Problems

Dynamics and Moduli Spaces
Fall 2012

1. Let $f(x) = 2x \bmod 1$ on $S^1 = \mathbb{R}/\mathbb{Z}$. What are the periodic points of f? Show that periodic points are dense and that the inverse orbit of any point is dense (in S^1).

2. Give explicit examples of infinitely many rationals $x = p/q \in S^1$ (with $\gcd(p, q) = 1$) such that x is periodic under f and its orbit has size $O(\log q)$.

3. *Artin's conjecture* implies that 2 generates $(\mathbb{Z}/p)^*$ for infinitely many primes p. What does this say about periodic points for $f(x)$?

4. Show that for any $\epsilon > 0$ there exist infinitely many primes q such that $x = 1/q$ has period bigger than $q^{1/2-\epsilon}$.

5. Let $E \subset S^1$ be a closed set such that $f|E$ is injective and $f(E) = E$. Prove that E is finite.

6. Prove that for any rational p/q, there exists an $x \in S^1$ such that the points $x, f(x), f^2(x), \ldots, f^q(x) = x$ appear in the same cyclic order as the points $0, p/q, 2p/q, \ldots, qp/q = 0$. (Example: for $p/q = 1/3$ we can take $x = 1/7$.)

7. Show the set of points in S^1 with dense forward orbits is a dense G_δ of full measure.

8. Show that same is true with $f|S^1$ replaced with $f|J(f)$ for any rational map f of degree > 1.

9. Compute the function $c = \pi(\lambda)$ such that $z^2 + c$ is linearly conjugate to $\lambda z + z^2$. What is special about the values of c and λ where $\pi'(\lambda) = 0$?

10. Define the automorphism group of a polynomial of degree d, and show it is naturally a subgroup of $\mathbb{Z}/(d-1)$. Describe, up to conjugacy, all the cubic polynomials with symmetry group $\mathbb{Z}/2$.

11. What is the approximate value of the Hausdorff dimension of $J(z^2 + c)$ when $|c| \gg 0$?

12. Let $D_1, \ldots, D_d \subset \Delta$ be disjoint disks inside the unit disk, each with compact closure in Δ. Let $f_i : D_i \to \Delta$ be a Riemann mapping (conformal isomorphism), and define $F : \bigcup D_i \to D_i$ by $F|D_i = f_i$.

    Show that $K(F) = \bigcap F^{-n}(\Delta)$ is a Cantor set, and that $F|K(F)$ is topologically conjugate to the 1-sided shift on d symbols. (This is $[d]^{\mathbb{N}}$ with $\sigma(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$.)

13. The hyperbolic metric on $\mathbb{H}$ is given by $|dz|/y$. Choose a Möbius transformation $m : \mathbb{H} \to \Delta$ and use it to show the hyperbolic metric on Δ is given by $2|dz|/(1 - |z|^2)$.

    Using the covering map $\pi : \mathbb{H} \to \Delta^*$ given by $\pi(z) = e^{iz}$, prove that the hyperbolic metric on Δ^* is given by $|dz|/|z||\log|z||$.

14. Using the explicit metrics above, show that the inclusion $\iota : \Delta^* \to \Delta$ is a contraction for the hyperbolic metric, and an asymptotic isometric near S^1. (That is, show $\|D\iota(z)\| \to 1$ as $|z| \to 1$.)

15. Critique the following argument, and add details to make it correct.

    Let $\mathcal{F}$ be the family of analytic functions $f : U \to \mathbb{C}$ which omit 0 and 1. Since the universal cover of $\mathbb{C} - \{0, 1\}$ is isomorphic to the unit disk, $\mathcal{F}$ can be locally lifted to a family of bounded analytic functions, and therefore $\mathcal{F}$ forms a normal family.

16. Give an example of a Jordan curve $J \subset \mathbb{C}$ and a degree 2 expanding map $f : J \to J$ such that J is not a quasicircle. Here *expanding* means there is a $\lambda > 1$ such that $|f(x) - f(y)| > \lambda|x - y|$ whenever x, y are 2 distinct points in J that are close enough together.

    Hint: the cusp defined by $y^2 = x^3$ has unbounded turning at $(x, y) = (0, 0)$, so it cannot be part of a quasicircle; nevertheless it is invariant under $(x, y) \mapsto (4x, 8y)$.

