3)$$

for  $T(\tilde{\tau}) \cong T(\tilde{\lambda})$  as a  $\mathbb{Z}[H]$ -module.

Since  $\tau$  is  $\psi$ -invariant, there is an  $H$ -invariant collapsing map

$$\tilde{\psi}(\tilde{\tau}) \rightarrow \tilde{\tau}.$$

By expressing each edge in the target as a sum of the edges in the domain which collapse to it, we obtain a natural map of  $\mathbb{Z}[H]$ -modules

$$P_E : \mathbb{Z}[H]^E \rightarrow \mathbb{Z}[H]^E.$$

There is a similar map  $P_V$  on vertices.

We can regard  $P_E$  and  $P_V$  as matrices  $P_E(t), P_V(t)$  whose entries are Laurent polynomials in  $t = (t_1, \dots, t_b)$ . In the terminology of the Appendix, such a matrix is *Perron-Frobenius* if it has a power such that every entry is a nonzero Laurent polynomial with positive coefficients.

**Theorem 3.4**  *$P_E(t)$  is a Perron-Frobenius matrix of Laurent polynomials.*

**Proof.** For any  $e, f \in E$ , the matrix entry  $(P_E)_{ef}$  is a sum of monomials  $t^\alpha$  for all  $\alpha$  such that  $\tilde{\psi}(\alpha \cdot e)$  collapses to  $f$ . Thus each nonzero entry is a positive, integral Laurent monomial, and since  $\psi$  is pseudo-Anosov there is some iterate  $P_E^N(t)$  with every entry nonzero. ■

The matrices  $P_E(t)$  and  $P_V(t)$  are compatible with the presentation (3.3) for  $T(\tilde{\tau})$ , so we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathbb{Z}[H]^V & \longrightarrow & \mathbb{Z}[H]^E & \longrightarrow & T(\tilde{\tau}) & \longrightarrow & 0 \\ P_V(t) \downarrow & & P_E(t) \downarrow & & P(t) \downarrow & & \\ \mathbb{Z}[H]^V & \longrightarrow & \mathbb{Z}[H]^E & \longrightarrow & T(\tilde{\tau}) & \longrightarrow & 0. \end{array} \quad (3.4)$$

Here  $P(t) = \psi^*$  under the natural identification  $T(\tilde{\tau}) = T(\tilde{\lambda})$ .

The next result makes precise the fact that twisted cycles on  $\mathcal{L}$  correspond to  $\psi$ -invariant twisted cycles on  $\lambda$  (compare Proposition 2.1).

**Theorem 3.5** *There is a natural isomorphism*

$$T(\tilde{\mathcal{L}}) \cong \text{Coker}(uI - P(t))$$

as modules over  $\mathbb{Z}[G]$ .

Here  $uI - P(t)$  is regarded as an endomorphism of  $T(\tilde{\tau}) \otimes \mathbb{Z}[u]$  over  $\mathbb{Z}[G] = \mathbb{Z}[H] \otimes \mathbb{Z}[u]$ .

**Proof.** The lamination  $\mathcal{L}$  fibers over  $S^1$  with fiber  $\lambda$  and monodromy  $\psi : \lambda \rightarrow \lambda$ , so we can regard  $\tilde{\mathcal{L}}$  as  $\tilde{\lambda} \times \mathbb{R}$ , equipped with the action of  $G = H \oplus \mathbb{Z}\tilde{\Psi}$ . The product structure on  $\tilde{\mathcal{L}}$  gives an isomorphism  $T(\tilde{\mathcal{L}}) \cong T(\tilde{\lambda}) \cong T(\tilde{\tau})$  as modules over  $\mathbb{Z}[H]$ , so to describe  $T(\tilde{\mathcal{L}})$  as a  $\mathbb{Z}[G]$ -module we need only determine the action of  $u$  under this isomorphism. But  $u$  acts on  $\tilde{\lambda} \times \mathbb{R}$  by  $(x, t) \mapsto (\tilde{\psi}(x), t - 1)$ , so for any transversal  $T \in T(\tilde{\lambda})$  we have  $uT = \tilde{\psi}^*(T) = P(t)T$ , and the Theorem follows.  $\blacksquare$

**The determinant formula.** The main result of this section is:

**Theorem 3.6** *The Teichmüller polynomial of the fibered face  $F$  is given by:*

$$\Theta_F(t, u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))} \quad (3.5)$$

when  $b_1(M) > 1$ .

**Remarks.**

1. If  $b_1(M) = 1$  then the numerator must be multiplied by  $(u - 1)$  if  $\tau$  is orientable. Compare Corollary 2.4.
2. To understand the determinant formula, recall that by Theorem 3.3, the locus  $\Theta_F(t, u) = 0$  in  $\hat{G}$  consists of characters for which we have

$$\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0.$$

Now a cocycle for  $\mathcal{L}$  is the same as a  $\psi$ -invariant cocycle for  $\lambda$ , so we expect to have  $\Theta_F(t, u) = \det(uI - P(t))$ . But the module  $T(\tilde{\lambda})$  is not quite free in general, so we need the formula above to make sense of the determinant.

**Proof of Theorem 3.6.** To simplify notation, let  $A = \mathbb{Z}[G]$ , let  $T$  be the  $A$ -module  $T(\tilde{\lambda}) \otimes \mathbb{Z}[G]$ , and let  $P : T \rightarrow T$  be the automorphism  $P = \tilde{\psi}^*$ .

Let  $K$  denote the field of fractions of  $A$ . For each  $f \in A$ ,  $f \neq 0$ , we can invert  $f$  to obtain the ring  $A_f = A[1/f] \subset K$ , and there is a naturally determined  $A_f$ -module  $T_f$  with automorphism  $P_f$  coming from  $P$  (see e.g. [AM, Ch. 3]). The presentation (3.3) for  $T$  determines a presentation

$$A_f^V \xrightarrow{D_f} A_f^E \rightarrow T_f \rightarrow 0 \quad (3.6)$$



for  $T_f$ .

Now let  $\Theta = \Theta_F(t, u) \in A$  be the Teichmüller polynomial for  $(M, F)$  (defined by (3.1)), and define  $\Delta \in K$  by

$$\Delta = \Delta(t, u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))}.$$

Our goal is to show  $\Theta = \Delta$  up to a unit in  $A$ . The method is to show that  $\Theta = \Delta$  up to a unit in  $A_f$  for many different  $f$ . We break the argument up into 5 main steps.

**I.** *The map  $D_f : A_f^V \rightarrow A_f^E$  is injective, with a left inverse, whenever  $f = (t_i^2 - 1)g$  for some  $i$ ,  $1 \leq i \leq b$ , and some  $g \neq 0$  in  $A$ .*

It is enough to show that the transpose  $D_f^* : A_f^E \rightarrow A_f^V$  is surjective — then  $D_f^*$  has a right inverse, so  $D_f$  has a left inverse. We prefer to work with  $D_f^*$  since it behaves like a geometric boundary map.

Pick a vertex  $v$  of  $\tau$ . Since  $\psi$  is pseudo-Anosov,  $\lambda$  fills the surface  $S$ , and hence the train track  $\tau$  is connected and each component of  $S - \tau$  is a disk, possibly with a single puncture. Thus the natural map  $\pi_1(\tau, v) \rightarrow H_1(S, \mathbb{Z})$  is surjective. We can therefore choose a simplicial loop  $\gamma : S^1 \rightarrow \tau$  that maps to the basis element  $[t_i] \in H = \text{Hom}(H^1(S, \mathbb{Z})^\psi, \mathbb{Z})$ .

Let  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{\tau}$  be a lift of  $\gamma$  to a path in  $\tilde{\tau}$ . Then  $\tilde{\gamma}$  connects  $\tilde{v}$  to  $t_i\tilde{v}$ , for some lift  $\tilde{v} = (h, v) \in H \times V$  of  $v$ . Let  $e \in A^E$  denote the sum of edges occurring in  $\tilde{\gamma}$ , where the edges along the path are assigned weights  $\pm 1$ , and the sign changes exactly when  $\tilde{\gamma}$  makes an illegal turn (one not permitted by the switching conditions of  $\tilde{\tau}$ ). We then have

$$D^*[e] = \pm(t_i \pm 1)hv \in A^V.$$

Since  $f = (t_i^2 - 1)g$  is divisible by  $(t_i \pm 1)$ , and  $h$  is a unit in  $A$ , the factor  $\pm(t_i \pm 1)h$  above is a unit in  $A_f$ , and thus the basis element  $v$  of  $A_f^V$  is in the image of  $D_f^*$ . Since  $v \in V$  was arbitrary,  $D_f^*$  is surjective, and therefore  $D_f$  is injective, with a left inverse.

**II.** *If  $T_f$  is a free  $A_f$ -module and  $D_f$  has a left inverse, then  $\Theta = \Delta$  up to a unit in  $A_f$ .*

Indeed, if  $T_f$  is free then

$$T_f \xrightarrow{uI - P} T_f \rightarrow T(\tilde{\mathcal{L}})_f \rightarrow 0$$

presents  $T(\tilde{\mathcal{L}})_f$  as a quotient of free modules. It is not hard to check that the formation of the Fitting ideal commutes with the inversion of  $f$ , and thus  $(\Theta) \subset A_f$  is the smallest principal ideal containing the Fitting ideal of

$T(\tilde{\mathcal{L}})_f$ . From the presentation of  $T(\tilde{\mathcal{L}})_f$  above, we have  $\Theta = \det(uI - P(t))$  up to a unit in  $A_f$ .

To bring  $\Delta$  into play, note that by (left) invertibility of  $D_f$  we have an exact sequence:

$$0 \rightarrow A_f^V \xrightarrow{D_f} A_f^E \rightarrow T_f \rightarrow 0.$$

Since  $T_f$  is free, this sequence splits, and thus  $P_E$  can be expressed as a block triangular matrix with  $P_V$  and  $P$  on the diagonal. Therefore

$$\det(uI - P_V(t)) \det(uI - P(t)) = \det(uI - P_E(t)),$$

which gives  $\Theta = \Delta$  up to a unit in  $A_f$ .

**III.** *The set*

$$I' = \{f \in A : T_f \text{ is free and } D_f \text{ has a left inverse}\}$$

generates an ideal  $I \subset A$  containing  $(t_i^2 - 1)$  for  $i = 1, \dots, b$ .

Let  $f = (t_i^2 - 1)$ , so  $D_f$  has a left inverse. Then the  $|V| \times |V|$ -minors of  $D_f$  generate the ideal (1) in  $A_f$ .

Consider a typical minor  $(V \times E')$  of  $D$  with determinant  $g \neq 0$ , where  $E = E' \sqcup E''$ . Set  $h = fg$ . Then the composition

$$A_h^V \xrightarrow{D_h} A_h^E \rightarrow A_h^{E'}$$

is an isomorphism (since its determinant is now a unit). Therefore the projection  $A_h^{E''} \rightarrow T_h$  is an isomorphism, so  $T_h$  is free.

Since the minor determinants  $g$  generate the ideal (1) in  $A_f$ , we conclude that  $f = (t_i^2 - 1)$  belongs to the ideal  $I$  generated by all such  $h = fg$ .

**IV.** *There are  $a, c \in A$  such that  $\text{rad}(a) \supset I$ ,  $\text{rad}(c) \supset I$  and*

$$a\Theta = c\Delta. \tag{3.7}$$

Here  $\text{rad}(a)$  denotes the radical of the principal ideal  $(a)$ , and similarly for  $\text{rad}(c)$ . Since  $A$  is a unique factorization domain,  $\text{rad}(a)$  and  $\text{rad}(c)$  are also principal.

Write  $\Delta/\Theta = a/c \in K$  as a ratio of  $a, c \in A$  with no common factor. By definition, for any  $f \in I'$  we have  $\Theta = \Delta$  up to a unit in  $A_f$ ; therefore  $a/c = d/f^n$  for some unit  $d \in A^*$  and  $n \in \mathbb{Z}$ . Since  $\text{gcd}(a, c) = 1$ , either  $f^n \in (c)$  and  $(a) = (1)$  or vice-versa (depending on the sign of  $n$ ). In either case, we have  $f \in \text{rad}(a) \cap \text{rad}(c)$ . As  $f \in I'$  was arbitrary, the ideals  $\text{rad}(a)$  and  $\text{rad}(c)$  both contain  $I'$ , and hence they contain  $I$ .

