

# Crepant resolutions of Weierstrass models with torsion

Julian Salazar

[jsalazar01@college.harvard.edu](mailto:jsalazar01@college.harvard.edu)

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Thesis advisor: Mboyo Esole (Northeastern)

Departmental advisor: Noam Elkies

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## Abstract

We perform the first computation of the Euler characteristics of all crepant resolutions of Weierstrass models for elliptic  $n$ -folds with torsion  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , and  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , in terms of  $n$  and the base, using a strategy valid for any singular fibration of projective hypersurfaces that admits a nice crepant resolution. We resolve our Weierstrass models while preserving the base (in particular, non-normal crossings in the discriminant locus), and study collisions to retrieve singular, possibly non-Kodaira fibers. Significant exposition is included (at the level of an intermediate course in algebraic geometry) with a focus on computation, spanning from the classic theory of elliptic surfaces up to recent analytic pushforward formulas in intersection theory. Applications to string theory (via gauge groups in F-theory) and Calabi-Yau geometry (via Hodge numbers in the threefold case) are also discussed.

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# 1 Introduction

In this undergraduate thesis, we perform a novel and comprehensive study of Weierstrass models for elliptic  $n$ -folds for certain torsion groups; that is, elliptic fibrations with Mordell-Weil group  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , or  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  over a smooth complex projective variety of (complex) dimension  $n - 1$ . We construct a crepant resolution for a general Weierstrass model of each group, and examine how each resolution's singular fibers degenerate over loci of higher codimension. We then compute each resolution's Euler characteristic as a generating function valid for all  $n$  using the work of Aluffi, Esole, and others. By a result of Batyrev, this value is invariant across all crepant resolutions. The Weierstrass models we consider specialize to  $G$ -models in F-theory and invariant Hodge numbers in the Calabi-Yau threefold case, while the strategy generalizes to other singular fibrations of projective hypersurfaces.

To this end, we motivate the problem from the lenses of math and physics, and introduce both classic and contemporary notions from the theory of elliptic surfaces, complex algebraic geometry, and intersection theory that enable our results. This work is organized as follows:

- **§1: Introduction.** This work is overviewed and placed in context. We summarize our strategy and main results, and state our viewpoint in presenting this work.

These sections review classic material, giving motivations and examples relevant to our work:

- **§2: Weierstrass models of elliptic fibrations.** Elliptic curves, fibrations, and Kodaira's classification are reviewed; in parallel, we develop their Weierstrass models, on which Tate's algorithm can be applied.
- **§3: Intersection theory and cohomology.** We recall the theory of Chern classes and adjunction, and sketch the homological perspectives which bridge topology and algebraic geometry.

These sections examine more specialized and recent results that enable our strategy:

- **§4: Torsion and non-Kodaira fibers.** Weierstrass models of elliptic fibrations with torsion are discussed, along with the current theory of singular fibers in higher codimension.
- **§5: Pushforwards along crepant resolutions.** The theory of resolutions is reviewed, along with the results of Batyrev, Aluffi, and others which empower our general computation of invariants.

These sections represent the bulk of the novel work in this thesis:

- **§6-8: Cases.** Using the ideas from preceding sections, we execute our strategy for studying the degenerations of each torsional Weierstrass model and their crepant resolutions.
- **§9: Applications.** We generalize our strategy to models of the same general form, specialize our results to current work in Calabi-Yau geometry and F-theory, and suggest future avenues of research.

## 1.1 History and motivation

Few objects in mathematics enjoy the broad ubiquity and relevance that *elliptic curves* do. The layman might define these as smooth curves of points  $(x, y)$  on a plane given by equations of the form  $y^2 = x^3 + ax + b$ . The algebraic geometer might protest, saying this description is only affine, that any smooth plane cubic is an elliptic curve, that quartic models exist, and so on. The complex geometer might decry the use of coordinates, defining elliptic curves as compact Riemann surfaces of genus 1. The topologist might simply ignore the differential structure and see a 2-torus, the product of two circles  $T^2 = S^1 \times S^1$ . The number

theorist sighs, complaining that we never specified the base field, the distinguished point, or the group law!

In this thesis, we utilize all of these perspectives; in fact, we will find the layman’s definition the most pertinent! We are studying *elliptic fibrations* (which can be understood as parameterized families of elliptic curves) and their concrete realizations in the form of *Weierstrass models* (such as  $y^2 = x^3 + ax + b$ ), in the presence of *torsion* (elements of finite order under the group law).

## From mathematics to physics

Elliptic curves arise across a wide breadth of scientific fields. Beyond mathematics, they also feature in computer science, having found great use in applied computation and cryptography, leading to improved pseudo-random number generators, integer factorization techniques, and the rise of elliptic curve cryptography for public-key encryption online. However, their most remarkable use comes from theoretical physics via string theory. In his landmark 1996 paper [Vaf96], Cumrun Vafa proposed a novel framework called *F-theory*, which is a particular limiting form of Witten’s foundational M-theory. Both constitute frameworks aiming to unify quantum mechanics with Einstein’s theory of general relativity, one of the central problems of modern physics.

F-theory is a busy area of research; it provides some of the particular phenomenology that a quantum theory of gravity would require, along with constructions of explicit, semi-realistic vacua. It utilizes elliptic fibrations to model aspects of type IIB string theory (a 10-dimensional theory linked by duality to M-theory), whereby an elliptic  $n$ -fold is used to compactify type IIB theories down to  $12 - 2n$  (real) dimensions [Boy09], with  $n = 3, 4$  of particular research interest. This application is interesting as it gives physical interpretation to the various perspectives elliptic curves and fibrations have inspired across mathematics.

From geometry, elliptic fibrations and Weierstrass models like those in this thesis are concrete testbeds on which particular phenomenologies are studied; a model’s geometric singular fibers (generalizing those from Kodaira’s classification of singular fibers on elliptic surfaces) and the manner in which they degenerate in higher codimension have physical implications [Wei10]. From arithmetic, the data provided by the Mordell-Weil group of the fibration, in particular its torsion subgroup, are related to the gauge group of the theory, among other things [dBDH<sup>+</sup>00] (a desirable case would reproduce the Standard Model’s  $SU(3) \times SU(2) \times U(1)$ ). In all, these relationships give the general motivation for our work; we make more specific connections in Section 9.