17. Show that the degree 1 polynomials $f_p(z) = \lambda_p z$ for $\lambda_p = \exp(2\pi i p/7)$ are not topologically conjugate, for $p = 1, 2$.

18. Let $f : X \to Y$ be a local homeomorphism between (Hausdorff) locally compact topological spaces. Prove that f is a covering map if it is proper. Is the converse true?

19. Give an example of a surjective holomorphic map $f : X \to X$ which is a local homeomorphism but which is not proper, for $X = \mathbb{C}$ and Δ.

20. Let X be a compact Riemann surface of genus $g \geq 2$. Show that any nonconstant holomorphic map $f : X \to X$ is actually an automorphism of finite order.

21. Let f be a rational map of degree $d > 1$, and let U be a connected open subset of $\widehat{\mathbb{C}}$. Let $P(f)$ denote the postcritical set (the closure of the forward orbits of the critical points of f). Prove that the following two conditions are equivalent.

    (i) $U \cap P(f) = \emptyset$. (ii) For any $n > 0$ and any component V of $f^{-n}(U)$, the map $f^n : V \to U$ is a covering map.

    (In particular, if U is a disk, then there is a univalent branch of f^{-n} sending U to V.)

22. Let $f : \Delta \to \mathbb{C}$ be a holomorphic map with $f'(0) = 1$. Show there exists an $\epsilon > 0$ such that if $|f''/f'| \leq \epsilon$ throughout the disk, then $f(\Delta)$ is convex and f is one-to-one.

23. Prove that the forms defined by $D(f) = \log f'$, $N(f) = (f''/f')dz$ and $S(f) = ((f''/f')' - 1/2(f''/f')^2)\, dz^2$ all satisfy the cocycle condition

    $$C(f \circ g) = C(g) + g^*(C(f)).$$

24. Let $f : X \to X$ be a *recurrent* self-covering map of a hyperbolic Riemann surface. This means there exists a z such that $f^n(z)$ does not tend to infinity, i.e. the forward orbit of z returns infinitely often to a compact set K.

    Show that either (i) f has finite order or (ii) f is an irrational rotation of a disk, a punctured disk or an annulus.

25. Let $f_\lambda(z) = \lambda z + z^2$. Suppose $0 < |\lambda| < 1$. When restricted to its Julia set, is $f_\lambda(z)$ generally *more expanding* or *less expanding* than $f_0(z) = z^2$ on $J(f_0) = S^1$?

    (There are at least two answers here, depending on whether one takes 'generally' to mean 'for random z with respect to harmonic measure' or 'for random z with respect to Hausdorff measure'.)

    Answer the preceding question when $|\lambda|$ is large.

26. Let $f(z) = z + z^2$. (i) Prove that any triangle with one vertex at $z = 0$ meets $J(f)$. (ii) Does the $K(f)$ contains a (small) disk in the upper halfplane, resting on the real axis at $z = 0$?

27. Formulate and prove a theorem of the following form: *A topological conjugacy between expanding dynamical systems is always Hölder continuous.*

28. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^\alpha$ for $x \geq 0$ and $f(x) = -x^\beta$ for $x < 0$, where $0 < \alpha, \beta$. Prove that f is quasisymmetric iff $\alpha = \beta$.

29. *Prove that there exists a map $f : S^1 \to S^1$ such that both f and f^{-1} are Hölder continuous, but f cannot arise as the boundary correspondence across a Jordan curve in $\widehat{\mathbb{C}}$. (Hint: use the map from the previous exercise.)

30. Prove that a topological conjugacy between two expanding, $C^{1+\alpha}$ maps on S^1 is quasisymmetric.

31. Let $B_\alpha(z) = z(z+\alpha)/(1+\overline{\alpha}z)$, $|\alpha| < 1$, be a proper, degree two map of the unit disk to itself. Find explicitly the rational map $f(z)$ (suitably normalized) obtained by gluing $B_\alpha(z)$ to $B_\beta(z)$.

32. Let E be the closure of the set of $(x, y) \in [0, 1] \times [0, 1]$ such that, if we expressed $x = 0.x_1x_2\ldots$ and $y = 0.y_1y_2\ldots$ in base 3 and base 2 respectively, then $(x_i, y_i) = (0, 0), (1, 1)$ or $(2, 0)$.

    Compute the box dimension (metric dimension) δ of E.

    Remark: it can be shown that $\mathrm{H.dim}(E) = \log_2(1 + 2^{\log_3 2}) < \delta$.

