

# Moduli spaces in genus zero and inversion of power series

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1 October 2012

Let  $\mathcal{M}_{0,n}$  denote the moduli space Riemann surfaces of genus 0 with  $n$  ordered marked points. Its Deligne-Mumford compactification  $\overline{\mathcal{M}}_{0,n}$  is naturally partitioned into connected strata of the form

$$S \cong \mathcal{M}_{0,n_1} \times \cdots \times \mathcal{M}_{0,n_s},$$

indexed by the different topological types of stable curves with  $n$  marked points. The stable curves in the stratum above have  $s$  irreducible components and  $s - 1$  nodes; thus  $\sum n_i = n + 2s - 2$ .

This note provides a short proof of the following result, which shows that the universal formula for inversion of power series is encoded in the stratification of moduli space.

**Theorem 1** *The formal inverse of  $f(x) = x - \sum_2^\infty a_n x^n / n!$  is given by  $g(x) = x + \sum_2^\infty b_n x^n / n!$ , where*

$$b_n = \sum a_{n_1} \cdots a_{n_s} \times \left( \begin{array}{c} \text{the number of strata } S \subset \overline{\mathcal{M}}_{0,n+1} \\ \text{isomorphic to } \mathcal{M}_{0,n_1+1} \times \cdots \times \mathcal{M}_{0,n_s+1} \end{array} \right).$$

That is,  $g(f(x)) = x$ .

Here the coefficients of  $f(x)$  and  $g(x)$  are regarded as elements of the polynomial ring  $\mathbb{Q}[a_2, a_3, \dots]$ , and the sum is over all  $s \geq 1$  and all multi-indices  $(n_1, \dots, n_s)$  with  $n_i \geq 2$ .

Using basic properties of the Euler characteristic, we obtain:

**Corollary 2 (Getzler)** *The generating functions*

$$f(x) = x - \sum_{n=2}^{\infty} \chi(\mathcal{M}_{0,n+1}) \frac{x^n}{n!} \quad \text{and} \quad g(x) = x + \sum_{n=2}^{\infty} \chi(\overline{\mathcal{M}}_{0,n+1}) \frac{x^n}{n!}$$

are formal inverses of one another.

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\*Research supported in part by the NSF. 2000 Mathematics Subject Classification: 32G15.

It is easy to see that  $a_n = \chi(\mathcal{M}_{0,n+1}) = (-1)^n(n-2)!$ , using the fibration  $\mathcal{M}_{0,n+1} \rightarrow \mathcal{M}_{0,n}$ . Thus by formally inverting  $f(x)$ , one can readily compute

$$\langle \chi(\overline{\mathcal{M}}_{0,n})_{n=3}^\infty = \langle 1, 2, 7, 34, 213, 1630, 14747, 153946, 1821473, \dots \rangle.$$

Corollary 2 is a consequence of [Ge1, Thm. 5.9], stated explicitly in [LZ, Rmk. 4.5.3]. The development in [Ge1] uses operads and yields more information, such as Betti numbers for  $\overline{\mathcal{M}}_{0,n}$ . Theorem 1 shows that Corollary 2 holds for any generalized Euler characteristic on the Grothendieck ring of varieties over  $\overline{\mathbb{Q}}$  (cf. [Bi]).

The proof of Theorem 1 will be based on simple properties of trees. Its aim is to provide an elementary entry point to the enumerative combinatorics of moduli spaces.

**Trees.** A tree  $\tau$  is a finite, connected graph with no cycles; its vertices will be denoted  $V(\tau)$ . The degree function  $d : V(\tau) \rightarrow \mathbb{N}$  gives the number of edges incident to each vertex. To each tree we associate the monomial

$$A(\tau) = \prod_{V(\tau)} A_{d(v)-1}$$

in the polynomial ring  $\mathbb{Z}[A_1, A_2, A_3, \dots]$ , with the convention  $A_0 = 1$ .

A tree is *stable* if it has no vertices of degree 2. An *endpoint* of  $\tau$  is a vertex with  $d(v) = 1$ . We say  $\tau$  is *rooted* if it has a distinguished endpoint (the root). The number of endpoints of  $\tau$ , other than its root, will be denoted  $N(\tau)$ . We always assume  $\tau$  has *at least* one edge, so  $N(\tau) \geq 1$ ; and the tree with *just one edge* is considered stable.