V. We have  $\Theta = \Delta$  up to a unit in  $A$ .

Let  $(p)$  be the smallest principal ideal satisfying

$$(p) \supset I \supset (t_1^2 - 1, \dots, t_b^2 - 1)$$

(the second inclusion by (III) above). If the rank  $b$  of  $H^1(S, \mathbb{Z})^\psi$  is 2 or more, then  $\gcd(t_1^2 - 1, \dots, t_b^2 - 1) = 1$  and thus  $(p) = (1)$ . Since the principal ideals  $\text{rad}(a)$  and  $\text{rad}(c)$  contain  $I$ , they are both equal to  $(1)$  as well. Hence the constants  $a$  and  $c$  appearing in (3.7) are units in  $A$ , and we are done.

To finish, we treat the case  $b = 1$ . In this case we have  $(p) \supset (t_1^2 - 1)$ , so we can only conclude that  $\Theta = \Delta$  up to a factors of  $(t_1 - 1)$  and  $(t_1 + 1)$ .

But  $\Delta$  and  $\Theta$  have no such factors. Indeed,  $\Delta$  is a ratio of monic polynomials of positive degree in  $u$ , so it has no factor that depends only on  $t_1$ .

Similarly, if we specialize to  $(t_1, u) = (1, n)$  (by a homomorphism  $\phi : A \rightarrow \mathbb{Z}$ ), then  $P : T \rightarrow T$  becomes an endomorphism of a finitely-generated abelian group, and  $T(\mathcal{L}) = \text{Coker}(uI - P)$  specializes to the group  $K = \text{Coker}(nI - P)$ . For  $n \gg 0$ , the image of  $(uI - P)$  has finite index in  $T$ , so  $K$  is a finite group. Thus  $(\phi(\Theta)) = (n)$ , the annihilator of  $K$ ; in particular,  $\phi(\Theta) \neq 0$ . This shows  $(t_1 - 1)$  does not divide  $\Theta$ . The same argument proves  $\gcd(\Theta, t_1 + 1) = 1$ , and thus  $\Theta = \Delta$  up to a unit in  $A$ . ■

**Notes.** The train track  $\tau$  in Figure 5 provides a typical example where the module  $T(\tilde{\tau})$  is not free over  $\mathbb{Z}[H]$ . Indeed, letting  $H = H_1(S, \mathbb{Z}) \cong \mathbb{Z}^3$ , we showed in §2 that the dimension of

$$Z_1(\tau, \mathbb{C}_\rho) = \text{Hom}(T, \mathbb{C}_\rho)$$

jumps at  $\rho = (-1, -1, -1)$ , while its dimension would be constant if  $T$  were a free module. Thus  $f \in \mathbb{Z}[H]$  must vanish at  $\rho = (-1, -1, -1)$  for  $T(\tau)_f$  to be free — showing the ideal  $I$  in the proof above contains  $(t_1 + 1, t_2 + 1, t_3 + 1)$ .

## 4 Symplectic symmetry

In this section we show the characteristic polynomial of a pseudo-Anosov map  $\psi : S \rightarrow S$  is symmetric. This symmetry arises because  $\psi$  preserves a natural symplectic structure on  $\mathcal{ML}(S)$ .

We then show the Teichmüller polynomial  $\Theta_F$  packages all the characteristic polynomials of fibers  $[S] \in \mathbb{R}_+ \cdot F$ , and thus  $\Theta_F$  is also symmetric.

**Symmetry.** Let  $\lambda$  be the expanding lamination of a pseudo-Anosov mapping  $\psi : S \rightarrow S$ . The *characteristic polynomial* of  $\psi$  is given by  $p(k) = \det(kI - P)$ , where

$$P : Z_1(\lambda, \mathbb{R}) \rightarrow Z_1(\lambda, \mathbb{R})$$

is the induced map on cycles,  $P = \psi_*$ .

**Theorem 4.1** *The characteristic polynomial  $p(k)$  of a pseudo-Anosov mapping is symmetric; that is,  $p(k) = k^d p(1/k)$  where  $d = \deg(p)$ .*

**Proof.** Since  $\psi$  is pseudo-Anosov, each component of  $S - \lambda$  is an ideal polygon, possibly with one puncture. Since these polygons and their ideal vertices are permuted by  $\psi$ , we can choose  $n > 0$  such that  $\psi^n$  preserves each complementary component  $D$  of  $S - \lambda$  and fixes its ideal vertices.

By Theorem 2.2, there is a natural isomorphism  $Z_1(\lambda, \mathbb{R}) \cong Z_1(\tau, \mathbb{R})$ , where  $\tau$  is a  $\psi$ -invariant train track carrying  $\lambda$ . By [HP, Thm. 1.3.6], there exists a *complete* train track  $\tau'$  containing  $\tau$ . The train track  $\tau$  is completed to  $\tau'$  by adding a maximal set of edges joining the cusps of the complementary regions  $S - \tau$ . Since  $\psi^n$  fixes these cusps,  $\psi^n(\tau')$  is carried by  $\tau'$ .

Now recall that the vector space  $Z_1(\tau', \mathbb{R})$  can be interpreted as a tangent space to  $\mathcal{ML}(S)$ , and hence it carries a natural *symplectic form*  $\omega$ . If  $\tau'$  is orientable (which only happens on a punctured torus), then  $\omega$  is just the pullback of the intersection form on  $S$  under the natural map

$$Z_1(\tau', \mathbb{R}) \rightarrow H_1(S, \mathbb{R}).$$

If  $\tau'$  is nonorientable, then  $\omega$  is defined using the intersection pairing on a covering of  $S$  branched over the complementary regions  $S - \tau'$ ; see [HP, §3.2].

For brevity of notation, let

$$(V \subset V') = (Z_1(\tau, \mathbb{R}) \subset Z_1(\tau', \mathbb{R})),$$

and let

$$P = \psi_* : V \rightarrow V, \quad Q = (\psi^n)_* : V' \rightarrow V';$$

then  $P^n = Q|_V$ .

Both  $P$  and  $Q$  respect the symplectic form  $\omega$  on  $V'$ . If  $(V, \omega)$  is symplectic — that is, if  $\omega|_V$  is non-degenerate — then  $P$  is a symplectic matrix and the symmetry of its characteristic polynomial  $p(k)$  is immediate. Unfortunately,  $(V, \omega)$  need not be symplectic — for example,  $V$  may be odd-dimensional.

To handle the general case, we first decompose  $V'$  into generalized eigenspaces for  $Q$ ; that is, we write

$$\begin{aligned} V' \otimes \mathbb{C} &= \oplus V_\alpha \\ &= \oplus_\alpha \cup_1^\infty \text{Ker}(\alpha I - Q)^i. \end{aligned}$$

Grouping together the eigenspaces with  $|\alpha| = 1$ , we get a  $Q$ -invariant decomposition  $V' = U \oplus S$  with

$$\begin{aligned} U \otimes \mathbb{C} &= \oplus_{|\alpha|=1} V_\alpha \quad \text{and} \\ S \otimes \mathbb{C} &= \oplus_{|\alpha|\neq 1} V_\alpha. \end{aligned}$$

For  $x \in V_\alpha$  and  $y \in V_\beta$ , the fact that  $Q$  preserves  $\omega$  implies

$$\omega(x, y) = \omega(Qx, Qy) = 0$$

unless  $\alpha\beta = 1$ . Thus  $U$  and  $S$  are  $\omega$ -orthogonal, and therefore  $(U, \omega)$  and  $(S, \omega)$  are both symplectic.

Since  $\psi^n$  fixes all the edges in  $\tau' - \tau$ ,  $Q$  acts by the identity on  $V'/V$ . Therefore  $S$  is a subspace of  $V$ , and

$$V = S \oplus (U \cap V) = S \oplus W.$$

Since  $P^n = Q$ , the splitting  $V = S \oplus W$  is preserved by  $P$ ;  $P|_S$  is symplectic; and the eigenvalues of  $P|_W$  are roots of unity. Therefore

$$p(k) = \det(kI - P|_S) \cdot \det(kI - P|_W).$$

The first term is symmetric because  $P|_S$  is a symplectic matrix, and the second term is symmetric because the eigenvalues of  $P|_W$  lie on  $S^1$  and are symmetric about the real axis. Thus  $p(k)$  is symmetric.  $\blacksquare$

**Characteristic polynomials of fibers.** We now return to the study of the Teichmüller polynomial  $\Theta_F = \sum a_g \cdot g \in \mathbb{Z}[G]$ . Given  $\phi \in H^1(M, \mathbb{Z}) = \text{Hom}(G, \mathbb{Z})$ , we obtain a polynomial in a single variable  $k$  by setting

$$\Theta_F(k^\phi) = \sum a_g k^{\phi(g)}.$$

Recall that  $\mathcal{L}$  denotes the mapping torus of the expanding lamination  $\lambda$  of any fiber  $[S] \in \mathbb{R}_+ \cdot F$  (Corollary 3.2); and  $\mathcal{L}$  is transversally orientable iff  $\lambda$  is.

**Theorem 4.2** *The characteristic polynomial of the monodromy of a fiber  $[S] = \phi \in \mathbb{R}_+ \cdot F$  is given by*

$$p(k) = \Theta_F(k^\phi) \cdot \begin{cases} (k-1) & \text{if } \mathcal{L} \text{ is transversally orientable,} \\ 1 & \text{otherwise,} \end{cases}$$

up to a unit  $\pm k^n$ .

**Proof.** Let  $t_i, u \in G$  be a basis adapted to the splitting  $G = H \oplus \mathbb{Z}$  determined by the choice of a lift of the monodromy,  $\tilde{\psi} : \tilde{S} \rightarrow \tilde{S}$ . Then  $\phi(t_i) = 0$  and  $\phi(u) = 1$ , so  $k^\phi : G \rightarrow \mathbb{C}^*$  has coordinates  $(t, u) = (1, k) \in \widehat{G}$ . Thus

$$\begin{aligned} \Theta_F(k^\phi) &= \Theta_F(1, u)|_{u=k} \\ &= \det(kI - P_E(1)) / \det(kI - P_V(1)) \end{aligned}$$

by the determinant formula (3.5).

Applying the functor  $\text{Hom}(\cdot, \mathbb{R})$  to the commutative diagram (3.4), with  $t = 1$ , we obtain the adjoint diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_1(\tau, \mathbb{R}) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(1)^*} & \mathbb{R}^V & \longrightarrow & \mathbb{R}^m & \longrightarrow & 0 \\ & & \downarrow P(1)^* & & \downarrow P_E(1)^* & & \downarrow P_V(1)^* & & \downarrow \text{id} & & \\ 0 & \longrightarrow & Z_1(\tau, \mathbb{R}) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(1)^*} & \mathbb{R}^V & \longrightarrow & \mathbb{R}^m & \longrightarrow & 0. \end{array}$$

Here  $m = 1$  if  $\mathcal{L}$  (and hence  $\tau$ ) is orientable, and  $m = 0$  otherwise (compare Corollary 2.4).

Since the rows of the diagram above are exact, the characteristic polynomial of  $P = P(1)^*$  is given by the alternating product

$$p(k) = \frac{\det(kI - P_E(1)) (k-1)^m}{\det(kI - P_V(1))} = \Theta_F(k^\phi) (k-1)^m.$$

■

**Corollary 4.3** *The Teichmüller polynomial is symmetric; that is,*

$$\Theta_F = \sum a_g \cdot g = \pm h \sum a_g \cdot g^{-1}$$

for some unit  $\pm h \in \mathbb{Z}[G]$ .

