

**Dimitar P. Jetchev**  
**Geometric Topology Tutorial**  
**Solution Set # 1**

**Problem 1.** *Using the Jordan separation theorem, prove that the utility graph does not embed in the plane.*

**Solution:** Assume that the utility graph  $K_{3,3}$  with  $V = \{A_1, A_2, A_3, B_1, B_2, B_3\}$  and edges  $E = \{(A_i, B_j) : 1 \leq i, j \leq 3\}$  is planar. Consider the loop  $L = (A_1, B_1, A_2, B_2)$ . By Jordan separation theorem,  $\mathbb{R}^2 - L$  is disconnected. This implies that the vertices  $A_3$  and  $B_3$  are either both inside, or outside of  $L$ . Suppose  $A_3$  and  $B_3$  are both inside the loop  $L$ . Look at the loops  $L' = (B_1, A_3, B_2, A_1)$  and  $L'' = (B_1, A_3, B_2, A_2)$ . Again, we apply the separation theorem for each one of these loops. Since  $B_3$  is interior to precisely one of these loops (say  $L'$ ), then there is no way to connect it to  $A_2$ , without crossing an edge of  $L'$ . The case when  $A_3$  and  $B_3$  are both outside  $L$  is done in exactly the same way.

**Remark.** Another way to prove that  $K_{3,3}$  is not planar is by Euler characteristic formula for planar graphs, combined with some inequalities. We discussed this in details in section. □

**Problem 2.** *Read paragraph 2 on p.21 of Stillwell carefully, and apply it to see how the orientation  $(P_0, P_1, P_2)$  of a triangle induces an orientation of its edges. Does the result look right? How can you fix it so it makes sense?*

**Solution:** The problem is that the orientation, obtained by simply omitting the vertices is not cyclic. For instance, if  $(P_0, P_1, P_2)$  is a 3-simplex, then by what Stillwell writes, all edges  $(P_i, P_j)$  are oriented in increasing order, i.e. from  $P_i$  to  $P_j$ , where  $i < j$ . This, however, doesn't give us a cyclic orientation. In order to fix this problem, we include the factor  $(-1)^i$ , where  $i$  is the index of the removed vertex. For instance, by removing  $P_1$ , the edge obtained will be oriented from  $P_2$  to  $P_0$ . □

**Problem 3.** *Draw the Cayley graphs of the groups with presentation  $\langle a, b : a^2, b^2 \rangle$  and  $\langle c, d : c^2, (cd)^2 \rangle$ . Then use Tietze moves to prove these groups are isomorphic.*

**Solution:** The following sequence of Tietze moves transforms one of the groups to the other one:

$$\begin{aligned} \langle c, d : c^2, (cd)^2 \rangle &\simeq \langle c, d, e : c^2, (cd)^2, e = cd \rangle \text{ (type 2 move)} \\ &\simeq \langle c, d, e : (cd)^2, c^2, e = cd, d = ce, e^2 \rangle \text{ (type 1 move)} \\ &\simeq \langle c, d, e : (cd)^2, c^2, d = ce, e^2 \rangle \text{ (type } 1^{-1} \text{ move)} \\ &\simeq \langle c, e : c^2, e^2 \rangle \text{ (type } 2^{-1} \text{ move)}, \end{aligned}$$

as we need. □

**Problem 4.** *Describe a knotted closed loop that can be traced by walking along the stairways and halls of the Science Center, without crossing your path.*

**Solution (due to Prof. McMullen):** Two people  $A$  and  $B$  walk from up the main stairs from floor 1 to floor 3, arriving outside 309 and holding hands. They start with their backs to us as seen from the yard, in position  $A$ - $B$ . They arrive on the third floor facing us in position  $B$ - $A$ . Now  $B$  goes down one wing of the Science Center toward Littauer, and  $A$  goes down the other wing to the new History of Science department. They go down the stairs in their wings, arrive on the first floor, and rejoin. Now they are in position  $B$ - $A$  with their backs to us. So the two paths they have traced join together to form a closed path, which is in fact a knot. The knot is simply the closure of a braid on two strands with an odd number of twists. The number of twists comes out to be 3, which gives the trefoil.  $\square$

**Problem 5.** Show how to switch one crossing of the  $5_2$  knot so the result  $K$  is still knotted. What knot do you get? Use Reidemeister moves to justify your answer.

**Solution:** Using the illustration of  $5_2$  on p.280 in Adams' book, we can simply switch the right-most undercrossing with an overcrossing. It is a simple exercise to give the sequence of Reidemeister moves that transforms this knot into the trefoil  $3_1$ .  $\square$

**Problem 6.** Let  $N$  be your student ID number (or any other random number with 9 digits). Draw the knot  $9_{18}$ , and label the crossings  $1, 2, \dots, 9$  from top to bottom. Make a new knot projection  $K$  by switching the  $i$ -th crossing of  $9_{18}$  whenever the  $i$ -th digit of  $N$  is odd. (For example, if  $N = 617495277$ , then you should switch crossings 2, 3, 5, 6, 8, 9.) Now, simplify  $K$  as much as possible, and locate an equivalent knot in the knot tables. Show  $K$  and  $L$  are the same knot by describing Reidemeister moves that transform  $K$  into  $L$ .