## This work in context

This thesis is the natural continuation of a program that began with the study of elliptic surfaces:

- **Euler characteristic ( $\chi_{\text{top}}$ ):** Kodaira first computed  $\chi_{\text{top}}$  in the case of smooth elliptic surfaces. By the work of Sethi, Vafa, and Witten, this moved forward to the case of  $\chi_{\text{top}}$  for various smooth Calabi-Yau elliptic threefolds and fourfolds [SVW96]. Formulas in low dimension for smooth non-Weierstrass models were considered in weighted projective space by Klemm, Yau, et al. [KLY98], for a  $D_5$  fibration by Esole, Fullwood, and Yau [EFY15], etc. The singular model case was finally considered by Grassi and Morrison [GM03] for Calabi-Yau Weierstrass threefolds, but only through an elaborate excision argument (see Section 3.2 for a preview). The key innovation comes from Aluffi and Esole [AE09], who used the functoriality of the Segre class to compute Euler characteristics for arbitrary dimension via generating functions, in the smooth Weierstrass model case, and without the Calabi-Yau condition. In this thesis, along with the related paper [EJK17], this technique of analytic pushforwards (aided also by [Alu10]; see Section 5.3) is combined with crepant resolutions to get dimension- and resolution-independent generating functions for singular models as well.

- **Torsion** ( $MW_{\text{tors}}(\varphi)$ ): Past work typically required the Mordell-Weil group  $MW(\varphi)$  to be trivial. A non-Weierstrass model has been explored that encodes a rank-1 group [EKY14]; another might explore  $MW_{\text{tors}}(\varphi) \cong \mathbb{Z}/2\mathbb{Z}$ -torsion (e.g., [MMTW14]) but without computing invariants. Non-trivial torsion is not only mathematically natural, but also essential for certain gauge groups to arise in F-theory (e.g.,  $SO(n)$ ; see Section 9.2). Here for the first time (esp.  $\mathbb{Z}/3\mathbb{Z}$ ,  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ ), we consider classical Weierstrass models which *only* encode torsion, study their fibers and degenerations, then compute the Euler characteristic of their crepant resolutions. In particular, the effect of torsion on  $\chi_{\text{top}}$  is entirely expressed by the parameterization (though the story is different for the Hodge numbers  $h^{p,q}$ ; see Section 9.3).
- **Singular fibers** ( $\varphi^{-1}(p)$ ,  $p \in \Delta$ ): The types and degeneration of the fiber components in a base-preserving crepant resolution are important in F-theory, as they reflect the physics of the given model (see e.g., [EY13], [EFY15], and Section 9.2); here, we study these interactions for our models. Kodaira and Néron classified the types of singular fibers that occur in smooth minimal elliptic surfaces. Miranda extended this to threefolds [Mir83] and Szydło to  $n$ -folds [Szy99], but only for  $\Delta$  with only simple normal crossings (Section 4.3). Otherwise, their resolutions modify the base to construct these crossings, which affects the Euler characteristic and the physical interpretation in F-theory. A general classification does not exist, even for Weierstrass models and crepant resolutions; here, we encounter a non-Kodaira fiber not previously known to arise from Weierstrass models in codimension 2.

## 1.2 Method of approach

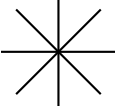
Here is the strategy we will expound upon in this thesis. We then apply it in Section 6 to Section 8:

- 1) For each torsion group, we construct a singular Weierstrass model  $\psi : \mathcal{W}_0 \rightarrow B$  (Section 2.2) that parameterizes the torsion group in generality (Section 4.2).
- 2) We apply Tate’s algorithm to preemptively detect the singular fibers over codimension 1 (Section 2.3).
- 3) We take successive crepant blowups  $\pi_k : X_k \rightarrow X_{k-1}$  of the ambient projective bundle to derive crepant resolutions  $\pi \circ \pi_1 \circ \cdots \circ \pi_m \circ \iota_m : \mathcal{W} = \mathcal{W}_m \rightarrow B$  for our Weierstrass models (Section 5.1).
- 4) By direct inspection, we observe how the fibers degenerate in higher codimension (Section 4.3).
- 5) Through adjunction (Section 3.1) and analytic pushforward formulas (Section 5.3), we express  $c(\mathcal{T}_{\mathcal{W}}) \sim [\mathcal{W}]$  in terms of  $c(\mathcal{T}_B)$ .
- 6) By the Poincaré-Hopf theorem (as a corollary of Hirzebruch-Riemann-Roch; see Section 3.3), this expression as a Chern polynomial gives a generating function  $\chi_t(\mathcal{W})$  with a coefficient for each  $\dim \mathcal{W}$ .
- 7) By the work of Batyrev (Section 5.2), all crepant resolutions of our models have the same Euler characteristic, generalizing our computation.

*Remark.* Two brief observations:

- Our approach to computing invariants for crepant resolutions is kindred to [EJK17], whose authors (including my advisor, Mboyo Esole) worked over the past year in parallel to my research. They focused on Weierstrass models that have trivial torsion (along with a handful of  $\mathbb{Z}/2\mathbb{Z}$ -torsion models).
- One does not need to study collisions to derive  $\chi_{\text{top}}$ —see Section 8.1 for an example where we justify the smoothness of a resolution without fiber analysis. We do not even need to be elliptic—see Section 9.1 which generalizes the solution. Our explicit study of fibers and resolutions are a related but independent endeavour, resembling in spirit the studies performed in [ESY15, §2] on  $SU(n)$ -models.

### 1.3 Summary of results

Torsion	Model	codim. 1	fiber	Resolution	codim. 2	fiber
$\mathbb{Z}/2\mathbb{Z}$	$y^2 = x(x^2 + a_2x + a_4)$	$a_4 = 0$	$I_2^{\text{ns}}$	$X_0 \xleftarrow{\langle x, y e_1 \rangle} X_1$	$a_2 = a_4 = 0$	III
		$a_2^2 - 4a_4 = 0$	$I_1$			
$\mathbb{Z}/3\mathbb{Z}$	$y^2 + a_1xy + a_3y = x^3$	$a_3 = 0$	$I_3^{\text{s}}$	$X_0 \xleftarrow{\langle x, y, a_3 e_1 \rangle} X_1$	$a_1 = a_3 = 0$	IV
		$a_1^3 - 27a_3 = 0$	$I_1$			
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$y^2 = xuv$	$\beta_2 = 0$	$I_2^{\text{ns}}$	$X_0 \xleftarrow{\langle x, y e_1 \rangle} X_1$	$\beta_2 = \gamma_2 = 0$	
	$(u = x - \beta_2)$	$\gamma_2 = 0$	$I_2^{\text{ns}}$	$X_1 \xleftarrow{\langle y_1, u, v e_2 \rangle} X_2$		
	$(v = x - \gamma_2)$	$\beta_2 - \gamma_2 = 0$	$I_2^{\text{ns}}$			

**Table 1.1:** Our general models for each torsion group, along with its Kodaira fibers, an explicit resolution via crepant blowups of the total space, the degenerate fiber, and respective base loci (Section 6 to Section 8).