**Proof.** Since  $\mathbb{R}_+ \cdot F \subset H^1(M, \mathbb{R})$  is open, we can choose  $[S] = \phi \in \mathbb{R}_+ \cdot F$  such that the values  $\phi(g)$  over the finite set of  $g$  with  $a_g \neq 0$  are all distinct. Then symmetry of  $\Theta_F$  follows from symmetry of the characteristic polynomial  $p(k) = \Theta_F(k^\phi) = \sum a_g k^{\phi(g)}$ . ■

**Notes.** Although the characteristic polynomial  $f(u) = \det(uI - P)$  of a pseudo-Anosov mapping  $\psi$  is always symmetric,  $f(u)$  may factor over  $\mathbb{Z}$  into a product of non-symmetric polynomials. In particular, the minimal polynomial of a pseudo-Anosov expansion factor  $K > 1$  need *not* be symmetric. For example, the largest root  $K = 1.83929\dots$  of the non-symmetric polynomial  $x^3 - x^2 - x - 1$  is a pseudo-Anosov expansion factor; see [AY], [Fr4, §5].

## 5 Expansion factors

In this section we study the expansion factor  $K(\phi)$  for a cohomology class  $\phi \in \mathbb{R}_+ \cdot F$ , and prove it is strictly convex and determined by  $\Theta_F$ .

**Definitions.** Let  $[S] = \phi \in \mathbb{R}_+ \cap F$  be a fiber with monodromy  $\psi$  and expanding measured lamination  $\Lambda \in \mathcal{ML}(S)$ . The *expansion factor*  $K(\phi) > 1$  is the expanding eigenvalue of  $\psi : \mathcal{ML}(S) \rightarrow \mathcal{ML}(S)$ ; that is, the constant such that

$$\psi \cdot \Lambda = K(\phi)\Lambda.$$

The function

$$L(\phi) = \log K(\phi)$$

gives the *Teichmüller length* of the unique geodesic loop in the moduli space of Riemann surfaces represented by

$$\psi \in \text{Mod}(S) \cong \pi_1(\mathcal{M}_{g,n}).$$

(Compare [Bers].)

**Theorem 5.1** *The expansion factor satisfies*

$$K(\phi) = \sup \{k > 1 : \Theta_F(k^\phi) = 0\} \tag{5.1}$$

for any fiber  $[S] = \phi \in \mathbb{R}_+ \cdot F$ .

**Proof.** By Theorem 4.2,  $p(k) = \Theta_F(k^\phi)$  is the characteristic polynomial of the map

$$P : Z_1(\lambda, \mathbb{R}) \rightarrow Z_1(\lambda, \mathbb{R})$$

determined by monodromy of  $S$ , and the largest eigenvalue of  $P$  is  $K(\phi)$ , with eigenvector the expanding measure associated to  $\Lambda$ . ■

Since the right-hand side of (5.1) is defined for real cohomology classes, we will use it to extend the definition of  $K(\phi)$  and  $L(\phi)$  to the entire cone  $\mathbb{R}_+ \cdot F$ . Then we have the homogeneity properties:

$$\begin{aligned} K(a\phi) &= K(\phi)^{1/a}, \\ L(a\phi) &= a^{-1}L(\phi). \end{aligned}$$

Here is a useful fact established in [Fr2, Thm. F].

**Theorem 5.2 (Fried)** *The expansion factor  $K(\phi)$  is continuous on  $F$  and tends to infinity as  $\phi \rightarrow \partial F$ .*

Next we derive some convexity properties of the expansion factor. These properties are illustrated in Figure 8 of §11.

**Theorem 5.3** *For any  $k > 1$ , the level set*

$$\Gamma = \{\phi \in \mathbb{R}_+ \cdot F : K(\phi) = k\}$$

*is a convex hypersurface with  $\mathbb{R}_+ \cdot \Gamma = \mathbb{R}_+ \cdot F$ .*

**Proof.** By homogeneity,  $\Gamma$  meets every ray in  $\mathbb{R}_+ \cdot F$ , and thus  $\mathbb{R}_+ \Gamma = \mathbb{R}_+ \cdot F$ . For convexity, it suffices to consider the level set  $\Gamma$  where  $\log K(\phi) = 1$ .

Choose a fiber  $[S] \in \mathbb{R}_+ \cdot F$  and a lift  $\tilde{\psi}$  of its monodromy. Then we obtain a splitting  $H^1(M, \mathbb{R}) = H^1(S, \mathbb{R})^\psi \oplus \mathbb{R}$  and associated coordinates  $(s, y)$  on  $H^1(M, \mathbb{R})$  and  $(t, u) = (e^s, e^y)$  on  $\widehat{G} = \exp H^1(M, \mathbb{R})$ .

By the determinant formula (3.5),  $\Theta_F(t, u)$  is the ratio between the characteristic polynomials of  $P_E(t)$  and  $P_V(t)$ . By Theorem 3.4,  $P_E(t)$  is a Perron-Frobenius matrix of Laurent polynomials; let  $E(t) > 1$  denote its leading eigenvalue for  $t \in \mathbb{R}_+^b$ . Since  $P_V(t)$  is simply a permutation matrix, we have  $\Theta_F(t, E(t)) = 0$  for all  $t$ . By Theorem A.1 of the Appendix,  $y = \log E(e^s)$  is a convex function of  $s$ , so its graph  $\Gamma'$  is convex.

To complete the proof, we show  $\Gamma' = \Gamma$ . First note that  $\Gamma' \subset \Gamma$ . Indeed, if  $\phi = (s, y) \in \Gamma'$ , then  $\Theta_F(e^s, e^y) = 0$  and so  $K(\phi) \geq e$ . But by Theorem A.1, the ray  $\mathbb{R}_+ \cdot \phi$  meets  $\Gamma'$  at most once; since  $u = E(t)$  is the *largest* zero of  $\Theta_F(t, u)$ , we have  $K(\phi) = e$ , and thus  $(s, u) \in \Gamma$ .

Since  $\Gamma'$  is a graph over  $H^1(S, \mathbb{R})$ , it is properly embedded in  $H^1(M, \mathbb{R})$ ; but  $\Gamma$  is connected, so  $\Gamma = \Gamma'$ . ■



**Corollary 5.4** *The function  $y = 1/\log K(\phi)$  on the cone  $\mathbb{R}_+ \cdot F$  is real-analytic, strictly concave, homogeneous of degree 1, and*

$$y(\phi) \rightarrow 0 \quad \text{as } \phi \rightarrow \partial F.$$

**Proof.** The homogeneity of  $y(\phi)$  follows from that of  $K(\phi)$ .

Let  $\Gamma$  be the convex hypersurface on which  $\log K(\phi) = 1$ . Since  $\Gamma$  is a component of the analytic set  $\Theta_F(e^\phi) = 0$ , and  $K(\phi)$  is homogeneous,  $K(\phi)$  is real-analytic.

To prove concavity, let  $\phi_3 = \alpha\phi_1 + (1 - \alpha)\phi_2$  be a convex combination of  $\phi_1, \phi_2 \in \mathbb{R}_+ \cdot F$ , and let  $y_i = 1/\log K(\phi_i)$ , so  $y_i^{-1}\phi_i \in \Gamma$ . By convexity of  $\Gamma$ , the segment  $[y_1^{-1}\phi_1, y_2^{-1}\phi_2]$  meets the ray through  $\phi_3$  at a point  $p$  which is farther from the origin than  $y_3^{-1}\phi_3$ . Since

$$p = \frac{\alpha y_1 (y_1^{-1} \phi_1) + (1 - \alpha) y_2 (y_2^{-1} \phi_2)}{\alpha y_1 + (1 - \alpha) y_2} = \frac{\phi_3}{\alpha y_1 + (1 - \alpha) y_2},$$

we find

$$y_3^{-1} \leq (\alpha y_1 + (1 - \alpha) y_2)^{-1}$$

and therefore  $y(\phi)$  is concave.

Finally  $y(\phi)$  converges to zero at  $\partial F$  by Theorem 5.2, so by real-analyticity it must be strictly concave. ■

### Notes.

1. The concavity of  $1/\log K(\phi)$  was established by Fried; see [Fr2, Thm. E], [Fr4, Prop. 8], as well as [LO] and [Mat]. Our proof of concavity is rather different and uses only general properties of Perron-Frobenius matrices (presented in the Appendix).
2. By Corollary 5.4, the expansion factor  $K(\phi)$  assumes its minimum at a unique point  $\phi \in F$ , providing a *canonical center* for any fibered face of the Thurston norm ball.

**Question.** Is the minimum always achieved at a *rational* cohomology class?

## 6 The Thurston norm

Let  $F \subset H^1(M, \mathbb{R})$  be a fibered face of the Thurston norm ball. In this section we use the fact that  $K(\phi)$  blows up at  $\partial F$  to show one can compute

the cone  $\mathbb{R}_+ \cdot F$  from the polynomial  $\Theta_F$ . This observation is conveniently expressed in terms of a second norm on  $H^1(M, \mathbb{R})$  attached to  $\Theta_F$ .

**Norms and Newton polygons.** Write the Teichmüller polynomial  $\Theta_F \in \mathbb{Z}[G]$  as

$$\Theta_F = \sum a_g \cdot g.$$

The *Newton polygon*  $N(\Theta_F) \subset H_1(M, \mathbb{R})$  is the convex hull of the finite set of integral homology classes  $g$  with  $a_g \neq 0$ . We define the *Teichmüller norm* of  $\phi \in H^1(M, \mathbb{R})$  (relative to  $F$ ) by:

$$\|\phi\|_{\Theta_F} = \sup_{a_g \neq 0 \neq a_h} \phi(g - h).$$

The norm of  $\phi$  measures the length of the projection of the Newton polygon,  $\phi(N(\Theta_F)) \subset \mathbb{R}$ . Multiplication of  $\Theta_F$  by a unit just translates  $N(\Theta_F)$ , so the Teichmüller norm is well-defined.

**Theorem 6.1** *For any fibered face  $F$  of the Thurston norm ball, there exists a face  $D$  of the Teichmüller norm ball,*

$$D \subset \{\phi : \|\phi\|_{\Theta_F} = 1\},$$

*such that  $\mathbb{R}_+ \cdot F = \mathbb{R}_+ \cdot D$ .*

**Proof.** Pick a fiber  $[S] \in \mathbb{R}_+ \cdot F$  with monodromy  $\psi$ . Choose coordinates  $(t, u) = (e^s, e^y)$  on

$$H^1(M, \mathbb{R}_+) \cong \exp(H^1(S, \mathbb{R})^\psi \oplus \mathbb{R}),$$

and let  $E(t)$  be the leading eigenvalue of the Perron-Frobenius matrix  $P_E(t)$ . As we saw in §5, we have  $\mathbb{R}_+ \cdot \Gamma = \mathbb{R}_+ \cdot F$ , where  $\Gamma$  is the graph of the function

$$y = f(s) = \log E(e^s).$$

Now the determinant formula (3.5) shows  $\Theta_F(t, u)$  is a factor of  $\det(uI - P_E(t))$  with  $\Theta_F(t, E(t)) = 0$ , so by Theorem A.1(C) of the Appendix,  $\mathbb{R}_+ \cdot \Gamma$  coincides with the dual cone  $C(u^d)$  of the leading term  $u^d$  of  $\Theta_F(t, u)$ . Equivalently,  $\mathbb{R}_+ \cdot \phi$  meets the graph of  $f(s)$  iff  $\phi$  achieves its maximum on  $N(\Theta_F)$  at the vertex  $v \in N(\Theta_F)$  corresponding to  $u^d$ .

Since  $\Theta_F$  is symmetric (Corollary 4.3), so is its Newton polygon, and thus the unit ball  $B$  of the Teichmüller norm is dual to the convex body  $N(\Theta_F)$ . Under this duality, the linear functionals  $\phi$  achieving their maximum at  $v$  correspond to the cone over a face  $D \subset B$ ; and therefore

$$\mathbb{R}_+ \cdot F = C(u^d) = \mathbb{R}_+ \cdot D.$$

■

**Skew norms.** Although in some examples the Thurston and Teichmüller norms actually agree (see §11), in general the norm faces  $F$  and  $D$  of Theorem 6.1 are skew to one another.