33. Prove that a quasisymmetric map $f : S^1 \to S^1$ is Hölder continuous.

34. Prove that the quasisymmetric homeomorphisms of S^1 form a group.

35. Give an example of a homeomorphism $f : S^1 \to S^1$ such that f and f^{-1} are both Hölder continuous, but f is not quasisymmetric.

36. Let $K \cong S^1$ be the Koch snowflake curve. Show there is a degree 4, piecewise linear covering map $f : K \to K$ with $|f'(z)| = 3$ almost everywhere. Is this more or less expanding that $z^4|S^1$?

37. Let $\sigma : S^1 \to S^1$ be the squaring map, acting on $\mathcal{H} = L^2(S^1)$ preserving the unit measure dm.

    Describe all the closed subspaces of $\mathcal{H}$ that are invariant under by σ and S^1.

38. Verify that σ has rapid decay of correlations: for any $f \in \mathcal{H}$, $\langle f, f \circ \sigma^i \rangle \to 0$ exponentially fast.

39. Let $\mathcal{H}_0$ denote the subspace where $\int f\, dm = 0$. The *variance norm* on $\mathcal{H}_0$ is defined by $V(f) = \lim(1/n)\|S_n(f)\|_2^2$, where $S_n(f) = \sum_0^{n-1} f \circ \sigma^i$.

    Show $V(f) = \langle f, f \rangle + \sum_1^\infty \langle f, f \circ \sigma^i \rangle$.

40. (i) Compute $V(z^n)$ for all $n \in \mathbb{Z}$.

    (ii) Compute $V(z^a, z^b)$ for the bilinear form underlying $V(f)$.

41. Give an example of a function of mean zero in $L^2(S^1)$ such that $V(f) = \infty$. *Do the same with $f \in C(S^1)$.

42. Consider the Ruelle operator, normalized so that $L_\phi(1) = 1$, with associated invariant measure m. Let $C_0(X) = \{f \in C(X) \,:\, \int f\, dm = 0\}$. Show that the spectral radius of $L_\phi | C_0(X)$ is one, even though this operator has no eigenvalues of norm 1.

43. Let $X(L)$ be the complete Riemann surface homeomorphic to a pair of pants, with cuffs all of the same length L. Prove there is a conformal map $X(L) \to X(L')$ in the obvious homotopy class iff $L' < L$.

44. Use the preceding discussion to show that the Hausdorff dimension of the limit set $\Lambda(L)$ is a monotone function of L. (Hint: every geodesic gets shorter under an inclusion.)

45. (Lakes of Wada). Given an explicit example of three disjoint open topological disks in the plane that have the same boundary.

    Show that such an example cannot exist if the boundary is locally connected.

46. Give a closed formula for the set of points of period n for the polynomial $f(z) = z^2 - 2$.

47. Find a procedure $B = f(A)$ to associated a set B of 2-points on $\widehat{\mathbb{C}}$ to any 3 points A on $\widehat{\mathbb{C}}$, such that $f(\gamma A) = \gamma f(A)$ for any $\gamma \in \operatorname{Aut} \widehat{\mathbb{C}}$.

    Show that A consists of the roots of $p(z) = z^3 + az + b$, then B consists of the roots of $q(z) = 3az^2 + 9bz - a^2$.

48. Show that Newton's method $f_p(z)$ for $p(z)/q(z)$ is an expanding rational map of degree 4, and that the conjugacy class of f_p within rational maps of degree 4 does not depend on the cubic p.

49. Show that $f_p^n(z)$ converges to a root of p for almost every $z \in \widehat{\mathbb{C}}$.

50. There is a unique hyperbolic metric on open intervals $I \subset \mathbb{R}$ such that every fractional linear tranformation $I \to J$ is an isometry, and such that $I = (0, \infty)$ has the metric dx/x.

    Find the hyperbolic metric on (a, b).

51. Show that if $f : I \to I$ has positive Schwarzian, then f is a contraction for the hyperbolic metric.

52. Let K be an open convex subset of $\mathbb{RP}^n$. The *Hilbert metric* on K is defined by the requirement that on any maximal interval linear interval $I \subset K$, the inclusion $I \subset K$ is an isometry for the hyperbolic metric on I.

    Show that the Hilbert metric satisfies the triangle inequality.