A *ribbon tree* is a rooted stable tree equipped with a cyclic ordering of the edges incident to each vertex. A ribbon structure records the same information as a planar embedding  $\tau \hookrightarrow \mathbb{R}^2$  up to isotopy.

A *marked tree* is a rooted stable tree equipped with a labeling of its endpoints by the integers  $1, 2, \dots, N(\tau) + 1$ . We require that the root is labeled 1.

**Theorem 3** *The formal inverse of  $F(x) = x - \sum_2^\infty A_n x^n$  is given by*

$$G(x) = \sum_{\text{ribbon } \tau} A(\tau) x^{N(\tau)}. \tag{1}$$

Here the sum is taken over all ribbon trees, up to isomorphism.

**Proof.** Suppose we are given ribbon trees  $\tau_1, \dots, \tau_d$  with  $d \geq 2$ . We can then construct a new ribbon tree  $\tau$  by identifying the roots of these trees

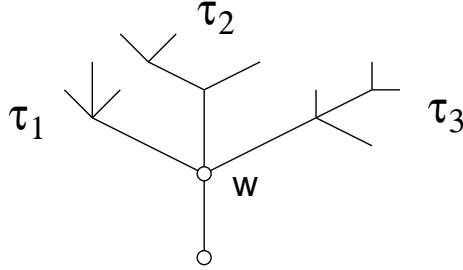


Figure 1. Three ribbon trees grafted together at their roots.

with a single vertex  $w$ , and adding a new edge leading from  $w$  to the root of  $\tau$  (see Figure 1). The ribbon structure at  $w$  is determined by the ordering of the trees  $(\tau_i)$ , and by the condition that the root of  $\tau$  lies between  $\tau_d$  and  $\tau_1$ .

Conversely, any ribbon tree with  $N(\tau) \geq 2$  is obtained by applying this construction to the subtrees  $(\tau_1, \dots, \tau_d)$  leading away from the edge adjacent to its root. Taking into account the vertex  $w$  of degree  $d + 1$  where these trees are attached, we find:

$$A(\tau)x^{N(\tau)} = A_d \prod_{i=1}^d A(\tau_i)x^{N(\tau_i)}.$$

But the right hand side above is precisely one of the terms occurring in the expression  $A_d G(x)^d$ . Summing over all possible values for  $d = d(w)$  we obtain

$$G(x) = x + \sum_{d=2}^{\infty} A_d G(x)^d,$$

where the first term accounts for the unique tree with  $N(\tau) = 1$ . Rearranging terms gives  $F(G(x)) = x$ . ■

**Corollary 4** *The formal inverse of  $f(x) = x - \sum_2^{\infty} a_n x^n / n!$  is given by*

$$g(x) = \sum_{\text{marked } \tau} a(\tau) \frac{x^{N(\tau)}}{N(\tau)!}, \quad (2)$$

where  $a(\tau) = \prod_{V(\tau)} a_{d(v)-1}$  and  $a_0 = 1$ .

**Proof.** The number of ribbon structures on a given stable rooted tree  $\tau$  is given by  $\prod(d(v) - 1)!$ . The group  $\text{Aut}(\tau)$  acts freely on the space of ribbon structures, so  $\tau$  contributes  $\prod(d(v) - 1)!/|\text{Aut}(\tau)|$  identical terms to equation (1) for  $G(x)$ . Similarly,  $\tau$  contributes  $N(\tau)!/|\text{Aut}(\tau)|$  terms to equation (2) for  $g(x)$ . Setting  $A_n = a_n/n!$ , we find  $F(x) = f(x)$  and

$$G(x) = \sum_{\text{marked } \tau} \frac{\prod(d(v) - 1)!}{N(\tau)!} A(\tau)x^{N(\tau)} = g(x),$$

so  $f(g(x)) = F(G(x)) = x$ . ■

**Remark.** The same reasoning shows (2) can be rewritten as

$$f^{-1}(x) = \sum_{\text{stable } \tau} \frac{N(\tau) + 1}{|\text{Aut}(\tau)|} a(\tau)x^{N(\tau)}.$$

For example, using the trees shown in Figure 2 we find

$$f^{-1}(x) = x + \frac{a_2}{2}x^2 + \frac{(a_3 + 3a_2^2)}{6}x^3 + \frac{(a_4 + 10a_2a_3 + 15a_2^3)}{24}x^4 + O(x^5).$$

For a quite different approach to Corollary 4, see [Ge2, Thm 1.3].

