Dynamics and algebraic integers: Perspectives on Thurston’s last theorem

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There exists a post-critically finite map $f: I \to I$ with entropy $h(f) = \log(\lambda)$

For any Perron number $\lambda$, there exists an $f : I \to I$ with $h(f) = \log(\lambda)$

Thesis: This result is an instance of the h-principle.

Defn. "Every continuous section of a suitable partial differential relation in a jet bundle is homotopic to a holonomic section."

Theorem (Thurston, Entropy in dimension one, 2014)

There exists a post-critically finite map $f : I \to I$ with entropy $h(f) = \log(\lambda)$

$\lambda$ is a weak Perron number.

Examples of the h-principle

There exists a continuous, nowhere differentiable function.

Weierstrass
A closed hyperbolic surface embeds isometrically ($C^1$) into $\mathbb{R}^3$. 

(Nash-Kuiper)

The sphere can be turned inside-out. 

(Thurston)

There exists a path-isometry $f : S^2 \rightarrow \mathbb{R}^2$.

(Length of $\gamma = \text{length of } f \circ \gamma$.) 

(Gromov)

A manifold admits a codimension 1 foliation $\iff \chi(M) = 0$ 

(Thurston)

(Lawson, Haefliger, Bott)
There exists a Perron-Frobenius matrix $P$ in $M_n(\mathbb{Z})$ with spectral radius $\rho(P) = \lambda$ if and only if $\lambda$ is a Perron number. Perron-Frobenius means $P_{ij} \geq 0$ and $P^k_{ij} > 0$ for some $k$.

Constructing $P$ from $\lambda$:

1. $\deg(\lambda) = d$
2. $T$ in $M_d(\mathbb{Z})$
3. $Tv = \lambda v$
4. Find convex $n$-gon $Q$, $T(Q) \subset Q$
5. PF matrix $P$ in $M_n(\mathbb{R})$, $\rho(P) = \lambda$.

Even for cubic $\lambda$, we may be forced to take $n \gg 0$. (M, Lind)

**Theorem** (Thurston)

There exists a post-critically finite map $f : I \rightarrow I$ with entropy $h(f) = \log(\lambda)$ if and only if $\lambda$ is a (weak) Perron number. Why is entropy is Perron? Markov partitions

Golden mean example:

For any pcf $f$, $h(f) = \log(\text{Perron})$.

\[
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\]

\[
\lambda = \rho(P) = \frac{1 + \sqrt{5}}{2}
\]
Constructing $f$ from $\lambda$

For any Perron number $\lambda$, there exists an $f : I \to I$ with $h(f) = \log(\lambda)$

$\{\text{Perron numbers } \lambda\} = \bigcup M_k$

$M_k = \{\lambda \text{ arising from } f \text{ with } k \text{ laps}\}$

STABLE
At any stage in the construction one can increase # laps or size of post-critical set.

[Proof. First step is Lind's theorem.]

Quadratic Entropy Problem

Describe $M_2 = \{\lambda \text{ arising from quadratic } f\}$.

$\lambda$ is in $M_2 \iff 0$ has finite orbit under $T_\lambda(x) = \lambda(1 - |1-x|)$

Q. Does there exist an algorithm to test if $\lambda$ is not in $M_2$?

Known: Pisots $\subset M_2$

Unknown (e.g.): Is Lehmer’s number in $M_2$?

Surface Entropy Conjecture

There exists a pseudo-Anosov map $f : S \to S$ with $h(f) = \log(\lambda)$

$\iff \lambda$ is a biPerron number.

STABLE (open)

(Fried 1985, Thurston)

Minimum entropy problem for surfaces

$log(\delta_g) = \min \{h(f) : f \text{ pseudo-Anosov on a surface of genus } g\}$

= length of shortest geodesic on $M_g$.

Example: $\delta_1 = \text{root of } t^2 - 3t + 1$  

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  

$\delta_2 = \text{root of } t^4 - t^3 - t^2 - t + 1$  

(Cho-Ham)

$Lanneau–Thiffeault$

Problem: Determine $\delta_g$ for all $g$. 
**Minimum entropy problem for surfaces**

**Known:** \( \delta_g = 1 + O(1/g) \)  
(Penner, 1991)

**Question:** Does \( \lim (\delta_g)^g \) exist?  
(M, 2000)

**Conjecture:** \( \lim (\delta_g)^g = \delta_1 \)  
(E. Hironaka, 2010)

**Theorem:** \( \limsup (\delta_g)^g \leq \delta_1 \)  
(E. Hironaka, 2010)

(Aaber-Dunfield, Kin, ...)

**Q.** Where do \( f \) with \( h(f) = O(1/g) \) come from?

**A.** Fibered 3-manifolds.

\( M = \text{hyperbolic 3-manifold fibering over } S^1, \quad b_1(M) > 1 \)

\( \Rightarrow M \text{ fibers in infinitely many ways} \)

\( \Rightarrow \text{infinitely many } f : S \to S \text{ whose} \)

mapping torus is \( M \).

- Among these are \( f \) with \( h(f) = O(1/g) \)  
(M, 2000)

- All such \( f \) arise from finitely many \( M \).