Torsion model	Generating function $\chi_t(\mathscr{W})$	$\dim \mathscr{W}$	Euler characteristic $\chi_{\text{top}}(\mathscr{W})$
$\mathbb{Z}/2\mathbb{Z}$	$\frac{12Lt^2}{1+4Lt} c_t(\mathcal{T}_B)$	2	$12L$
		3	$12c_1L - 48L^2$
		4	$12c_2L - 48c_1L^2 + 192L^3$
		5	$12c_3L - 48c_2L^2 + 192c_1L^3 - 768L^4$
$\mathbb{Z}/3\mathbb{Z}$	$\frac{12Lt^2}{1+3Lt} c_t(\mathcal{T}_B)$	2	$12L$
		3	$12c_1L - 36L^2$
		4	$12c_2L - 36c_1L^2 + 108L^3$
		5	$12c_3L - 36c_2L^2 + 108c_1L^3 - 324L^4$
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\frac{4Lt^2(3+5Lt)}{(1+2Lt)^2} c_t(\mathcal{T}_B)$	2	$12L$
		3	$12c_1L - 28L^2$
		4	$12c_2L - 28c_1L^2 + 64L^3$
		5	$12c_3L - 28c_2L^2 + 64c_1L^3 - 144L^4$

**Table 1.2:** Euler characteristics for any crepant resolution over  $B$  of the corresponding torsional Weierstrass model  $\mathscr{W}$  inside  $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \rightarrow B$ . Here,  $c_i = c_i(\mathcal{T}_B)$  and  $L = c_1(\mathcal{L})$ . See also Section 3.3, Section 5.3.

Torsion model	Gauge group	CY-3 $h^{1,1}$	CY-3 $h^{1,2}$	CY-3 $\chi_{\text{top}}$	CY-4 $\chi_{\text{top}}$
$\mathbb{Z}/2\mathbb{Z}$	$\text{SO}(3) \cong \text{SU}(2)/\mathbb{Z}/2\mathbb{Z}$	$12 - K^2$	$12 + 17K^2$	$-36K^2$	$-12c_2K + 144K^3$
$\mathbb{Z}/3\mathbb{Z}$	$\text{PSU}(3) \cong \text{SU}(3)/\mathbb{Z}/3\mathbb{Z}$	$12 - K^2$	$12 + 11K^2$	$-24K^2$	$-12c_2K + 72K^3$
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$	$13 - K^2$	$13 + 7K^2$	$-16K^2$	$-12c_2K + 36K^3$

**Table 1.3:** Invariants of  $\mathscr{W}$  with a view towards string theory (Section 9): the F-theoretic gauge group, along with Euler characteristics and Hodge numbers in the Calabi-Yau elliptic 3-, 4-fold cases.



## 1.4 Viewpoint and conventions

This thesis is aimed at the level of an early graduate student who has taken a *first course in algebraic geometry* (e.g., at the level of [Sha13]), and who is comfortable with *vector bundles, differential forms, and basic algebraic topology*. Technical scheme-theoretic language is largely restricted to proofs. In fact, full proofs are relatively eschewed, in favor of intuitions and computed examples. For those less comfortable, the intuitions and strategy remain intact if all intervening exposition is imagined over  $\mathbb{C}$  and bundles are viewed analytically, thanks to GAGA (Theorem 3.38).

*Remark.* In addition to new results, we aim to preserve the spirit of *math theses as original expositions*; we use elliptic curves and fibrations as a vehicle to introduce the student to more advanced concepts in algebraic geometry (Chern classes, adjunction, resolutions of singularities) in a very concrete way (as a playground for computational examples), up to areas of original research! This reflects my own growth path in the subject over the past two years, for which I am grateful to my advisors.

**Definition 1.4.** We take the following conventions and definitions. The standard reference is [Har77]:

- An **algebraic set** over a field  $K$  is a reduced algebraic  $K$ -scheme; an **(algebraic) variety** is an irreducible algebraic set. For those unfamiliar, it suffices to consider **quasiprojective varieties**, which are varieties that are open subsets of projective varieties (generalizing affine and projective varieties) [Sha13]. **Subvarieties** are always assumed to be *closed*.
- A **curve** is an algebraic set of dimension 1. A **surface** is an algebraic set of dimension 2. A **hypersurface** in a variety is an algebraic subset of codimension 1.
- A **line bundle**  $\mathcal{L}$  over a variety  $X$  is an invertible sheaf on  $X$ , i.e., it is coherent and invertible (up to isomorphism) with respect to the tensor product of  $\mathcal{O}_X$ -modules, where the structure sheaf (trivial line bundle)  $\mathcal{O}_X$  is the identity. A **vector bundle** is a locally free coherent sheaf. For projective varieties  $X$ ,  $\mathcal{O}_X(-1)$  is the **tautological line bundle**. We write  $\mathcal{O}_X(d)$  for  $\mathcal{O}_X(-1)^{\vee \otimes d}$ .
- A **projective bundle** over a variety  $X$  is a map  $\pi : Y \rightarrow X$  that locally trivializes to products with projective space (i.e., at all  $p$  we locally and compatibly have  $\pi^{-1}(U) \cong U \times \mathbb{P}^r$ ). One can think analogically to the classical notion of a vector bundle.
- The **projectivization**  $\mathbb{P}(\mathcal{F})$  of a vector bundle  $\mathcal{F}$  over  $X$  is constructed by projectivizing each fiber space into its *lines* (this is opposite of [Har77]’s convention). Formally,  $\mathbb{P}(\mathcal{F}) = \text{Proj}(\text{Sym } \mathcal{F}^\vee) \rightarrow X$ . The following results hold (see [EH16, §3, §9]):
  - Every projective bundle  $\pi : Y \rightarrow X$  is a projectivization  $\mathbb{P}(\mathcal{F})$  for some vector bundle  $\mathcal{F}$  over  $X$ .
  - Given  $\mathbb{P}(\mathcal{F})$ , there is a **tautological subbundle**  $S \subseteq \pi^* \mathcal{F}$  whose fibers are the lines projectivizing to the point in  $\mathbb{P}(\mathcal{F})$ . One sees that  $S \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)$ , and we get the exact **universal sequence** of  $\mathbb{P}(\mathcal{F})$ :

$$0 \rightarrow S \rightarrow \pi^* \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0.$$

where  $\mathcal{Q} \cong \pi^* \mathcal{F} / S$  is the **universal quotient bundle**.

- Dually, we deduce that **global coordinates** exist on  $\mathbb{P}(\mathcal{F})$ ; that is, a restriction to  $\mathcal{O}_{\mathbb{P}^r}(1)$  for every fiber of  $\pi$ .

## 2 Weierstrass models of elliptic fibrations

In this section we review the theory of elliptic curves and fibrations, especially Kodaira’s classification of singular fibers over codimension 1. This is done in conjunction with deriving the notion of a Weierstrass model, on which one can solve for Kodaira’s singular fibers using Tate’s algorithm.