53. Show that if $f : K \to K$ is a projective linear map, and $f(K)$ has compact closure in K, then f is a strict contraction for the Hilbert metric. Deduce that f has a unique fixed point.

54. Using the preceding results to deduce that a matrix $A_{ij} > 0$ has a unique positive eigenvector on $\mathbb{R}^n$, and that the iterates of any positive vector converge projectively to this eigenvector.

55. Prove that $C^\alpha(X)$ is a Banach algebra, for any compact metric space X. Here $\|\phi\| = \sup |\phi(x)| + \sup |\phi(x) - \phi(y)|/d(x, y)^\alpha$. One must show $C^\alpha(X)$ is a Banach space and $\|\phi\psi\| \leq \|\phi\| \cdot \|\psi\|$.

56. Let $f : S^2 \to S^2$ be an orientation–preserving diffeomorphism of finite order. Suppose f fixes 3 points. Show that f is the identity.

57. Let $\Phi_c : \mathbb{C} - \overline{\Delta} \to \mathbb{C} - K(f_c)$ be the uniformizing map for the outside of the Julia set of $f_c(z) = z^2 + c$. Find the power series in $1/z$ which represents $d^2\Phi_c/dc^2$ at $c = 0$. (We know $d\Phi_c/dc = -z(\sum_1^\infty a_{2^i}/z^{2^i})$.)

58. Let $\Phi_c : \mathbb{C} - \overline{\Delta} \to \mathbb{C} - K(f_c)$ be the uniformizing map for $f_c(z) = z^d + cq(z)$, where $\deg q(z) = d - 2$. Find $d\Phi_c/dz$ at $c = 0$.

59. The value $\lambda(E)$ of the cross-ratio of an ordered 4-tuple $E \subset \widehat{\mathbb{C}}$ lies in $X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$. Let us say an orientation-preserving homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *cross-ratio bounded* if

$$\sup_E d(\lambda(E), \lambda(f(E))) < \infty,$$

where distance is measured in the hyperbolic metric on X.

Show that if f arises as the time t map of a holomorphic motion of the whole sphere, then f is cross-ratio bounded.

60. Show that if f is cross-ratio bounded, then f is quasiconformal.

61. *Show that if f is quasiconformal, then f is cross-ratio bounded.

62. Give an example of a finite graph $E \subset \widehat{\mathbb{C}}$ and an embedding $f : E \to \widehat{\mathbb{C}}$ such that f cannot be extended to the sphere locally but not globally. (The local condition means that for all $x \in E$ there is a neighborhood $U \subset E$ of x such that $f|U$ extends to the sphere.) Find an additional local condition that characterizes when f extends globally.

63. Show that $V(z) = \sum_0^\infty 2^{-n} z^{2^n}$, as a function on S^1, is continous but nowhere differentiable. Show however that $V(z)$ is nearly Lipschitz, in the sense that it has an $x|\log x|$ modulus of continuity.

64. Look up the definition of a function in Zygmund class. Show that Lipschitz functions are Zygmund and Zygmund functions are in C^α for all $\alpha < 1$; in fact, they have an $x|\log x|$ modulus of continuity.

65. Show that $V(z)$ is in the Zygmund class.

66. Show that a bounded vector field on the plane with an $x|\log x|$ modulus of continuity is uniquely integrable.

67. Let $E = \mathbb{C}^*/q^{\mathbb{Z}}$ be the elliptic curve associated to a complex number with $0 < |q| < 1$. Define $f : \mathbb{C}^* \to \mathbb{C}^*$ by $f(x + iy) = Kx + K^{-1}iy$, $K > 1$. Then $f(qz) = qf(z)$, so f descends to a map $F : E \to E$ which is homotopic to the identity. In local complex coordinates, F is an affine stretch, yet F is not the map of least conformal distortion in its homotopy class — the identity map is. Does this contradict Teichmüller's theorem?

68. Let $\Sigma(\Gamma)$ be the shift associated to a finite graph Γ. Show that the following are equivalent: (i) $\Sigma(\Gamma)$ has a dense (forward) orbit; (ii) any two vertices in Γ are connected by a directed path; (iii) Γ is a connected union of directed cycles.

69. Let Γ be a nonelementary Kleinian group. (i) Prove that Γ contains a hyperbolic element. (ii) Prove that Λ is perfect. (iii) Prove that Λ is the closure of the set of repelling fixed points of elements of Γ.