Here is a construction showing that  $F$  and  $D$  carry different information in general. Let  $\lambda \subset S$  be the expanding lamination of a pseudo-Anosov mapping  $\psi$ , and let  $\mathcal{L} \subset M$  be its mapping torus. Assume  $b_1(M) \geq 2$ .

Assume moreover that  $\psi$  has a fixed-point  $x$  in the center of an ideal  $n$ -gon of  $S - \lambda$ , with  $n \geq 3$ . (In the measured foliation picture,  $x$  is an  $n$ -prong singularity.) Then the mapping torus of  $x$  gives an oriented loop  $X \subset M$  transverse to  $S$ . Construct a 3-dimensional submanifold

$$M' \xrightarrow{i} M$$

by removing a tubular neighborhood of  $X \subset M$ , small enough that we still have  $\mathcal{L} \subset M'$ . Let  $S' = S \cap M'$ ; it is a fiber of  $M'$ .

Let  $F$  and  $F'$  be the faces of the Thurston norm balls whose cones contain  $[S]$  and  $[S']$ . We wish to compare the norms of  $\phi$  and  $\phi' = i^*(\phi)$  for  $\phi \in \mathbb{R}_+ \cdot F$ .

First, the Teichmüller norms agree: that is,

$$\|\phi'\|_{\Theta_{F'}} = \|\phi\|_{\Theta_F}. \quad (6.1)$$

Indeed, the mapping torus of the expanding lamination is  $\mathcal{L}' = \mathcal{L}$  for both  $M'$  and  $M$ , and therefore  $i_*(\Theta_{F'}) = \Theta_F$ , which gives (6.1).

On the other hand, the Thurston norms satisfy

$$\|\phi'\|_T = \|\phi\|_T + \phi(X). \quad (6.2)$$

Indeed, let  $[R] = \phi$  be a fiber in  $M$  and let  $[R'] = [R \cap M']$  be the corresponding fiber in  $M'$ . Then we have

$$\begin{aligned} \|\phi'\|_T &= |\chi(R')| = |\chi(R - X)| = |\chi(R)| + |R \cap X| \\ &= \|\phi\|_T + \phi(X). \end{aligned}$$

By (6.1) and (6.2), the Teichmüller and Thurston norms can agree on at most one of the cones  $\mathbb{R}_+ \cdot F$  and  $\mathbb{R}_+ \cdot F'$ . With an appropriate choice of  $X$ , one can construct examples where the Thurston norm is not even a constant multiple of the Teichmüller norm on  $\mathbb{R}_+ \cdot F$ .

**Notes.**

1. Theorem 6.1 provides an effective algorithm to determine a fibered face  $F$  of  $M$  from a single fiber  $S$  and its monodromy  $\psi$ .

The first step is to find a  $\psi$ -invariant train track  $\tau$ . Bestvina and Handel have given an elegant algorithm to find such a train track, based on entropy reduction [BH]. Versions of this algorithm have been implemented by T. White, B. Menasco – J. Ringland, T. Hall and P. Brinkman; see [Bri].

Once  $\tau$  is found, it is straightforward to compute the matrices  $P_E(t)$  and  $P_V(t)$  giving the action of  $\tilde{\psi}$  on  $\tilde{\tau}$ . The determinant formula

$$\Theta_F(t, u) = \det(uI - P_E(t)) / \det(uI - P_V(t))$$

then gives the Teichmüller polynomial for  $F$ , and the Newton polygon of  $\Theta_F$  determines the cone  $\mathbb{R}_+ \cdot F$  as we have seen above. Finally  $F$  itself can be recovered as the intersection of  $\mathbb{R}_+ \cdot F$  with the unit sphere  $\|\phi\|_A = 1$  in the *Alexander norm* on  $H^1(M, \mathbb{R})$  (see §7).

2. For any fiber  $[S] \in \mathbb{R}_+ \cdot F$  with expanding lamination  $\lambda$ , we have

$$\|[S]\|_{\Theta_F} = -\chi(\lambda),$$

where the Euler characteristic is computed with Čech cohomology. To verify this equation, use the determinant formula for  $\Theta_F$  and observe that  $\chi(\lambda) = \chi(\tau) = |V| - |E|$ .

## 7 The Alexander norm

In this section we show that a fibered face  $F$  can be computed from the Alexander polynomial of  $M$  when  $\lambda$  is transversely orientable.

**The Alexander polynomial and norm.** Assume  $b_1(M) > 1$ , let  $G = H_1(M, \mathbb{Z})/\text{torsion}$ , and let  $\hat{G} = \text{Hom}(G, \mathbb{C}^*)$ .

Recall that the Teichmüller polynomial of a fibered face defines, via its zero set, the largest hypersurface  $V \subset \hat{G}$  such  $\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0$  for all  $\rho \in V$  (Theorem 3.3). Similarly, the *Alexander polynomial* of  $M$ ,

$$\Delta_M = \sum a_g \cdot g \in \mathbb{Z}[G],$$

defines the largest hypersurface on which  $\dim H^1(M, \mathbb{C}_\rho) > 0$ . (See [Mc, Cor 3.2]). The *Alexander norm* on  $H^1(M, \mathbb{R})$  is defined by

$$\|\phi\|_A = \sup_{a_g \neq 0 \neq a_h} \phi(g - h).$$

(By convention,  $\|\phi\|_A = 0$  if  $\Delta_M = 0$ .)

**Theorem 7.1** *Let  $F$  be a fibered face in  $H^1(M, \mathbb{R})$  with  $b_1(M) \geq 2$ . Then we have:*

1.  $F \subset A$  for a unique face  $A$  of the Alexander norm ball, and
2.  $F = A$  and  $\Delta_M$  divides  $\Theta_F$  if the lamination  $\mathcal{L}$  associated to  $F$  is transversally orientable.

**Remark.** Transverse orientability of  $\mathcal{L}$  is equivalent to transverse orientability of the invariant lamination  $\lambda \subset S$  for a fiber  $S \in \mathbb{R}_+ \cdot F$ , together with the condition that that monodromy preserves the orientation of  $\lambda$ .

**Proof of Theorem 7.1.** In [Mc] we show

$$\|\phi\|_A \leq \|\phi\|_T$$

for all  $\phi \in H^1(M, \mathbb{R})$ , with equality if  $\phi$  comes from a fibration  $M \rightarrow S^1$ ; this gives part (1) of the Theorem.

For part (2), pick a fiber  $[S] \in \mathbb{R}_+ \cdot F$  with monodromy  $\psi$  and invariant lamination  $\lambda$ . Let  $(t, u)$  be coordinates on the character variety  $\widehat{G}$  adapted to the splitting  $G = H \oplus \mathbb{Z}$  coming from the choice of a lift  $\tilde{\psi}$  of  $\psi$ .

If  $\mathcal{L}$  is transversally orientable, then  $\lambda$  is carried by an orientable train track  $\tau$ . Since  $\tau$  fills the surface  $S$ , we obtain a surjective map:

$$\pi : Z_1(\tau, \mathbb{C}_t) \cong H_1(\tau, \mathbb{C}_t) \twoheadrightarrow H_1(S, \mathbb{C}_t) \quad (7.1)$$

for any character  $t \in \widehat{H}$ .

Let  $P(t)$  and  $Q(t)$  denote the action of  $\tilde{\psi}$  on  $Z_1(\tau, \mathbb{C}_t)$  and  $H_1(S, \mathbb{C}_t)$  respectively. Fixing a nontrivial character  $t$ , we have

$$\Delta_M(t, u) = \det(uI - Q(t)) \quad \text{and} \quad \Theta_F(t, u) = \det(uI - P(t))$$

up to a unit in  $\mathbb{Z}[G]$ . By (7.1),  $\Delta_M(t, u)$  is a divisor of  $\Theta_F(t, u)$ . It follows that  $\Delta_M$  divides  $\Theta_F$  (using an algebraic argument as in §3 to lift the divisibility to  $\mathbb{Z}[G]$ ).

The action of  $\psi$  on  $\text{Ker}(\pi)$  corresponds to the action of  $\psi$  by permutations on the components of  $S - \tau$ , so it does not include the leading eigenvalue  $E(t)$  of  $P(t)$ . Therefore  $\Delta_M(t, E(t)) = 0$ , so we can apply Theorem A.1(C) of the Appendix to conclude that there is a face  $A$  of the Alexander norm ball with  $\mathbb{R}_+ \cdot A = \mathbb{R}_+ \cdot F$  (just as in Theorem 6.1). By (1) we have  $F \subset A$ , and therefore  $F = A$ . ■

**Note.** Dunfield has given an example where the fibered face  $F$  is a *proper* subset of the Alexander face  $A$ ; see [Dun].

## 8 Twisted measured laminations

In this section we add another interpretation to the Teichmüller polynomial, by showing  $\Theta_F$  determines the eigenvalues of  $\psi \in \text{Mod}(S)$  on the space of twisted (or affine) measured laminations  $\mathcal{ML}_s(S)$ . We will establish:

**Theorem 8.1** *A pseudo-Anosov mapping  $\psi : S \rightarrow S$  has a unique pair of fixed-points*

$$[\Lambda_+], [\Lambda_-] \in \mathbb{P}\mathcal{ML}_s(S)$$

for any  $s \in H^1(S, \mathbb{R})^\psi$ . The supporting geodesic laminations  $(\lambda_+, \lambda_-)$  of  $(\Lambda_+, \Lambda_-)$  coincide with the expanding and contracting laminations of  $\psi$  respectively, and we have

$$\psi \cdot \Lambda_+ = k\Lambda_+$$

where  $k > 0$  is the largest root of the equation  $\Theta_F(e^s, k) = 0$ .

We emphasize that the action of  $\psi$  on  $\mathcal{ML}_s(S)$  depends on a choice of lift  $\tilde{\psi}$ , which make  $\Theta_F$  into a function of two variables as in the preceding section.  $\mathcal{ML}_s(S)$ . Fix a cohomology class  $s \in H^1(S, \mathbb{R})$ . We can interpret  $s$  as a homomorphism

$$s : H_1(S, \mathbb{Z}) \rightarrow \mathbb{R},$$

determining an element  $t \in H^1(S, \mathbb{R}_+)$  by

$$t = e^s : H_1(S, \mathbb{Z}) \rightarrow \mathbb{R}_+ = SL_1(\mathbb{R}).$$

Thus  $s$  (or  $t$ ) gives  $\mathbb{R}$  the structure of a module  $\mathbb{R}_s$  (or  $\mathbb{R}_t$ ) over the ring  $\mathbb{Z}[H_1(S, \mathbb{Z})]$ .

The space of *twisted measured laminations*,  $\mathcal{ML}_s(S)$ , is the set of all  $\Lambda = (\lambda, \mu)$  such that:

- $\lambda \subset S$  is a compact geodesic lamination,
- $\mu \in Z_1(\lambda, \mathbb{R}_s)$  is a cycle, and
- $\mu(T) > 0$  for every nonempty transversal  $T$  to  $\lambda$ .

Here  $\mu$  can be thought of as a transverse measure taking values in a fixed flat  $\mathbb{R}$ -bundle  $L_s \rightarrow S$ . For  $s = 0$ , the bundle  $L_s$  is trivial, so  $\mathcal{ML}_0(S)$  reduces to the space of ordinary measured laminations  $\mathcal{ML}(S)$ . Let  $\mathbb{P}\mathcal{ML}_s(S) = \mathcal{ML}_s(S)/\mathbb{R}_+$  denote the projective space of rays in  $\mathcal{ML}_s(S)$ .