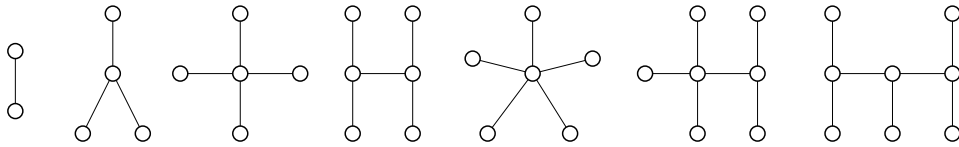


Figure 2. The stable trees with  $N(\tau) \leq 4$ .

**Proof of Theorem 1.** A stable curve  $X \in \overline{\mathcal{M}}_{0,n+1}$  of genus zero determines a marked tree  $t(X)$  whose interior vertices correspond to the irreducible components of  $X$ , and whose edges correspond to its nodes and labeled points. Conversely, any marked tree with  $N(\tau) \geq 2$  can be realized by a stable curve, so the map

$$\tau \mapsto S(\tau) = \{X \in \overline{\mathcal{M}}_{0,N(\tau)+1} : t(X) \cong \tau\}$$

gives a bijection between marked trees with  $N(\tau) \geq 2$  and the strata of moduli spaces. The desired inversion formula now follows from the preceding corollary. ■

**Proof of Corollary 2.** Let  $a_n = \chi(\mathcal{M}_{0,n+1})$ . It is known that  $\chi(X - Y) + \chi(Y) = \chi(X)$  whenever  $Y$  is a closed subvariety of a complex variety  $X$  [Ful, p.141, note 13], and that  $\chi(A \times B) = \chi(A) \times \chi(B)$ . The first property implies that  $\chi(\overline{\mathcal{M}}_{0,n+1})$  is the sum of the Euler characteristics of its strata  $S$ , and the second implies that

$$\chi(S) = a_{n_1} \cdots a_{n_s}$$

whenever  $S \cong \mathcal{M}_{0,n_1+1} \times \cdots \times \mathcal{M}_{0,n_s+1}$ . Thus the stated relationship between generating functions follows from Theorem 1.  $\blacksquare$

**Moduli space over  $\mathbb{R}$ .** The real points of moduli space  $\mathcal{M}_{0,n}(\mathbb{R})$  form a submanifold with  $(n-1)!/2$  connected components, each homeomorphic to  $\mathbb{R}^{n-3}$ . Let  $M_n$  be the component of  $\mathcal{M}_{0,n}(\mathbb{R})$  where the marked points can be chosen to lie in  $\mathbb{R}$ , with  $x_1 < x_2 < \cdots < x_n$ . Let  $\overline{M}_n$  be the closure of  $M_n$  in  $\overline{\mathcal{M}}_{0,n}$ . The strata of  $\overline{M}_n$  are encoded by ribbon trees, since the cyclic ordering of the points  $(x_i)$  is preserved under stable limits (cf. [De]). Thus in this setting, Theorem 3 yields:

**Corollary 5** *The formal inverse of  $F(x) = x - \sum_2^\infty A_n x^n$  is given by  $G(x) = x + \sum_2^\infty B_n x^n$ , where*

$$B_n = \sum A_{n_1} \cdots A_{n_s} \times \left( \begin{array}{l} \text{the number of strata } S \subset \overline{M}_{0,n+1} \\ \text{isomorphic to } M_{n_1+1} \times \cdots \times M_{n_s+1} \end{array} \right).$$

**Notes and references.** A compendium of results on trees, generating functions and inversion can be found in [St, Ch. 5]; see also [Ca]. For background on the many connections between graphs and moduli space, see e.g. [ACG, Ch. XVIII], [LZ] and the references therein.

I would like to thank S. Koch for useful conversations related to this note.

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