### 2.1 Elliptic curves

We define our fundamental building block of study:

**Definition 2.1.** An **elliptic curve**  $(E, O)$  over a field  $K$  is a curve that is smooth, projective, irreducible, and of genus 1, with a **distinguished  $K$ -point**  $O$ . We say

$$\varphi : (E, O) \rightarrow (E', O')$$

is a **morphism (of elliptic curves)** if it is a morphism that preserves base points, i.e.,  $\varphi(O) = O'$ . These elliptic curves are **isomorphic** if there exists a  $\varphi$  that is bijective and whose inverse is also a morphism.

*Remark.* The *(arithmetic) genus* of a variety  $X$  is an invariant defined later in Definition 3.41. For smooth curves over  $\mathbb{C}$ , it corresponds to the number of holes  $X$  has when viewed as a Riemann surface. The genus 1 case is when  $X$  “looks like” a torus; see Figure 2.6.

Definition 2.1 appears to be a far cry from the definition we alluded to in Section 1.1 as curves with equation  $y^2 = x^3 + ax + b$ , even if we pass to the homogeneous form

$$y^2z = x^3 + axz^2 + bz^3.$$

For one, being a projective curve does not induce the ambient projective space to be  $\mathbb{P}^2$  with homogeneous coordinates  $[x, y, z]$ , nor does it induce the equation of any particular cubic. Let us belabor the relationship between our formal coordinate-free definition and this informal one, as it is this relation which generalizes to the notion of *Weierstrass models* of elliptic fibrations that is central to this thesis:

**Definition 2.2.** A curve  $C \subseteq \mathbb{P}^2$  defined by a **Weierstrass equation** over  $K$  is the projective variety described by a homogeneous equation of the form

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

where  $[x, y, z]$  are the coordinates of  $\mathbb{P}^2$  and  $a_1, \dots, a_6 \in K$ . For a given Weierstrass equation, one typically defines the following quantities [Sil09, §III]:

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, & b_4 &= a_1a_3 + 2a_4, & b_6 &= a_3^2 + 4a_6, \\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \\ c_4 &= b_2^2 - 24b_4, & c_6 &= -b_2^3 + 36b_2b_4 - 216b_6, \\ \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, & j &= c_4^3/\Delta. \end{aligned}$$

We call  $\Delta$  the **discriminant** and  $j$  the  **$j$ -invariant** of a Weierstrass equation.

To retrieve the more concise form, we require  $\text{char}(K) \neq 2, 3$ . The following two linear transformations on  $\mathbb{P}^2$  become isomorphisms:

$$[x, y, z] \mapsto [2x, y - a_1x - a_3z, 2z], \quad \text{then} \quad [x, y, z] \mapsto [3x - 9b_2z, y, 108z],$$

as one verifies with determinants. These maps eliminate the  $y$  and  $x^2$  terms respectively, transforming our original equation to  $y^2z = x^3 - 27c_4xz^2 - 54c_6z^3$ . Finally, away from  $z \neq 0$ , we can pass to affine coordinates via  $[x/z, y/z, 1] \mapsto (x, y)$  to get the **simplified Weierstrass equation**

$$y^2 = x^3 + ax + b, \quad (\text{where } a = -27c_4 \text{ and } b = -54c_6)$$

in  $\mathbb{P}^2 \setminus \{z = 0\} \cong \mathbb{A}^2$ , as long as we remember the point  $[0, 1, 0]$  in the  $z = 0$  locus, which is unique by inspection and was preserved by our transformation. By construction, this curve is isomorphic to our original  $C$ , and so we can use both forms interchangeably when, e.g.,  $\text{char}(K) = 0$ . One verifies that

$$\Delta = \frac{1}{1728}(c_4^3 - c_6^2) \propto 4a^3 + 27b^2.$$

An interesting property of this five-parameter family of curves (or two, after simplification) is that almost every curve defined by a Weierstrass equation is an elliptic curve:

**Proposition 2.3.** *The curve  $C$  defined by a Weierstrass equation is an elliptic curve  $(C, [0, 1, 0])$  if and only if  $\Delta \neq 0$ .*

*Proof.* By construction,  $(C, [0, 1, 0])$  is a projective variety with distinguished  $K$ -point. To show  $C$  is smooth, rewrite the equation as a homogeneous polynomial

$$F(x, y, z) = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3,$$

so that our curve is  $C = \{F(x, y, z) = 0\}$ . Then  $\frac{\partial F}{\partial z}(0, 1, 0) = 1$ , so  $[0, 1, 0] \in C$  is smooth. Pass to affine coordinates by defining  $f(x, y) = F(x, y, 1)$ , then solve

$$f(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0,$$

to conclude  $C$  is smooth everywhere if and only if  $\Delta = 0$ . Finally, note that  $C$  is a plane curve of degree 3; that is, the degree of its homogeneous polynomial  $F$  in  $\mathbb{P}^2$  is 3. Then  $C$  is a genus  $g = \frac{1}{2}(3-1)(3-2) = 1$  curve by the genus-degree formula (Proposition 3.20), which we prove later.  $\square$

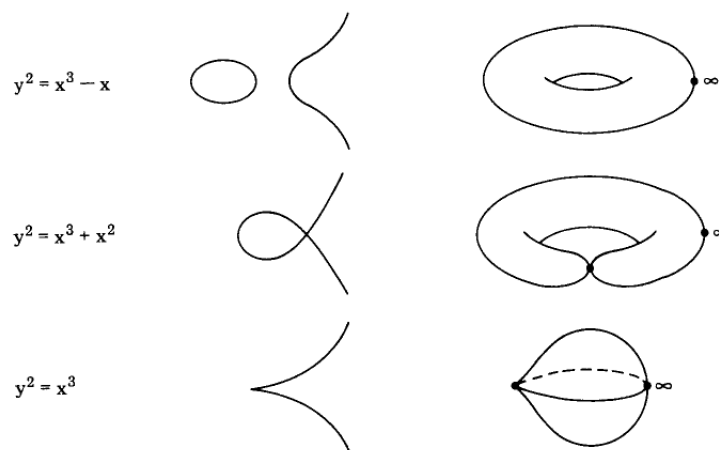
*Remark.* In fact, the same argument shows every irreducible smooth plane cubic (with distinguished point) is an elliptic curve. This is also why we call  $\Delta$  the discriminant, as it is related to the discriminant of the Weierstrass equation as a cubic polynomial in  $x$ .