Using train tracks, one can give  $\mathcal{ML}_s(S)$  local charts and a topology. A basic result from [HO] is:

**Theorem 8.2 (Hatcher-Oertel)** *The spaces  $\mathcal{ML}_s(S)$  form a fiber bundle over  $H^1(M, \mathbb{R}_+)$ . In particular,  $\mathcal{ML}_s(S) \cong \mathbb{R}^n$  for all  $s$ .*

**Perron-Frobenius eigenvectors.** Let  $\psi : S \rightarrow S$  be a pseudo-Anosov mapping with monodromy  $\psi$  and expanding lamination  $\lambda$  carried by an invariant train track  $\tau$ . Fix a lift  $\tilde{\psi}$  of  $\psi$  to  $\tilde{S}$ . As in (3.4), we then obtain a matrix

$$P_E(t) : \mathbb{Z}[H]^E \rightarrow \mathbb{Z}[H]^E$$

describing the action of  $\tilde{\psi}$  on the edges of  $\tilde{\tau}$ , and  $P_E(t)$  is a Perron-Frobenius matrix of Laurent polynomials by Theorem 3.4. We can think of  $P_E(t)$  as a map

$$P_E : H^1(S, \mathbb{R}_+)^{\psi} \rightarrow \text{End}(\mathbb{R}^E),$$

whose values are traditional Perron-Frobenius matrices over  $\mathbb{R}$ .

As in §4, we can apply the functor  $\text{Hom}(\cdot, \mathbb{R}_t)$  to (3.4) to obtain the adjoint diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_1(\tau, \mathbb{R}_t) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(t)^*} & \mathbb{R}^V \\ & & \downarrow P(t)^* & & \downarrow P_E(t)^* & & \downarrow P_V(t)^* \\ 0 & \longrightarrow & Z_1(\tau, \mathbb{R}_t) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(t)^*} & \mathbb{R}^V. \end{array} \quad (8.1)$$

For each  $t$ , the largest eigenvalue  $E(t)$  of  $P_E(t)^*$  is positive and simple, with a positive eigenvector  $\mu(t) \in \mathbb{R}^E$ .

**Theorem 8.3** *For each  $t \in H^1(S, \mathbb{R}_+)$ , the leading eigenvalue  $u = E(t)$  of  $P_E(t)^*$  is the largest root of the polynomial equation*

$$\Theta_F(t, u) = 0,$$

*and its positive eigenvector  $\mu(t)$  belongs to  $Z_1(\tau, \mathbb{R}_t)$ .*

**Proof.** First suppose  $t = 1$  is the trivial cohomology class. Then  $P_E(1)$  is an integral Perron-Frobenius matrix, and hence  $u = E(1) > 1$  is the largest root of the polynomial  $\det(uI - P_E(1))$ . On the other hand,  $P_V(1)$  is a permutation matrix, with eigenvalues on the unit circle, so  $\det(uI - P_V(1))$  has no root at  $u = E(1)$ . Since Theorem 3.6 expresses  $\Theta_F(1, u)$  as the ratio of these two determinants,  $E(1)$  is the largest root of the polynomial  $\Theta_F(1, u) = 0$ .

To see  $\mu(1)$  is a cycle, just note that  $D(1)^*\mu(1) = 0$  because (8.1) is commutative and  $P_V(1)$  has no eigenvector with eigenvalue  $E(1)$ .

The same reasoning applies whenever  $E(t)$  is not an eigenvalue of  $P_V(t)$ , and thus the Theorem holds for generic  $t$ . By continuity, it holds for all  $t \in H^1(S, \mathbb{R}_+)$ . ■

**Proof of Theorem 8.1.** Suppose  $\psi \cdot \Lambda = E\Lambda$ . As we saw in Corollary 3.2, the only possibilities for the support of  $\Lambda$  are the expanding and contracting geodesic laminations  $\lambda_+, \lambda_-$  of  $\psi$ . In the case  $\Lambda = (\lambda_+, \mu)$ , positivity of  $\mu$  on transversals implies  $\mu$  is a positive eigenvector of  $P_E(t)^*$ ,  $t = e^s$ , under the isomorphism

$$Z_1(\lambda_+, \mathbb{R}_t) = Z_1(\tau, \mathbb{R}_t).$$

Since  $P_E(t)^*$  is a Perron-Frobenius matrix, its positive eigenvector is unique up to scale, and thus  $k = E(t)$ . By Theorem 8.3,  $k$  is the largest root of  $\Theta_F(t, k) = \Theta_F(e^s, k) = 0$ . ■

**Corollary 8.4** *Let  $k(s)$  be the eigenvalue of*

$$\psi : \mathcal{ML}_s(S) \rightarrow \mathcal{ML}_s(S)$$

*at  $\Lambda_+$ . Then  $\log k(s)$  is a convex function on  $H^1(S, \mathbb{R})^\psi$ .*

**Proof.** Apply Theorem A.1 of the Appendix. ■

**Notes.**

1. It can happen that  $\psi \cdot \Lambda_+ = k(s)\Lambda_+$  with  $0 < k(s) < 1$ , even though  $\Lambda_+ \in \mathcal{ML}_s(S)$  is supported on the expanding lamination of  $\psi$ . Indeed,  $k(s)$  depends on the choice of a lift  $\tilde{\psi}$  of  $\psi$ , and changing this lift by  $h \in H$  changes  $k(s)$  to  $e^{\phi(h)}k(s)$ .
2. **Question.** Given a Riemann surface  $X \in \text{Teich}(S)$ , is there a natural isomorphism  $\mathcal{ML}_s(S) \cong Q_s(X)$  between the space of twisted measured laminations and the space of twisted quadratic differentials, defined as holomorphic sections of  $K(X)^2 \otimes L_s$ ? Hubbard and Masur established this correspondence in the untwisted case [HM].
3. The existence of a fixed-point for  $\psi$  on  $\mathcal{ML}_s(S)$  is also shown in [Oe2, Prop. 2.3].



## 9 Teichmüller flows

We now turn to the study of measured foliations  $\mathcal{F}$  of  $M$ .

Assume  $M$  is oriented and  $\mathcal{F}$  is transversally oriented; then the leaves of  $\mathcal{F}$  are also oriented. Measured foliations so oriented correspond bijectively to closed, nowhere-vanishing 1-forms  $\omega$  on  $M$ , and we let  $[\mathcal{F}] = [\omega] \in H^1(M, \mathbb{R})$ . A flow  $f : M \times \mathbb{R} \rightarrow M$  has *unit speed* (relative to  $\mathcal{F}$ ) if it is generated by a vector field  $v$  with  $\omega(v) = 1$ . Such a flow preserves the foliation  $\mathcal{F}$  and its transverse measure.

In this section we prove:

**Theorem 9.1** *Let  $F$  be a fibered face of the Thurston norm ball for  $M$ . Then any  $\phi \in \mathbb{R}_+ \cdot F$  determines:*

- a measured foliation  $\mathcal{F}$  of  $M$  with  $[\mathcal{F}] = \phi$ ,
- a complex structure  $J$  on the leaves of  $\mathcal{F}$ , and
- a unit-speed Teichmüller flow

$$f : (M, \mathcal{F}) \times \mathbb{R} \rightarrow (M, \mathcal{F})$$

with stretch factor  $K(f_t) = K(\phi)^{|t|}$ .

The data  $(\mathcal{F}, J, f)$  is unique up to isotopy.

The idea of the proof is to use the results on twisted measured laminations in §8 to construct the analytic structure  $(\mathcal{F}, J, f)$  from the purely combinatorial information provided by the cohomology class  $\phi$ .

**From measured laminations to quadratic differentials.** As usual we choose a fiber  $[S] \in \mathbb{R}_+ \cdot F$  with monodromy  $\psi$  and expanding and contracting laminations  $\lambda_{\pm}$ . Choose a lift  $\tilde{\psi}$  of  $\psi$  to the  $H$ -covering space  $\tilde{S}$  of  $S$ , and write

$$G = H_1(M, \mathbb{Z})/\text{torsion} = H \oplus \mathbb{Z}\tilde{\psi}.$$

Let  $G$  act on  $\tilde{S}$  by

$$(h, i) \cdot s = \tilde{\psi}^i(h(s));$$

this action embeds  $G$  into the mapping-class group  $\text{Mod}(\tilde{S})$ .

**Theorem 9.2** *There exist measured laminations  $\tilde{\Lambda}_{\pm} \in \mathcal{ML}(\tilde{S})$ , supported on  $\tilde{\lambda}_{\pm}$ , such that for all  $g \in G$  we have*

$$g \cdot \tilde{\Lambda}_{\pm} = K^{\pm\phi(g)} \tilde{\Lambda}_{\pm}, \tag{9.1}$$

where  $K = K(\phi)$  is the expansion factor of  $\phi$ .

**Proof.** Writing  $\phi = (s, y)$ , the condition  $K = K(\phi)$  means  $y > 0$  is the largest solution to the equation  $\Theta_F(K^s, K^y) = 0$ . By Theorem 8.1 there exists a twisted measured lamination  $\Lambda_+ \in \mathcal{ML}_{s \log K}(S)$ , supported on  $\lambda_+$ , with  $\psi \cdot \Lambda_+ = K^y \Lambda_+$ . The lift of  $\Lambda_+$  to  $\tilde{S}$  then gives a lamination  $\tilde{\Lambda}_+$  satisfying (9.1).

To construct  $\Lambda_-$ , note that  $K(\phi) = K(-\phi)$  because the expansion and contraction factors of a pseudo-Anosov mapping are reciprocal. Thus the same construction applied to  $-\phi$  yields  $\tilde{\Lambda}_-$  satisfying (9.1). ■

Although  $\text{int}(\tilde{S})$  has infinite topological complexity, it has a natural quasi-isometry type coming from the lift of a finite volume hyperbolic metric on  $\text{int}(S)$ . Complex structures compatible with this quasi-isometry type are parameterized by the (infinite-dimensional) Teichmüller space  $\text{Teich}(\tilde{S})$ .

**Theorem 9.3** *There is a Riemann surface  $X \in \text{Teich}(\tilde{S})$  and a holomorphic quadratic differential  $q(z) dz^2$  on  $X$  such that:*

1.  $G \subset \text{Mod}(\tilde{S})$  acts by commuting Teichmüller mappings  $g(x)$  on  $X$ , preserving the foliations of  $q$ , and
2. The map  $g(x)$  stretches the vertical and horizontal leaves of  $q$  by  $(K^{-\phi(g)}, K^{+\phi(g)})$ , where  $K = K(\phi)$ .

**Proof.** Integrating the transverse measures on  $\tilde{\Lambda}_\pm$ , we will collapse their complementary regions and obtain a map  $f : \tilde{S} \rightarrow X$ .

On any small open set  $U_\alpha \subset \tilde{S}$ , we can introduce local coordinates  $(u, v)$  such that  $u$  and  $v$  are constant on the leaves of  $\tilde{\Lambda}_-$  and  $\tilde{\Lambda}_+$  respectively. Then there is a continuous map

$$f_\alpha : U_\alpha \rightarrow \mathbb{C}$$

given by  $f_\alpha(u, v) = x(u) + iy(v)$ , where  $x(u)$  and  $y(v)$  are monotone functions whose distributional derivatives  $(x'(u), y'(v))$  are the transverse measures for  $(\tilde{\Lambda}_-, \tilde{\Lambda}_+)$ . The coordinate  $z_\alpha = f_\alpha$  is unique up to

$$z_\alpha \mapsto \pm z_\alpha + b; \tag{9.2}$$

the sign ambiguity arises because the laminations are not oriented.

Since the coordinate change (9.2) is holomorphic, we can assemble the charts

$$V_\alpha = f_\alpha(U_\alpha)$$

to form a Riemann surface  $X$ . The forms  $dz_\alpha^2$  on  $U_\alpha$  are invariant under (9.2), so they patch together to yield a holomorphic quadratic differential  $q$  on  $X$ . Finally the maps  $f_\alpha$  piece together to give the collapsing map  $f : \tilde{S} \rightarrow X$ .

The construction of  $f : \tilde{S} \rightarrow X$  is functorial in the measured laminations  $(\tilde{\Lambda}_-, \tilde{\Lambda}_+)$ . That is, if we apply the same construction to  $(a^{-1}\tilde{\Lambda}_-, a^{+1}\tilde{\Lambda}_+)$ , we obtain a new marked surface  $f' : \tilde{S} \rightarrow X'$  and a unique map  $F : X \rightarrow X'$  such that  $F \circ f = f'$ . Moreover  $F$  is a Teichmüller mapping, stretching the vertical and horizontal leaves of  $q$  by  $a^{-1}$  and  $a^{+1}$  respectively.