70. Prove that any nonelementary Kleinian group Γ contains a free group on 2 generators.

71. Prove that any nonelementary Kleinian group Γ is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$. (This means the only Lie group of positive dimension containing Γ is $\mathrm{PSL}_2(\mathbb{C})$.)

72. Let $q \in Q(\mathbb{H})$ be a holomorphic quadratic differential. Prove there exists a conformal map $f : \mathbb{H} \to \widehat{\mathbb{C}}$ with Schwarzian derivative $Sf = q$. (Hint: write f as the ratio of two solutions to the differential equation $y'' + aqy = 0$, for suitable $a \in \mathbb{R}$.)

73. Let Γ be a Kleinian group and suppose $X = \Omega/\Gamma$ has a cusp. Show that Γ has a parabolic element.

74. Let $f : \Delta \to \Delta$ be a holomorphic function. Suppose the radial limit of $f(z)$ vanishes on a set $E \subset S^1$ of positive measure. Prove that f vanishes identically. (Hint: use the fact that $\log |f(z)|$ is subharmonic.)

75. Let $\rho_n : \pi_1(\Sigma_g) \to \Gamma_n$ be a sequence of quasifuchsian groups in a Bers slice which converge to a totally degenerate group Γ. Show that the limit sets satisfy $\Lambda(\Gamma_n) \to \Lambda(\Gamma)$ in the Hausdorff topology.

76. (Kodaira surface.) Give an example of a compact Riemann surface X and a nonconstant holomorphic map $f : X \to \mathcal{M}_g$. (Hint: send $p \in X$ to a Riemann surface Y which is a canonical covering space of X branched over p.)

77. Compute the representation of $\pi_1(X)$ into $\mathrm{SL}_2(\mathbb{Z})$ coming from monodromy in the family of Riemann surfaces $y^2 = x(x-1)(x-t)$ over the space $X = \mathbb{C} - \{0, 1\}$.

78. Compute the monodromy around $t = 0$ of the family of Riemann surfaces given by $y^2 = x(x^n - 1)(x - t)$, both as an element of Mod_g and of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

79. Give an example of a nonconstant holomorphic curve $f : Z \to \mathcal{M}_g$ such that the monodromy of f acts *reducibly* on the homology of the surface of genus g (even though it is *irreducible* as a subgroup of the mapping–class group.)

    (Hint: such examples arise easily from a Lefschetz pencil on a surface X with $b_1(X) \neq 0$. As an explicit example, one might take a product of elliptic curves $X = E \times X$, together with the map $p : X \to \mathbb{P}^1$ given by $p(z, w) = \wp(z)\wp(w)$, and consider this surface as a family of Riemann surfaces of genus g over $\mathbb{P}^1 - D$, for a suitable finite set D.)

80. (Boundary of Bers' embedding.) Show that for any analytic variety $W \subset \partial\mathcal{T}_g \subset \mathbb{C}^{3g-3}$, there is a holomorphic function f on $\mathbb{C}^{3g-3}$ such that $f(W) = 0$ and f is nowhere zero on $\mathcal{T}_g$.

81. (i) Show the Siegel space $\mathfrak{H}_g$ can be naturally realized as a bounded domain in $\mathbb{C}^N$, $N = g(g+1)/2$. (ii) Does the preceding result hold for analytic subvarieties $W \subset \partial\mathfrak{H}_g$?

82. Show that in general, a nonconstant holomorphic map $f : Z \to \mathcal{A}_g$ is not determined by its monodromy. Where does the proof of rigidity for $\mathcal{M}_g$ break down for $\mathcal{A}_g$?

83. Construct a non-constant holomorphic map $f : \Delta \to \Delta$ such that $\lim_{r \to 1} f(re^{it}) = 0$ for a dense set of $t \in \mathbb{R}$. (This shows the boundary values of f on a dense set cannot determine f.)

84. Let $p : X \to Y$ be a nonconstant holomorphic map between hyperbolic Riemann surfaces, with at least one critical point. Show that its lift $P : \Delta \to \Delta$ to the universal covers of X and Y does not extend continuously to S^1.

85. Given an example of a pair of distinct holomorphic curves $f_i : B_i \to \mathcal{M}_g$, $i = 1, 2$ with the same monodromy.

86. Given an example of a continuous map of degree zero $f : \Sigma_2 \to \Sigma_1$, from a surface of genus two to a surface of genus one, such that f is surjective on π_1.

    What does the harmonic representative of f look like?