Meanwhile, when  $\Delta = 0$ , one encounters two types of singular points  $p$ . The first is the *node*, where  $X$  looks like  $xy = 0$  at  $p$ ; the second is the *cusp*, where  $X$  looks like  $y^2 - x^3 = 0$  at  $p$ . The node gives two tangent lines at  $p$ , while the cusp gives one (with multiplicity 2); see Figure 2.6. Formally:

**Definition 2.4.** Let  $p \in X$ , and let  $\hat{\mathcal{O}}_{X,p}$  denote the completion of its local ring. We say  $p$  is a **node** if  $\hat{\mathcal{O}}_{X,p} \cong K[[x, y]]/\langle xy \rangle$ , and a **cusp** if  $\hat{\mathcal{O}}_{X,p} \cong K[[x, y]]/\langle y^2 - x^3 \rangle$ .

**Corollary 2.5.** *When  $\Delta = 0$ , the curve defined by a Weierstrass equation is singular at only one point. When  $c_4 = 0$ , the point is a cusp; otherwise it is a node. These **nodal** and **cuspidal curves** are **rational**; that is, they are birational to  $\mathbb{P}^1$ . Elliptic curves are not rational curves.*

*Proof.* This requires a bit more work with the Weierstrass equation; see [Sil09, §III]. For the final claim,  $\mathbb{P}^1$  can be viewed as  $\{x = 0\} \subseteq \mathbb{P}^2$  (a plane line), so the genus-degree formula shows it has genus 0, unlike an elliptic curve  $E$  with genus 1. A birational map between smooth curves  $\mathbb{P}^1$  and  $E$  must induce an isomorphism, which is a contradiction since genus (Definition 3.41) is a topological invariant (for example, over  $\mathbb{C}$  we have  $\mathbb{P}^1$  as the Riemann sphere and  $E$  as a torus).  $\square$



**Figure 2.6:** Weierstrass equations giving elliptic, nodal, and cuspidal curves respectively. On the left they are viewed as varieties over  $\mathbb{R}$  in the affine patch  $\mathbb{RP}^2 \setminus \{z \neq 0\} \cong \mathbb{R}^2$ ; on the right they are considered over  $\mathbb{C}$  (with distinguished point “ $\infty$ ”) and then depicted as surfaces in  $\mathbb{R}^3$ . Original image from Frances Kirwan, *Complex Algebraic Curves* (1992).

To complete the relationship between this family of equations and the abstract definition of an elliptic curve, we show that:

**Proposition 2.7.** *For every elliptic curve  $(E, O)$  there exists an isomorphic elliptic curve  $(C, [0, 1, 0])$  in  $\mathbb{P}^2$  defined by a Weierstrass equation.*

*Proof.* This classic result follows from the Riemann-Roch theorem for algebraic curves, whose generalization due to Hirzebruch will be needed later. In particular, one consequence of Riemann-Roch is that since elliptic curves have genus 1, we have

$$\dim \mathcal{L}(n[O]) = n, \quad n > 0,$$

where  $\mathcal{L}(n[O]) = H^0(E, \mathcal{O}_E(n[O]))$  is the space ( $\mathcal{O}_E$ -module) of rational functions  $f \in K(E)$  whose only pole is at  $O$  with multiplicity  $\leq n$  (see Corollary 3.51; the longhand notation is from sheaf cohomology, as defined in Definition 3.37). There is an obvious inclusion  $\mathcal{L}(n[O]) \rightarrow \mathcal{L}((n+1)[O])$  as modules, and for technical reasons these modules are free (since  $E$  is projective, then  $\mathcal{O}_E \cong K'$  for some finite extension  $K'/K$ ).

Hence, we can iteratively choose a basis for  $\mathcal{L}(n[O])$ . In particular,  $\mathcal{L}([O]) = \langle 1 \rangle$  (where 1 is the constant function). We can choose a rational function  $x$  such that  $\mathcal{L}(2[O]) = \langle 1, x \rangle$ ;  $x$  must then have a pole of order 2 at  $O$ . Likewise, there is a  $y$  with a pole of order 3 at  $O$  such that  $\mathcal{L}(3[O]) = \langle 1, x, y \rangle$ . Since  $x^2$  has a pole of order 4, we can take it as the basis element introduced at  $\mathcal{L}(4[O])$ ; likewise with  $xy$  at  $\mathcal{L}(5[O])$  and  $x^3$  at  $\mathcal{L}(6[O]) = \langle 1, x, y, x^2, xy, x^3 \rangle$ . However,  $y^2$  also has a pole of order 6 at  $O$ , implying a non-trivial linear dependence:

$$a_0 y^2 = a_1 xy + a_2 x^2 + a_3 y + a_4 x + a_5 x^3 + a_6.$$

Since  $x^3, y^2$  have poles of order 6, we have  $a_0, a_5 \neq 0$ . Take  $y \mapsto \frac{a_0}{a_5} y, x \mapsto \frac{a_0}{a_5} x$  and rescale to get both  $y^2, x^3$  with coefficient 1; thus we can assume  $a_0 = a_5 = 1$ . Rearranging and changing the signs of  $a_i$  show that the affine Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x.$$

holds for every  $P \in E$ .

Let  $C' \subseteq \mathbb{P}^2$  be the affine curve this equation cuts out in  $z \neq 0$ ; adding in  $[0, 1, 0]$  (projective completion) gives the curve  $C$  defined by the Weierstrass equation. We want to show that

$$\varphi : (E, O) \rightarrow (C, [0, 1, 0]), \quad P \mapsto \begin{cases} [x(P), y(P), 1] & \text{if } P \in E \setminus \{O\} \\ [0, 1, 0] & \text{if } P = O \end{cases}$$

is an isomorphism. For a rational function  $f \in K(E)$  on projective curves  $E$ , the total multiplicity of all zeros equal that of all poles (i.e.,  $\deg(\operatorname{div}(f)) = 0$  as in [Sha13]). Hence  $x, y$  have zeros of total multiplicity 2 and 3 respectively, implying they are degree 2, 3 (surjective) maps. Hence the degree of  $P \mapsto [x(P), y(P), 1]$  is  $\gcd(2, 3) = 1$  and is thus birational. Then  $C$  cannot be singular, since by Corollary 2.5 a singular curve is birational to  $\mathbb{P}^1$  while an elliptic curve is not. Hence  $C$  is a smooth (elliptic) curve, and being a map of smooth curves, our birational map  $E \rightarrow C$  must induce the isomorphism  $\varphi$ .  $\square$

*Remark.* There are other families which exhibit all isomorphism classes of elliptic curves. Since the Weierstrass equation is a special case, the family of all plane cubics (Example 2.8) also have this property. An important case occurs when  $K = \mathbb{C}$ ; it can be shown that each elliptic curve is isomorphic to some  $(\mathbb{C}/\Lambda, 0)$ , where  $\Lambda$  is a lattice spanned by  $\{1, \tau\}$  with  $\operatorname{Im}(\tau) > 0$ . Topologically, it follows that complex elliptic curves are tori, as seen in Figure 2.6.