**Solution:** The odd digits in my student ID number are the first, the third and the eight. Luckily, after switching these three crossings, we get a knot, which is equivalent to the unknot.  $\square$

**Immersed loops.** A smooth map  $f : S^1 \rightarrow \mathbb{R}^2$  is an immersion if  $f'(x) \neq 0$  for all  $x$ . Two immersions  $f_0$  and  $f_1$  are regularly homotopic if they can be connected by a continuous family of immersions  $f_t, t \in [0, 1]$ . The image  $f(S^1) = L \in \mathbb{R}^2$  of an immersion in general position is like an (oriented) knot projection, where you forget about the over/under data at the crossings. It is a loop with traverse double points.

Two immersed loops  $L_0$  and  $L_1$  in general position are regularly homotopic iff one can be transformed into the other using Reidemeister moves II and III but not I.

**Problem 7.** Prove that the figure eight (an immersed loop with one crossing) is not regularly homotopic to the unloop.

**Solution:** Simply notice that the parity of the number of crossing of an immersed loop doesn't change under Reidemeister moves II, III. Since the figure "eight" has odd number of crossings, whereas the unknot has even number of crossings, then there is no sequence of Reidemeister moves II and III, transforming one of the knots

to the other one. □

**Problem 8.** Find an invariant  $n(L) \in \mathbb{Z}$  of oriented immersed loops that doesn't change under Reidemeister moves II and III. Using this, show there are infinitely many different regular homotopy classes of immersed loops.

**Solution (due to Ian Le):** One idea to produce an infinite family of non-equivalent loops is to use the winding number. To explain this notion in a better way, first assume that the curve is differentiable (which we can always do). Then orient the curve and look at the tangent vector as it traverses the curve. When one goes back to the original position, the moving tangent vector has rotated at  $2\pi n(L)$  for some  $n(L) \in \mathbb{Z}$  (Note that the winding number can be defined formally as in complex analysis, but we will omit the formal details). We can immediately notice that the winding number doesn't change under Reidemeister moves II and III (only move I changes it by  $+1$  or  $-1$ ). This in particular solves problem 7 in a different way, since the winding number of the unknot is 0, whereas that of the figure "eight" is 1. For this problem, simply notice that we can define immersed loops with any winding number  $n(L) \in \mathbb{Z}$  simply by performing  $n(L)$  consecutive Reidemeister I moves on the unknot and then forgetting about the over/under data. Thus, we get an infinite family.

**Remark.** Some people used other invariants as well, such as  $n(L) := S(L) - F(L)$ , where  $S(L)$  denotes the number of "smiles" (points, where the tangent vector is horizontal, points to the right, and the slope of this vector increases to zero as we move towards this point along the orientation) and  $F(L)$  is the number of frowns of the immersed loop (points, where the tangent vector is horizontal, points to the right and where the slope of this vector decreases to zero as we move towards this point along the orientation). □

Solutions to problem 9 and 10 are due to **Richard Rivero**.

**Problem 9.** Formulate conjectures about the classification of immersed loops in  $\mathbb{R}^2$  and  $S^2$  up to regular homotopy.

**Conjecture 1.** For  $n \in \mathbb{Z}$ ,  $n \geq 3$ , define the  $n$ -immersion to be the immersion constructed by attaching one small loop to each of the vertices of a regular  $n$ -gon. Furthermore, let the 1-immersion be the unloop and the 2-immersion loop be the figure "eight" loop.

There is a difference between immersed loops in  $\mathbb{R}^2$  and those in  $S^2$ . For immersed loops on the sphere, we have the freedom to unknot two small loops at a time via carrying pieces of the immersed loop through one of the poles into the opposite hemisphere, without Reidemeister I moves. This reduces significantly the distinct equivalence classes of immersed loops.

**Conjecture 2.** Every immersion on the 2-sphere is regularly isotopic to either

the unloop or the figure eight loop. In fact, all the odd  $n$ -immersions are regularly isotopic to the figure eight loop, and all the even  $n$ -immersions are regularly isotopic to the unloop.  $\square$

**Problem 10.** *Prove one of your conjectures.*

**Proof of Conjecture 2.** Assume that Conjecture 1 is true. Let  $L \subset S^2$  be an immersion. If necessary, move  $L$  so that it doesn't intersect the point  $p = (0, 0, 1) \in S^2$ . Now, consider the immersion  $L' \subset \mathbb{R}^2$  which is the image of  $L$  under stereographic projection. Use Conjecture 1 to classify  $L'$  as an  $n$ -immersion, and then use Reidemeister moves to transform  $L'$  into "standard" form  $L_n$  (i.e. a regular polygon with loops attached). Now, it is not hard to see that a Reidemeister move II applied to  $L'$  corresponds to a Reidemeister move II applied to  $L$  (and vice-versa), and similarly for Reidemeister III moves. Hence, by applying Reidemeister moves to  $L'$ , we have in fact also transformed  $L$  into  $L_n$ . Lastly, use the move described above in question 9 to undo loops in  $L_n$ , two at a time. If  $n$  is odd, then the figure eight will be the result, and if  $n$  is even, we get the unloop.  $\square$