Since  $g \in G$  multiplies the laminations  $(\tilde{\Lambda}_-, \tilde{\Lambda}_+)$  by  $(K^{-\phi(g)}, K^{+\phi(g)})$ , this functoriality provides the desired lifting of  $G$  to Teichmüller mappings on  $X$ .  $\blacksquare$

**Isotopy.** Finally we quote the following topological result of Blank and Laudenbach, recently treated by Cantwell and Conlon [LB], [NR], [CC]:

**Theorem 9.4** *Any two measured foliations  $\mathcal{F}, \mathcal{F}'$  representing the same cohomology class on  $M$  are isotopic.*

**Proof of Theorem 9.1.** We will construct  $(\mathcal{F}, J, f)$  from the Riemann surface  $X$ , its quadratic differential  $q$  and the action of  $G$  given by Theorem 9.3.

Let  $\tilde{\mathcal{F}}$  be the measured foliation of  $X \times \mathbb{R}$  with leaves  $X_r = X \times \{r\}$  and with transverse measure  $dr$ . Let  $\tilde{f}_t : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$  be the unit speed flow  $\tilde{f}_t(x, r) = (x, r + t)$ . Let  $\tilde{J}$  be the unique complex structure on  $T\tilde{\mathcal{F}}$  such that  $(X_0, \tilde{J}_0) = X$  and such that  $\tilde{f}_t : X_0 \rightarrow X_t$  is a Teichmüller mapping stretching the vertical and horizontal leaves of  $q$  by  $(K^{-t}, K^{+t})$ . Finally, let  $G$  act on  $X \times \mathbb{R}$  by

$$g \cdot (x, r) = (g(x), r + \phi(g)), \quad (9.3)$$

where  $g(x)$  is the Teichmüller mapping of  $X$  to itself provided by Theorem 9.3.

With this action,  $G$  preserves the structure  $(\tilde{\mathcal{F}}, \tilde{J}, \tilde{f}_t)$ , and therefore the quotient  $N = (X \times \mathbb{R})/G$  carries a measured foliation  $\mathcal{F}$ , a complex structure  $J$  on  $T\mathcal{F}$ , and a unit speed Teichmüller flow  $f_t : N \rightarrow N$ .

To complete the construction, we will show  $N$  can be identified with  $M$  in such a way that  $[\mathcal{F}] = \phi$ . To construct a homeomorphism  $N \cong M$ , first note that  $\phi$  pulls back to a trivial cohomology class on  $X \cong \tilde{S}$ , so there exists a smooth function  $\xi : X \rightarrow \mathbb{R}$  such that

$$\xi(h(x)) = \xi(x) + \phi(h)$$

for all  $h \in H \subset G$ . Set  $a = \phi(\tilde{\psi}) > 0$ , so  $\phi(h, i) = \phi(h) + ai$ . Then the homeomorphism of  $X \times \mathbb{R}$  given by

$$(x, r) \mapsto (x, ar + \xi(x))$$

conjugates the action of  $g = (h, i)$  by

$$g \cdot (x, r) = (g(x), r + i) \tag{9.4}$$

to the original action (9.3). Thus both actions have the same quotient space. On the other hand, the quotient of  $X \times \mathbb{R}$  by the action of  $G$  given by (9.4) is:

$$N = (X \times \mathbb{R})/G = ((X/H) \times \mathbb{R})/\mathbb{Z} \cong M,$$

because  $\mathbb{Z}$  acts on  $X/H \cong S$  by a map isotopic to  $\psi$ .

Thus we have identified  $N$  with  $M$ . It is easy to see that  $[\mathcal{F}] = \phi$  under this identification, so we have completed the construction of  $(\mathcal{F}, J, f)$ .

To prove uniqueness, the first step is to apply Theorem 9.4 to see that  $\phi$  determines  $\mathcal{F}$  up to isotopy. Then, given two Teichmüller flows  $f_1$  and  $f_2$  for the same foliation  $\mathcal{F}$ , we can pick a fiber  $S$  which is nearly parallel to the leaves of  $\mathcal{F}$  and transverse to both flows. Each flow determines, via its distortion of complex structure, a pair of  $\psi$ -invariant twisted measured laminations  $[\Lambda_{\pm}]$  for  $S$ . The uniqueness of  $(\mathcal{F}, J, f)$  then follows from the uniqueness of these twisted laminations, guaranteed by Theorem 8.1. ■

**Note.** Our original approach to Theorem 9.1 involved taking the geometric limit of the pseudo-Anosov flows known to exist for fibered classes in  $H^1(M, \mathbb{Q})$  by ordinary Teichmüller theory. An examination of the expansion factor  $K([\mathcal{F}])$  led to the more algebraic approach presented here.

## 10 Short geodesics on moduli space

Let  $S$  be a closed surface of genus  $g \geq 2$ , and let  $\mathcal{M}_g = \text{Teich}(S)/\text{Mod}(S)$  be its moduli space, endowed with the Teichmüller metric. Then closed geodesics on  $\mathcal{M}_g$  correspond bijectively to conjugacy classes of pseudo-Anosov elements  $\psi \in \text{Mod}(S) \cong \pi_1(\mathcal{M}_g)$ . The length  $L(\psi)$  of the geodesic for  $\psi$  is given by

$$L(\psi) = \log K(\psi),$$

where  $K(\psi) > 1$  is the pseudo-Anosov expansion factor for  $\psi$ . From [Pen2] we have:

**Theorem 10.1 (Penner)** *The length of the shortest geodesic on the moduli space  $\mathcal{M}_g$  of Riemann surfaces of genus  $g$  satisfies  $L(\mathcal{M}_g) \asymp 1/g$ .*

(Here  $A \asymp B$  means we have  $A/C \leq B \leq CA$  for a universal constant  $C$ .)

In this section we show any closed fibered hyperbolic 3-manifold with  $b_1(M) \geq 2$  provides a source of short geodesics on moduli space as above.

Indeed, let  $S \subset M$  be a fiber of genus  $g \geq 2$  with monodromy  $\psi$ . The assumption  $b_1(M) \geq 2$  is equivalent to the condition that  $\psi$  fixes a primitive cohomology class

$$\xi_0 \in H^1(S, \mathbb{Z}).$$

Let  $\tilde{S} \rightarrow S$  be the  $\mathbb{Z}$ -covering space corresponding to  $\xi_0$ , with deck group generated by  $h : \tilde{S} \rightarrow \tilde{S}$ , and let  $\tilde{\psi}$  be a lift of  $\psi$  to  $\tilde{S}$ .

**Theorem 10.2** *For all  $n$  sufficiently large,*

$$R_n = \tilde{S} / \langle h^n \tilde{\psi} \rangle$$

*is a closed surface of genus  $g_n \asymp n$ , and  $h : \tilde{S} \rightarrow \tilde{S}$  descends to a pseudo-Anosov mapping class  $\psi_n \in \text{Mod}(R_n)$  with*

$$L(\psi_n) = \frac{L(\psi)}{n} + O(n^{-2}) \asymp \frac{1}{g_n}. \quad (10.1)$$

**Proof.** Corresponding to the commuting maps  $\tilde{\psi}$  and  $h$  on  $\tilde{S}$ , we have a covering space

$$\tilde{M} = \tilde{S} \times \mathbb{R} \rightarrow M$$

with deck group  $\mathbb{Z}H \oplus \mathbb{Z}\tilde{\Psi}$ , where

$$\begin{aligned} H(s, t) &= (h(s), t) & \text{and} \\ \tilde{\Psi}(s, t) &= (\tilde{\psi}(s), t - 1). \end{aligned}$$

Define a map

$$(\phi, \xi) : H_1(M, \mathbb{Z}) \rightarrow \mathbb{Z}H \oplus \mathbb{Z}\tilde{\Psi} \rightarrow \mathbb{Z}^2$$

by sending  $H$  to  $(0, 1)$  and  $\tilde{\Psi}$  to  $(-1, 0)$ . Then the first factor  $\phi : H_1(M, \mathbb{Z}) \rightarrow \mathbb{Z}$  is the same as the cohomology class corresponding to the fiber  $S$ .

Now  $\phi$  belongs to the cone on a fibered face  $F$ , so  $\phi_n = n\phi + \xi$  also comes from a fibration  $\pi_n : M \rightarrow S^1$  for all  $n \gg 0$ . Since  $\mathbb{Z}\langle H^n \tilde{\Psi} \rangle$  corresponds to the kernel of  $\phi_n$ , the  $\mathbb{Z}$ -covering space  $M_n \rightarrow M$  corresponding to  $\pi_n$  is given by

$$M_n = \tilde{M} / \langle H^n \tilde{\Psi} \rangle \cong \tilde{S} / \langle h^n \tilde{\psi} \rangle \times \mathbb{R} = R_n \times \mathbb{R}.$$

Similarly, the monodromy of  $\pi_n$  is induced by the action of  $H^{-1}$  on  $\widetilde{M}$ , so it can be identified with  $\psi_n^{-1} : R_n \rightarrow R_n$  (up to isotopy).

Now  $\|\cdot\|_T$  is linear on  $\mathbb{R}_+ \cdot F$ , so we have

$$\|\phi_n\|_T = |\chi(R_n)| = 2g_n - 2 = n\phi(e) - \phi_0(e) \asymp n$$

for some  $e \in H_1(M, \mathbb{Z})$  (the Euler class). Finally the expansion factor is differentiable and homogeneous of degree  $-1$ , so we have

$$K(\psi_n) = K(\phi_n) = K(\phi)^{1/n} + O(n^{-2}),$$

giving (10.1). ■

### Notes.

1. The exchange of deck transformations and dynamics in the statement of Theorem 10.2 is often called *renormalization*. Compare [Y], where the same construction is used to analyze rotation maps.
2. It is easy to see that  $L(\mathcal{M}_1) = \log(3 + \sqrt{5})/2$  is the log of the leading eigenvalue of  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . For genus 2 we have  $L(\mathcal{M}_2) \leq 0.543533\dots = \log k$ , where  $k^4 - k^3 - k^2 - k + 1 = 0$  [Zh], and in general  $L(\mathcal{M}_g) \leq (\log 6)/g$  [Ba].
3. It can be shown that the minimal expansion factor  $K_n$  for an  $n \times n$  integral Perron-Frobenius matrix is the largest root of  $x^n = x + 1$ ; it satisfies  $K_n = 2^{1/n} + O(1/n^2)$ . The factor  $K_n$  is realized by the matrix

$$M_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \pmod n \\ 1 & \text{if } (i, j) = (1, 3) \\ 0 & \text{otherwise,} \end{cases}$$

which is the adjacency matrix of a cyclic graph with one shortcut; see Figure 6 for the case  $n = 8$ . (For a detailed development of the Perron-Frobenius theory, see [LM, §4].)

Since the expansion factor of  $\psi$  agrees with that of a Perron-Frobenius matrix attached to a train track with at most  $6g - 6$  edges, we have  $L(\mathcal{M}_g) \geq (\log 2)/(6g - 6)$ .

4. **Question.** Does  $\lim_{g \rightarrow \infty} g \cdot L(\mathcal{M}_g)$  exist? What is its value?

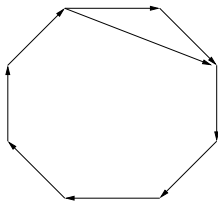


Figure 6. An 8-vertex graph in which the number of paths of length  $n$  grows as slowly as possible.

## 11 Examples: Closed braids

Closed braids provide a natural source of fibered link complements  $M^3 = S^3 - L(\beta)$ . In this section we present the computation of  $\Theta_F$  and the fibered face  $F \subset H^1(M, \mathbb{R})$  for some simple braids.

**Braids.** Let  $S = D^2 - \bigcup_1^n U_i$  be the complement of  $n$  disjoint round disks lying along a diameter of the closed unit disk  $D^2$ . Let  $\text{Diff}^+(S, \partial D)$  be the group of diffeomorphisms of  $S$  to itself, preserving orientation and fixing  $\partial D^2$  pointwise.