## 2.2 Elliptic fibrations

An important perspective in algebraic geometry is that of the *parameter space*. Informally, if one has a variety that is a “function” of its parameters (e.g., coefficients), then one might be able to treat its parameters as a variety in their own right, with fruitful results:

**Example 2.8.** **Plane cubics** are curves in  $\mathbb{P}^2$  with defining polynomials of degree 3. Their general form is:

$$\sum_{0 \leq i+j \leq 3} a_{i,j} x^i y^j z^{3-i-j} = a_{3,0} x^3 + a_{2,1} x^2 y + \cdots + a_{0,1} y z^2 + a_{0,0} z^3 = 0$$

There are 10 coefficients  $a_{i,j} \in K$ , where two sets of coefficients give the same curve when they are multiples of each other. Hence  $[a_{3,0}, \dots, a_{0,0}] \in \mathbb{P}^9$  is a natural parameter space for the plane cubics. A parameter

space of the simplified Weierstrass equation is then the affine subvariety of  $\mathbb{P}^9$  where the coefficients of  $y^2z$ ,  $x^3$  are equal and non-zero, the coefficients of  $xz^2$ ,  $z^3$  are arbitrary, and all other coefficients are zero. This corresponds to the set of tuples  $\{(a, b)\} \cong \mathbb{A}^2$  one could take for  $y^2 = x^3 + ax + b$ . Likewise,  $\{(a_1, a_2, a_3, a_4, a_6)\} \cong \mathbb{A}^5$  is a parameter space of the original Weierstrass equation.

However, the parameter space by itself does not encode the geometry of its corresponding cubics. We might have distinguished divisors (such as the hypersurface  $\Delta = 0$  in the Weierstrass parameter space), but the form of  $\Delta$  and the appearance of singular curves appeal to information outside of  $\mathbb{A}^5$ . We instead consider a variety that encodes all relevant information. In the simplified Weierstrass case, this is just be

$$\mathcal{E} = \{((a, b), [x, y, z]) \mid y^2z = x^3 + axz^2 + bz^3\} \subseteq \mathbb{A}^2 \times \mathbb{P}^2.$$

with the projection  $\varphi : \mathcal{E} \rightarrow \mathbb{A}^2$ . Here, the fibers  $\varphi^{-1}(a, b)$  are elliptic curves as long as  $\Delta(a, b) \neq 0$ . If we wanted all our fibers to be elliptic curves, we would have to consider  $\mathcal{E} \cap (\{\Delta \neq 0\} \times \mathbb{P}^2)$ , which is rather unnatural; furthermore, by keeping the *discriminant locus* intact, we can study exactly how our family fails to give smooth curves (Section 2.3, Section 4.3). This motivates the following set of definitions:

**Definition 2.9.** First, some important scheme-theoretic concepts:

- The **generic point** of a variety is the unique point (in the sense of Zariski topology) whose closure is the whole variety. The **generic fiber** is the fiber over this point. One may take the intuition that properties of the generic fiber are properties that hold for all fibers in some dense open set of the variety.
- A property of a variety over  $K$  is **geometric** if it remains true after a base change to the algebraic closure  $\bar{K}$ . For example, the affine variety  $\{x^2 + y^2 = 0\}$  over  $\mathbb{R}$  is *not geometrically irreducible* as it becomes two lines over  $\mathbb{C}$ . The projective variety  $\{y^2z = x^3 - xz^2\}$ , consisting of two disjoint components over  $\mathbb{R}$ , is *not geometrically disconnected*, since it becomes an elliptic curve over  $\mathbb{C}$  (see Figure 2.6).

**Definition 2.10.** A **genus-one fibration** over a variety  $B$  is a surjective, proper morphism  $\varphi : \mathcal{E} \rightarrow B$ , where the generic fiber is a geometrically connected, smooth projective curve of genus 1. We call  $B$  the **base** and  $\mathcal{E}$  the **total space**.

We provide this intermediate definition as it highlights our necessary distinction between a smooth projective curve of genus 1 and an elliptic curve: the latter has a distinguished  $K$ -point  $O$ . This requirement is unnecessary right now, but it becomes relevant when discussing the group law of an elliptic fibration (Section 4.1), as it is through this that we acquire an *elliptic fibration with torsion*. In fact, our notion of genus-one fibration is often taken to be the definition of an elliptic fibration in purely geometric contexts (e.g., the classification of surfaces).

However, it is not useful to simply choose an arbitrary distinguished point for each fiber to make them elliptic curves. Intuitively, we want the distinguished point to vary continuously over the base as well; formally, this requires the generic fiber to be an elliptic curve. We say that:

**Definition 2.11.** An **elliptic fibration** is a genus-one fibration  $\varphi : \mathcal{E} \rightarrow B$  equipped with a **zero section**  $\sigma : B \rightarrow \mathcal{E}$  that is rational ( $\sigma$  is a rational map such that  $\varphi \circ \sigma = \text{id}_B$  where defined).

For example, the family of plane cubics over  $\mathbb{C}$  can be described as a genus-one fibration  $\mathcal{E}$ , but any obvious (rational) assignment  $\sigma : \mathbb{P}^9 \rightarrow \mathcal{E}$  of distinguished points seems arbitrary. On the other hand, the family of simplified Weierstrass curves is an elliptic fibration in a natural way; the generic fiber is a smooth,

geometrically connected projective curve of genus 1 (explicitly,  $y^2 = x^3 + ax + b$  away from  $\Delta = 0$ ), with the zero section

$$\sigma : \mathbb{A}^2 \rightarrow \mathcal{E}, \quad (a, b) \mapsto ((a, b), [0, 1, 0]).$$

In this situation, we observe that  $\mathcal{E}$  has dimension 3, as the base has dimension 2 and the fibers are elliptic curves (dimension 1), making it an elliptic threefold. In general:

**Notation 2.12.** An **elliptic  $n$ -fold** is an elliptic fibration whose total space has dimension  $n$  (equivalently, whose base has dimension  $n - 1$ ). An elliptic twofold is also called an **elliptic surface**.