(Farb-Leininger-Margalit, 2011)

**A manifold \( M \) from \( \delta_1 \)**  
(E. Hironaka, 2010)

\[
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})
\]

\( = \text{Mod}_1 \cong \text{Mod}_{0,4} \)

\( \updownarrow \)

Braids on 3 strands

\( M^3 = S^3-L \text{ fibers over } S^1 \)

**Entropy of other fibrations** \( M^3 \to S^1 \)

\( H_1(M) \)

\[
\begin{array}{ccc}
-1 & & \\
1 & -1 & 1 \\
-1 & & \\
\end{array}
\]

\( L_{2g}(t) = t^2 - t^g(t+1+t^{-1}) + 1 \)
**Teichmüller polynomial:**

**specializations**

\[ L_4(t) = t^4 - t^3 - t^2 - t + 1 \]

⇒ minimal genus 2 example

\[ L_{2g}(t) \Rightarrow \text{genus } g \text{ examples with } \delta(f)^g - \delta_1, \]

in agreement with conjecture

\[ L_{2g}(t) = t^{2g} - X^g(X+1+X^{-1}) + 1 \approx t^{2g} - 3t^g + 1 \]

**Conclusion:** \( \limsup (\delta_g)^g \leq \delta_1 \)

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**Remaining problem:** lower bounds for \( \delta_g \).

**Is there a more tractable, algebraic problem?**

\[ [f : S \to S] \Rightarrow \]

train track map \([F : \tau \to \tau] \Rightarrow \]

Perron-Frobenius \( P \), \( \delta(f) = \rho(P) \).

**Symplectic structure on \( \mathbb{PML} \Rightarrow P \) is reciprocal.**

(eigenvalues invariant under \( t \to 1/t \))

Q. Can we give a lower bound on \( \rho(P) \)?

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**Minimum entropy problem for graphs**

**Theorem**

(Max, 2013)

\[ \text{Min } \rho(P) : P \text{ in } M_{2g}(\mathbb{Z}) \text{ is reciprocal and Perron-Frobenius} \]

\[ \Rightarrow \text{metrized directed graph } (\Gamma, \lambda) \]

\[ \text{Largest root of} \]

\[ L_{2g}(t) = t^{2g} - t^g(t+1+t^{-1}) + 1 \]

**Corollary**

\[ \rho(P)^g \geq \delta_1 \text{ for all } P \text{ and } g. \]

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**Scheme of the proof**

\( P \text{ in } M_n(\mathbb{Z}), \text{Perron-Frobenius} \)

\[ \Rightarrow \text{metrized directed graph } (\Gamma, \lambda) \]

\[ \#(\text{closed loops } \leq T) \propto \rho(P)^T \]

**Optimal metric on } \Gamma \Rightarrow \text{invariant } \lambda(\Gamma) \]

\[ \rho(P)^n \geq \lambda(\Gamma) \]

**Fact:** \( \{\Gamma : \lambda(\Gamma) < M\} \text{ is finite} \)

**Case at hand, take } M=8. \]

\[ (\delta_1)^2 = (\text{golden ratio})^4 = 6.854... \]
Table 2. Number of trivalent graphs with a given curve complex $G$.

List of $G$ with $\lambda(G) < 8$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\lambda(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nA_1$</td>
<td>$2 \leq n \leq 7$</td>
</tr>
<tr>
<td>$A_2^*$</td>
<td>4</td>
</tr>
<tr>
<td>$A_2^{**}$</td>
<td>$5.82... = 3+2\sqrt{2}$</td>
</tr>
<tr>
<td>$A_2^{***}$</td>
<td>$7.46... = 4+2\sqrt{3}$</td>
</tr>
<tr>
<td>$A_3^*$</td>
<td>$5.82... = 3+2\sqrt{2}$</td>
</tr>
<tr>
<td>$Y^*$</td>
<td>$7.46... = 4+2\sqrt{3}$</td>
</tr>
</tbody>
</table>

$G$ carries key info of $\Gamma$

New invariant: $\lambda(\Gamma) \geq \lambda(G)$

$\{G : \lambda(G) < 8\} = 11$

(RAAGs, Cartier-Foata 1968)

Scheme of the proof

Analyse each $\Gamma : \lambda(\Gamma) < 8$.

(2009)

$\Gamma \Rightarrow$ curve complex $G$

$V(G) =$ {simple loops $C$ in $\Gamma$}

$E(G) =$ {disjoint $(C,D)$}

$\lambda(\Gamma)$ (2013)

$\lambda(\Gamma)$ (2009)
**Example:** These $\Gamma$ all have $G = 4A_1 = \bullet \bullet \bullet$

$\Rightarrow$ rest of proof is now tractable.

**Concluding Remarks**

- $M$: Teichmüller polynomial
- $\Gamma$: Perron polynomial
- $G$: Clique polynomial

thermodynamic formalism
convexity of pressure

**Problem:** Determine $\delta_g$ for all $g$?

**SEMISTABLE**

**EPILOGUE**

SPOKEN BY PROSPERO

Now my charms are all o'erthrown,

* * *

Gentle breath of yours my sails
Must fill, or else my project fails,
Which was to please.

Better for more people to be involved... it's a fun and interesting topic.