The *braid group*  $B_n$  is the group of connected components of  $\text{Diff}^+(S, \partial D)$ . It has standard generators  $\sigma_i$ ,  $i = 1, \dots, n-1$ , which interchange  $\partial U_i$  and  $\partial U_{i+1}$  by performing a half Dehn twist to the left (see [Bi], [BZ]).

There is a natural map  $B_n \rightarrow \text{Mod}(S)$  sending a braid  $\beta \in B_n$  to a mapping class  $\psi \in \text{Mod}(S)$ . Moreover  $\beta$  determines a *canonical* lift  $\tilde{\psi}$  of  $\psi$  to the  $H$ -covering space of  $S$ , by the requirement that  $\tilde{\psi}$  fixes the preimage of  $\partial D^2$  pointwise.

There is a natural basis  $t_i = [\partial U_i]$  for  $H_1(S, \mathbb{Z})$ , on which  $\beta$  acts by  $\beta(t_i) = t_{\sigma_i}$ , and  $b = \text{rank } H$  is just the number of cycles of the permutation  $\sigma$ .

**Links.** Let  $M$  be the fibered 3-manifold with fiber  $S$  and monodromy  $\psi$ . There is a natural model for  $M$  as a link complement  $M = S^3 - L(\beta)$  in the 3-sphere. To construct the link  $L(\beta)$ , simply close the braid  $\beta$  while passing it through an unknot  $\alpha$  (see Figure 1 of §1). The surface  $S$  embeds into  $M$  as a disk spanning  $\alpha$ , punctured by the  $n$  strands of  $\beta$ .

The meridians of components of  $L(\beta)$  give a natural basis for  $H_1(M, \mathbb{Z})$ ; in particular the meridian of  $\alpha$  corresponds to the natural lifting  $\tilde{\psi}$  of  $\psi$ .

**Train tracks and braids on three strands.** We will now compute  $\Theta_F(t, u)$  and  $F$  in three examples, where  $F$  is the fibered face carrying  $S$ .

These examples all come from braids  $\beta$  in the semigroup of  $B_3$  generated

by  $\sigma_1$  and  $\sigma_2^{-1}$ . This semigroup is easy to work with because it preserves a pair of train tracks  $\tau_1, \tau_2$ , where  $\tau_1$  is shown in Figure 5 and  $\tau_2$  is the reflection of  $\tau_1$  through a vertical line.

As an additional simplification, each train track  $\tau_i$  is a spine for  $S$ , and thus the Thurston and Teichmüller norms agree in these examples: we have

$$\|\phi\|_T = |\chi(S)| = |\chi(\lambda)| = |\chi(\tau)| = \|\phi\|_{\Theta_F}$$

for all fibers  $[S] \in \mathbb{R}_+ \cdot F$  (see Note 2 of §6). In particular, the fibered face  $F$  coincides with a face of the Teichmüller norm ball, so it is easily computed from  $\Theta_F$ .

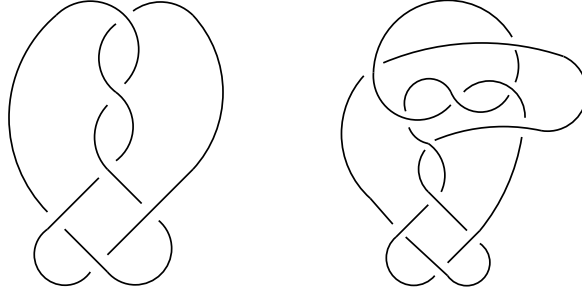


Figure 7. The links  $6_2^2 = L(\sigma_1\sigma_2^{-1})$  and  $9_{51}^2 = L(\sigma_1\sigma_2^{-3})$ .

**I. The simplest pseudo-Anosov braid.** For the first example, consider the simplest pseudo-Anosov braid,  $\beta = \sigma_1\sigma_2^{-1}$ . Its three strands are permuted cyclically, so  $H = \text{Hom}(H^1(S, \mathbb{Z})^\psi, \mathbb{Z})$  is of rank one, generated by  $t = t_1 + t_2 + t_3$ .

The train tracks  $\tau_1$  and  $\tau_2$  differ only in their switching conditions, so their vertex and edge modules  $\mathbb{Z}[t]^V, \mathbb{Z}[t]^E$  are naturally identified. Using this identification, we can express the action of  $\sigma_1, \sigma_2^{-1}$  on these modules as  $4 \times 4$  and  $6 \times 6$  matrices of Laurent polynomials.

Now the determinant formula gives  $\Theta_F$  as the characteristic polynomial for the action of  $\psi$  on the 2-dimensional subspace

$$\text{Ker } D(t)^* : \mathbb{Z}[t]^E \rightarrow \mathbb{Z}[t]^V.$$

By restricting  $\sigma_1$  and  $\sigma_2^{-1}$  to this subspace, and projecting to the coordinates for the edge subset  $E' = \{a, c\}$ , we obtain the simpler  $2 \times 2$  matrices:

$$\sigma_1(t) = \begin{pmatrix} t & t \\ 0 & 1 \end{pmatrix}, \quad \sigma_2^{-1}(t) = \begin{pmatrix} 1 & 0 \\ t^{-1} & t^{-1} \end{pmatrix}.$$



Restricting to  $\text{Ker } D(t)^*$  removes the factor of  $\det(uI - P_V(t))$  from  $\det(uI - P_E(t))$ , and therefore we have:

$$\Theta_F(t, u) = \det(uI - \beta(t)), \quad (11.1)$$

where  $\beta(t)$  is the appropriate product of the matrices above.

Setting  $\beta(t) = \sigma_1(t)\sigma_2^{-1}(t)$ , we find the Teichmüller polynomial is given by

$$\Theta_F(t, u) = 1 - u(1 + t + t^{-1}) + u^2.$$

Its Newton polygon is a diamond, and its norm is:

$$\|(s, y)\|_{\Theta_F} = \max(|2s|, |2y|).$$

(Here  $(s, y)$  denotes the cohomology class evaluating to  $s$  and  $y$  on the meridian of  $\alpha$  and  $\beta$  respectively.)

The fibered face  $F \subset H^1(M, \mathbb{R})$  is the same as the face of the Teichmüller norm ball meeting  $\mathbb{R}_+ \cdot [S] = \mathbb{R}_+ \cdot (0, 1)$ , and therefore

$$F = \{1/2\} \times [-1/2, 1/2]$$

in these  $(s, y)$ -coordinates.

The closed braid  $L(\beta)$  can be simplified to a projection with 6 crossings (see Figure 7), and it is denoted  $6_2^2$  in Rolfsen's tables [Rol]. In this projection, the two components of  $L(\beta)$  are clearly interchangeable. In fact, the Thurston norm ball for  $S^3 - L(\beta)$  has 4 faces, all fibered, and

$$\|(s, y)\|_T = \max(|2s|, |2y|).$$

for all  $(s, y) \in H^1(M, \mathbb{R})$ .

**II. The Thurston and Alexander norms.** The braid  $\beta = \sigma_1\sigma_2^{-3}$  also permutes its strands cyclically. By (11.1) in this case we obtain

$$\Theta_F(t, u) = t^{-2} - u(t + 1 + t^{-1} + t^{-2} + t^{-3}) + u^2.$$

Figure 8 shows the Teichmüller norm ball for this example in  $(s, y)$  coordinates, along with the graph  $y = \log k(s)$ , where  $k(s)$  eigenvalue of  $\psi$  on  $\mathcal{ML}_s(S)$  discussed in §8. The graph  $\Gamma$  is also the level set  $\log K(\phi) = 1$  of the expansion function on  $\mathbb{R}_+ \cdot F$ . This picture illustrates the fact that  $\Gamma$  is convex, that the cones over  $F$  and  $\Gamma$  coincide, and that  $K(\phi)$  tends to infinity at  $\partial F$ .

To compute the full Thurston norm ball for this example, we appeal to the inequality  $\|\phi\|_A \leq \|\phi\|_T$  between the Alexander and Thurston norms

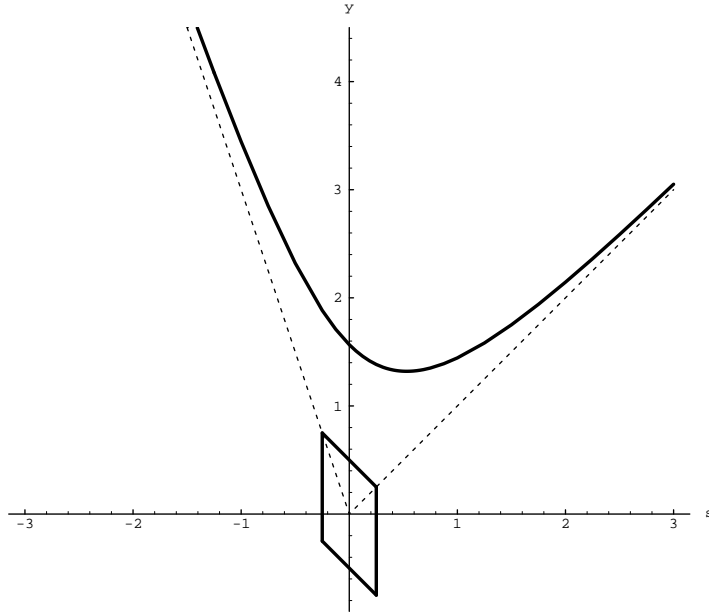


Figure 8. Norm ball and expansion factor.

(see §7). Because of this inequality, the two norms agree if they coincide on the extreme points of the Alexander norm ball. Now a straightforward computation gives

$$\Delta_M(t, u) = t^{-2} + u(t - 1 + t^{-1} - t^{-2} + t^{-3}) + u^2$$

in the present example. The polynomials  $\Delta_M$  and  $\Theta_F$  have the same Newton polygon, and thus the Alexander, Thurston and Teichmüller norms all coincide on  $F$ . But the endpoints of  $\pm F$  are the extreme points of the Alexander norm ball, and therefore

$$\|(s, y)\|_T = \|(s, y)\|_A = \max(|2s + 2y|, |4s|)$$

for all  $(s, y) \in H^1(M, \mathbb{R})$ .

For example, the simplest surface spanning both components of  $L(\beta)$  has genus  $g = 2$ , since  $\|(\pm 1, \pm 1)\|_T = 4$ .

Finally we remark that the closed braid  $L(\sigma_1\sigma_2^{-3})$  is actually the same as the link  $9_{51}^2$  of Rolfsen's tables (see Figure 7). We have thus established:

*The Thurston and Alexander norms coincide for the link  $9_{51}^2$ .*

In [Mc] we found that the two norms coincide for all examples in Rolfsen's table of links with 10 or fewer crossings, except  $9_{21}^3$ , and possibly  $9_{41}^2$ ,  $9_{50}^2$ ,  $9_{51}^2$ , and  $9_{15}^3$ . The link  $9_{51}^2$  can now be removed from the list of possible exceptions.

**III. Pure braids.** We conclude by discussing *pure braids*  $\beta$  in the semigroup generated by the full twists  $\sigma_1^2, \sigma_2^{-2}$ . A pure braid acts trivially on  $H_1(S, \mathbb{Z})$ , and thus the Thurston norm ball is 4-dimensional. We take  $(t_1, t_2, t_3, u)$  as a basis for  $H^1(M, \mathbb{Z})$ , where  $t_i$  is the meridian of the  $i$ th strand of  $\beta$  and  $u$  is the meridian of  $\alpha$ .