Remarkably, in the same way every elliptic curve is isomorphic to a curve given by a Weierstrass equation, it turns out that every elliptic fibration is isomorphic to a variety determined by a similarly-constrained equation:

**Definition 2.13.** A **Weierstrass model**, for a variety  $B$  with line bundle  $\mathcal{L}$ , is a hypersurface  $\mathcal{W}$  in the projective bundle (Definition 1.4)  $\pi : X_0 = \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \rightarrow B$  cut out by a section of the line bundle  $\mathcal{O}_{X_0}(3) \otimes \pi^* \mathcal{L}^{\otimes 6}$  of the form:

$$y^2 z + a_1 x y z + a_3 y z^2 = x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3,$$

where each factor denotes a section of an appropriate line bundle over  $X_0$ :

$$z \in H^0(X_0, \mathcal{O}_{X_0}(1)), \quad x \in H^0(X_0, \mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes 2}), \quad y \in H^0(X_0, \mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes 3}), \quad a_i \in H^0(B, \mathcal{L}^{\otimes i}).$$

For characteristics other than 2 and 3, we have the **simplified form**:

$$y^2 z = x^3 + f x z^2 + g z^3, \quad f \in H^0(B, \mathcal{L}^{\otimes 4}), \quad g \in H^0(B, \mathcal{L}^{\otimes 6}).$$

(In actuality, we are taking implicit pullbacks  $\pi^* a_i$ ,  $\pi^* f$ ,  $\pi^* g$  to make them sections over  $X_0$ .)

By virtue of being a section, and observing that the formula for  $\Delta$  holds fiberwise:

**Proposition 2.14.** A Weierstrass model is an elliptic fibration  $\psi : \mathcal{W} \xrightarrow{\iota} X_0 \xrightarrow{\pi} B$  with zero section  $\sigma : p \mapsto ([0, 0, 1], p)$ . It is smooth along  $z = 0$ .

Ultimately, the Weierstrass model is the correct generalization of the Weierstrass equation for arbitrary elliptic  $n$ -folds, as observed by Mumford and Suominen [MS72]:

**Theorem 2.15.** Every elliptic fibration  $\varphi : \mathcal{E} \rightarrow B$  with only elliptic curves as fibers is isomorphic over  $B$  to a Weierstrass model  $\psi : \mathcal{W} \xrightarrow{\iota} X_0 \xrightarrow{\pi} B$  (for  $\text{char}(K) \neq 2, 3$ , up to isomorphisms of the triple  $(\mathcal{L}, f, g)$ ).

*Proof sketch.* We first work locally (over an affine open set  $U \subseteq B$ ). Furthermore, since vector bundles are locally free sheaves, we can assume  $\mathcal{L} = \varphi_*(\mathcal{O}_{\mathcal{E}}(\sigma(B))/\mathcal{O}_{\mathcal{E}})$  is free on  $U$ , where  $\sigma$  is the zero section. In brief, we can view  $\mathcal{E}|_U \rightarrow U$  as an elliptic curve over the function field  $K(U)$ , and then apply the proof of Proposition 2.3 to conclude  $\mathcal{E}|_U$  is isomorphic to a Weierstrass model over  $U$ .

Then, one appeals to the existence of global coordinates on projective bundles. In particular, the proof of Proposition 2.3 shows that the bundle  $\varphi_*(\mathcal{O}_{\mathcal{E}}(3[\sigma(B)])) = \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}$  over  $B$  is locally generated by what we called  $1, x, y$ . Projectivize this direct sum of line bundles to get  $\pi : X_0 \rightarrow B$ , then take the pull-back. Then from the properties of the projective bundle (Definition 1.4), the sections  $1, x, y$  are canonically mapped to (global) sections  $H^0(X_0, \mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes k})$  for  $k = 0, 2, 3$ . (The  $a_i$  remain sections of  $\mathcal{L}^{\otimes i}$ .)

Our equation is now valid everywhere, defining  $\mathcal{W}$  as the vanishing locus of the Weierstrass equation. When  $\text{char}(K) \neq 2, 3$ , we can take the Weierstrass simplified form; one verifies that scaling  $(\mathcal{L}, f, g)$  is valid due to the equation's homogeneity.  $\square$

**Corollary 2.16.** Every elliptic fibration  $\varphi : \mathcal{E} \rightarrow B$  is birational over  $B$  to a Weierstrass model  $\psi : \mathcal{W} \xrightarrow{\iota} X_0 \xrightarrow{\pi} B$ .

## 2.3 Kodaira's classification of singular fibers

As observed in Section 2.2, not every fiber in an elliptic fibration is an elliptic curve. This is not a bug, but a feature! We call such fibers **singular fibers**. An interesting (and very unsolved) problem is classifying the singular fibers across all possible elliptic fibrations (after some “regularizing” requirements such as total space being smooth), even just in the case  $K = \mathbb{C}$ . We discuss the state-of-the-art in Section 4.3.

*Remark.* Note that the singular fibers of  $\varphi : \mathcal{E} \rightarrow B$  are *singular* in the sense that they are not elliptic curves. This does not (necessarily) mean that  $\mathcal{E}$  has a singular (i.e., non-smooth) point at the singular fiber's cusp, node, etc.

The reason one might expect a classification is that requiring the total space of an elliptic fibration to be a variety is a restrictive property, even before having additional constraints like the Weierstrass form (recall that once singular fibers are in play, Weierstrass models are only birational models by Corollary 2.16). For example, in a direct generalization of Corollary 2.5:

**Proposition 2.17.** *A smooth Weierstrass model  $\psi : \mathcal{W} \xrightarrow{\iota} X_0 \xrightarrow{\pi} B$  can only have nodal and cuspidal singular fibers.*

In the case of a singular (non-smooth) Weierstrass model, one might examine the smooth fibrations in its birational class. This culminated in the work of Kodaira and Néron, who independently classified all the possible singular, geometric fibers that occur in a smooth (minimal) elliptic surface. We need some terminology:

**Definition 2.18.** The **discriminant locus** of an elliptic fibration  $\varphi : \mathcal{E} \rightarrow B$  is the base hypersurface  $\Delta \subseteq B$  whose fibers are singular fibers. In the case of a Weierstrass model this is exactly the algebraic set  $\{\Delta = 0\}$  where  $\Delta$  is the discriminant.

**Definition 2.19.** The **geometric fiber** over the point  $p$  is  $\varphi^{-1}(p)$  after a base change to the algebraic closure  $\overline{K}$  (it is the fiber viewed as a variety over the algebraic closure; see the remark after Definition 2.10). When the geometric fiber differs in number of components from the fiber, we say the fiber is of **non-split type**, denoted with a superscript **ns**. Otherwise it is **split**, denoted with a superscript **s**.

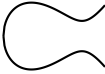
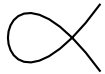

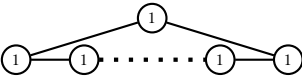


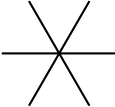
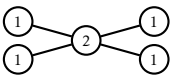
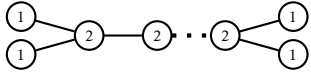
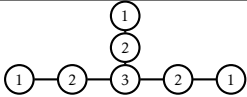
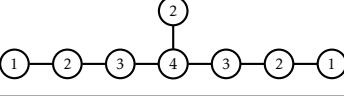
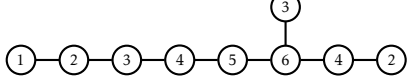
**Definition 2.20.** An elliptic  $n$ -fold  $\varphi : \mathcal{E} \rightarrow B$  is **minimal** if  $\mathcal{E}$  is smooth and no fibers contain a rational, **exceptional curve (of the first kind)** (i.e.,  $\deg[C]^2 = -1$  in the sense of Theorem 3.4).