By cutting down to the kernel of  $D(t)^*$  on  $\mathbb{Z}[H]^E$  as before, we obtain an action of the full twists on a rank 2 module over  $\mathbb{Z}[t_1, t_2, t_3]$ . Setting  $(t_1, t_2, t_3) = (a, b, c)$  to improve readability, we find that  $\sigma_1^2$  and  $\sigma_2^{-2}$  act on this module by:

$$\sigma_1^2 = \begin{pmatrix} ab & ab + b \\ 0 & 1 \end{pmatrix}, \quad \sigma_2^{-2} = \begin{pmatrix} 1 & 0 \\ b^{-1} + b^{-1}c^{-1} & b^{-1}c^{-1} \end{pmatrix}.$$

For a concrete example, we consider the pure braid  $\beta = \sigma_1^2 \sigma_2^{-6}$  whose link  $L(\beta)$  appears in Figure 1 of §1. Applying (11.1) with the matrices above, we find its Teichmüller polynomial is given by:

$$\Theta_F(a, b, c, u) = \frac{a}{b^2 c^3} - \frac{u}{b^3 c^3} (1 - b^4 c^3 (1 + c + ac) + (a + 1)b(1 + c)(1 + bc)(1 + b^2 c^2)) + u^2.$$

The projection of the fibered face  $F$  for this example to  $H^1(S, \mathbb{R})$  is shown in Figure 2 of §1.

Since the coefficient of  $u^0$  is  $ab^{-2}c^{-3} = t^{(1, -2, -3)}$ , we find the Thurston norm on  $\mathbb{R}_+ \cdot F$  is given by

$$\|(s, y)\|_T = -s_1 + 2s_2 + 3s_3 + 2y.$$

For example,  $\|(-1, 1, -1, 1)\|_T = 2$ , showing that  $L(\beta)$  is spanned by a Seifert surface of genus 0 running in alternating directions along the strands of  $\beta$ . It is interesting to locate this surface explicitly in Figure 1.

**Notes.**

1. For a general construction of pseudo-Anosov mappings, including the examples above as special cases, see [Pen1], [Fa].
2. The Thurston norm for the knot  $6_2^2$  is also discussed in [Fr1, p. 264] and [Oe2, Ex. 2.2].

## A Appendix: Positive polynomials and Perron-Frobenius matrices

This Appendix develops the theory of Perron-Frobenius matrices over a ring of Laurent polynomials. These results are used in §5–§8.

**Laurent polynomials.** Let  $(s_1, \dots, s_b)$  be coordinates for  $s \in \mathbb{R}^b$ , and let

$$(t_1, \dots, t_b) = (e^{s_1}, \dots, e^{s_b})$$

be coordinates for  $t = e^s$  in  $\mathbb{R}_+^b$ . An integral *Laurent polynomial*  $p(t)$  is an element of the ring  $\mathbb{Z}[t_1^{\pm 1}, \dots, t_b^{\pm 1}]$  generated by the coordinates  $t_i$  and their inverses. We can write such a polynomial as

$$p(t) = \sum_{\alpha \in A} a_\alpha t^\alpha, \tag{A.1}$$

where the exponents  $\alpha = (\alpha_1, \dots, \alpha_b)$  range over a finite set  $A \subset \mathbb{Z}^b$ , where  $t^\alpha = t_1^{\alpha_1} \cdots t_b^{\alpha_b}$ , and where the coefficients  $a_\alpha \in \mathbb{Z}$  are nonzero.

**Newton polygons.** The *Newton polygon*  $N(p) \subset \mathbb{R}^b$  of  $p(t) = \sum_A a_\alpha t^\alpha$  is the convex hull of the set of exponents  $A \subset \mathbb{Z}^b$ .

If we think of  $(s_i)$  as a basis for an abstract real vector space  $V$ , then  $N(p)$  also naturally resides in  $V$ . Each monomial  $t^\alpha$  appearing in  $p(t)$  determines an open *dual cone*  $C(t^\alpha) \subset V^*$  consisting of the linear maps  $\phi : V \rightarrow \mathbb{R}$  that achieve their maximum on  $N(p)$  precisely at  $\alpha$ . Equivalently,

$$C(t^\alpha) = \{\phi : \phi(\alpha) > \phi(\beta) \text{ for all } \beta \neq \alpha \text{ in } A\}.$$

**Positivity and Perron-Frobenius.** A Laurent polynomial  $p(t) \neq 0$  is *positive* if it has coefficients  $a_\alpha > 0$ .

Let

$$P(t) = P_{ij}(t) \in M_n(\mathbb{Z}[t_1^{\pm 1}, \dots, t_b^{\pm 1}])$$

be an  $n \times n$  matrix of Laurent polynomials, with each entry either zero or positive. If for some  $k > 0$ , *every* entry of  $P_{ij}^k(t)$  is a positive Laurent polynomial, we say  $P(t)$  is an (integral) *Perron-Frobenius* matrix. By convention, we exclude the case where  $n = 1$  and  $P(1) = [1]$ .

The matrix  $P(t)$  is a traditional Perron-Frobenius matrix for every fixed value  $t \in \mathbb{R}_+^b$ . In particular, the largest eigenvalue  $E(t)$  of  $P(t)$  is simple, real and positive [Gant]. Since  $P(1)$  is an integral matrix ( $\neq [1]$ ), we always have  $E(1) > 1$ .

The main result of this section is:

**Theorem A.1** Let  $E(t)$  be the leading eigenvalue of a Perron-Frobenius matrix  $P(t)$ . Then:

- (A) The function  $f(s) = \log E(e^s)$  is a convex function of  $s \in \mathbb{R}^b$ .
- (B) The graph of  $y = f(s)$  meets each ray from the origin in  $\mathbb{R}^b \times \mathbb{R}$  at most once.
- (C) The rays passing through the graph of  $y = f(s)$  coincide with the dual cone  $C(u^d)$  of the polynomial

$$\Theta_F(t, u) = u^d + b_1(t)u^{d-1} + \cdots + b_d(t),$$

for any factor  $\Theta_F(t, u)$  of  $\det(uI - P(t))$  satisfying  $\Theta_F(t, E(t)) = 0$ .

**Positivity and convexity.** In addition to Laurent polynomials, it is also useful to consider finite power sums  $p(t) = \sum a_\alpha t^\alpha$  with real exponents  $\alpha \in \mathbb{R}^b$ , and real coefficients  $a_\alpha \in \mathbb{R}$ . As for a Laurent polynomial, we say a nonzero power sum is *positive* if its coefficients are positive.

**Proposition A.2** If  $p(t) = \sum a_\alpha t^\alpha$  is a positive power sum, then

$$f(s) = \log p(e^s)$$

is convex function of  $s \in \mathbb{R}^b$ .

**Proof.** By restricting  $f(s)$  to a line and applying a translation, we are reduced to showing  $f''(0) \geq 0$  when  $p(t)$  is a power sum in one variable  $t$ . But then

$$f''(0) = \frac{(\sum a_\alpha)(\sum \alpha^2 a_\alpha) - (\sum \alpha a_\alpha)^2}{(\sum a_\alpha)^2} \geq 0,$$

by Cauchy-Schwarz. ■

**Proof of Theorem A.1 (A).** Since  $E(t)$  agrees with the spectral radius of  $P(t)$ , and  $P_{ij}(t) \geq 0$ , we have

$$E(t) = \lim_{n \rightarrow \infty} \left( \sum_{i,j} P_{ij}^n(t) \right)^{1/n}.$$

Therefore  $\log E(e^s) = \lim n^{-1} \log E_n(e^s)$ , where  $E_n(t) = \sum_{i,j} P_{ij}^n(t)$ . Since the nonzero entries of  $P(t)$  are positive,  $E_n(t)$  is a positive Laurent polynomial, and thus  $\log E_n(e^s)$  is convex by the preceding result. Therefore the limit  $f(s) = \log E(e^s)$  is also convex.

**Proof of (B).** Let  $(s, y)$  be coordinates on  $\mathbb{R}^b \times \mathbb{R}$ , and let  $R$  be a ray through the origin. (B) is immediate when  $R$  is contained in  $y$ -axis. Dispensing with that case, we can pass to functions of a single variable  $t = e^s$  by restricting to the plane spanned  $R$  and the  $y$ -axis, and we can assume  $R$  is the graph of a linear function of the form  $y = \gamma s$ , for  $s > 0$ .

Now the function  $f(s)$  is convex and real analytic. Thus  $f(s)$  is either strictly convex or affine ( $f(s) = as + b$ ).

To treat the affine case, note  $b = f(0) = \log E(1) > 0$ , since the leading eigenvalue of the integral Perron-Frobenius matrix  $P(1)$  is greater than one. Thus the equation  $y = \gamma s = f(s) = as + b$  has at most one solution, and we are done.

Now assume  $f(t)$  is strictly convex. Recall that  $f(t)$  is a limit of the convex functions  $f_n(t) = n^{-1} \log E_n(t)$ . If the ray  $R$  crosses the graph of  $y = f(s)$  twice, then it also crosses the graph of  $y = f_n(s)$  twice for some finite value of  $n$ .

Fixing such an  $n$ , let  $\beta_n = \beta/n$  where  $a_\beta t^\beta$  is the term with largest exponent appearing in the power sum  $E_n(t)$ . Then  $f'_n(s) \rightarrow \beta_n$  as  $s \rightarrow \infty$ , so by strict convexity we have  $f'_n(s) < \beta_n$  for all finite  $s$ . Since  $f_n(s)$  has more than one term, and  $a_\beta > 1$ , we also have:

$$f_n(s) = \frac{\log E_n(e^s)}{n} > \beta_n s + \frac{\log a_\beta}{n} \geq \beta_n s. \quad (\text{A.2})$$

Now suppose  $y = f_n(s)$  crosses the line  $y = \gamma s$  twice. Then by convexity, the slopes satisfy  $\beta_n > f'_n(s) > \gamma$  at the second intersection point. But (A.2) then implies  $f_n(s) > \gamma s$  for all  $s > 0$ , so in fact the ray  $y = \gamma s$  has *no* intersections with the graph of  $y = f_n(s)$ .

**Proof of (C).** Passing again to functions of a single variable  $t = e^s$ , we consider the condition that the ray  $y = \gamma s$ ,  $s > 0$ , passes through the graph of  $y = E(t)$ .

By assumption,  $u = E(t)$  is the largest root of the equation

$$\Theta_F(t, u) = \sum a_{\alpha i} t^\alpha u^i = u^d + b_1(t)u^{d-1} + \dots + b_d(t) = 0.$$

Since the coefficients  $b_i(t)$  are homogeneous of degree  $i$  in the roots of  $\Theta$ , we have

$$E(t) \asymp \sup |b_i(t)|^{1/i}.$$

In particular, as  $t \rightarrow +\infty$ ,  $E(t)$  grows like  $t^\beta$  with

$$\beta = \sup \alpha / (d - i), \quad (\text{A.3})$$

the sup taken over all monomials  $t^\alpha u^i$  appearing in  $\Theta$  other than  $u^d$ . Thus as  $s \rightarrow \infty$  the convex function  $y = f(s) = \log E(e^s)$  is asymptotic to a linear function of the form  $y = \beta s + \delta$ .

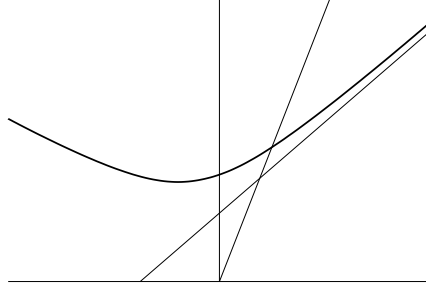


Figure 9. A ray crossing the eigenvalue graph  $y = f(s) = \log E(e^s)$ .

Now consider the ray  $R$  through  $(1, \gamma)$ , with equation  $y = \gamma s$ ,  $s > 0$ . By (B), this ray meets  $y = f(s)$  iff  $\gamma > \beta$  (see Figure 9). By (A.3), we have  $\gamma > \beta$  iff

$$d\gamma > \alpha + i\gamma$$

for all monomials  $t^\alpha u^i$  in  $\Theta$  other than  $u^d$ . Thus  $R$  meets  $y = f(s)$  iff the linear functional

$$\phi(\alpha, i) = 1 \cdot \alpha + \gamma \cdot i$$

achieves its maximum on the Newton polygon  $N(\Theta)$  at the vertex  $(\alpha, i) = (0, d)$  coming from  $u^d$ . This condition says exactly that  $R$  belongs to the dual cone  $C(u^d)$ . ■

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