However, Kodaira and Néron's result can be viewed in greater generality as the classification of geometric generic fibers over the components of the discriminant locus for any elliptic  $n$ -fold [Mir83]. The essential argument is the same as in Theorem 2.15; locally, the base looks like  $\mathbb{A}^{n-1}$ , and the fibration over this neighborhood can be thought of as an elliptic curve over the function field  $K(x_1, \dots, x_{n-1})$ , making it locally like an elliptic surface. Hence, studying the geometric generic fiber over each irreducible component of  $\Delta$  is analogous to studying the geometric fiber over a “point” in the base of an elliptic surface over the function field  $K(B)$  (instead of  $K$ ). This leads us to state Kodaira's classification in the following way:

**Theorem 2.21** (Kodaira's classification of singular fibers). *The possible geometric generic singular fibers of any minimal elliptic fibration  $\varphi : \mathcal{E} \rightarrow B$  over (codimension-1) divisors are curves whose components are rational curves over  $\overline{K}$  intersecting in the manner described by one of the rows in Table 2.22.*

*Proof sketch.* In algebraic combinatorics, one has the classic problem of solving for all irreducible graphs (allowing for node multiplicity) with maximal eigenvalue 2 in terms of their adjacency matrix. The graphs of the intersections depicted in Table 2.22 are examples of such undirected, extended Dynkin diagrams.



Kodaira symbol	Intersection diagram	Affine Dynkin diagram	$\nu_C(c_4)$	$\nu_C(c_6)$	$\nu_C(\Delta)$
$I_0$ (nonsing.)		-	$\geq 0$	$\geq 0$	0
$I_1$		-	0	0	1
$I_2$		$\tilde{A}_1$	0	0	2
$I_m, m \geq 1$		$\tilde{A}_{m-1}$	0	0	$m$
II		-	$\geq 1$	1	2
III		$\tilde{A}_1$	1	$\geq 2$	3
IV		$\tilde{A}_2$	$\geq 2$	2	4
$I_0^*$		$\tilde{D}_4$	$\geq 2$	$\geq 3$	6
$I_m^*, m \geq 1$		$\tilde{D}_{m+4}$	2	3	$m+6$
IV*		$\tilde{E}_6$	$\geq 3$	4	8
III*		$\tilde{E}_7$	3	$\geq 5$	9
II*		$\tilde{E}_8$	$\geq 4$	5	10

**Table 2.22:** The list of Kodaira singular fibers, with the associated configuration of rational curves, affine Dynkin diagram, and valuation criteria from Tate's algorithm over  $\text{char}(K) \neq 2, 3$ . See Section 2.3 for details.

We depicted either the generalized graph (numbers represent the node multiplicity), or in special cases, physical intersections of curves. By the Perron-Frobenius theorem, subtracting the matrix  $2I$  from this adjacency matrix gives a negative semi-definite matrix with a kernel of dimension 1 (corresponding to the unique dimension-1 eigenspace formerly of eigenvalue 2).

Let  $F = \sum n_i D_i$  be a fixed singular fiber, written as a formal sum of its irreducible component curves, over the generic point of its (codimension-1 in the base) singular locus. There is a notion of intersection number  $\deg[D_i][D_j]$  (induced from the Chow ring of  $\mathcal{E}$ ; see Theorem 3.4), and so one constructs a  $\mathbb{Z}$ -bilinear form on the abelian subgroup generated by the classes  $[D_i]$  giving the degree of their intersection. One can show (by the Hodge index theorem or otherwise) that for the fibration of curves in a surface [Mir89, §I.6], the intersection form is negative semi-definite with a kernel of dimension 1, spanned by  $[F] = \sum n_i [D_i]$  itself. Furthermore, one can show that the components have self-intersection  $\deg[D_i]^2 = -2$  and are rational.

Observe that the constraints on our modified adjacency matrix and those on our intersection form coincide, such as the main diagonal of  $-2$  entries. By construction, one can exhibit elliptic fibrations (in fact, Weierstrass models suffice) producing each of the singular fibers in Table 2.22, and so we are done.  $\square$

When the base is no longer a curve (i.e., in the case of general minimal elliptic  $n$ -folds), it becomes necessary for classification to consider generic fibers at the intersections of  $\Delta$ 's components, and beyond; that is, in higher codimension. More on this in Section 4.3. For now, observe that:

- The underlying Dynkin diagrams do not distinguish *completeness of intersection*. For example,  $IV$  and  $I_3$  both correspond to an intersection diagram of a triangle with nodes of multiplicity 1 ( $\tilde{A}_2$ ), each curve intersecting the other two transversally. However, the  $IV$  case is when all three intersection points coincide. Kodaira distinguishes between the two cases, and so do we (e.g., in Section 4.3, our non-Kodaira fiber has underlying diagram  $\tilde{A}_3$ , the same as the Kodaira fiber  $I_4$ ).
- Kodaira fibers classify *geometrically*. For example, in  $I_2$  we have two points of intersection in the geometric fiber; however, the original “arithmetic” fiber might see the two points as a single irreducible component. This is the distinction between split and non-split (Definition 2.19).
- The names given in Table 2.22 are actually those of the **affine Dynkin diagrams**, which correspond (after distinguishing a node) to our intersection pictures. Furthermore, for reasons beyond the scope of this thesis, they correspond to *affine Lie algebras*.

Following our theme, we once more pass from the abstract result to a computational realization. Kodaira's classification tells us what singular fibers ought to look like. However, if one is provided with an elliptic fibration, could one determine which of these singular fibers it has? In the case of Weierstrass models, the answer is a resounding yes. The following result is due to Tate:

**Theorem 2.23** (Tate's algorithm). *Given a Weierstrass model of an elliptic fibration (over a perfect field), one can compute the Kodaira type of the fiber (and whether the fiber is geometrically irreducible) over each component  $C \subseteq \Delta$ , via a deterministic procedure involving the valuations  $v_C(a_i)$  over the coefficients  $a_i$ .*

We omit the 11-step algorithm that Tate describes; see [Sil94, §IV.9] for a full description and details of the proof (in the form of a case-by-case argument). We will ultimately only be concerned with the case of  $\text{char}(K) = 0$ . Tate's original algorithm can also deduce if a fiber is split vs. non-split, but in practice we will just detect this *en ce moment*. Hence, we can pass to the simplified Weierstrass equation

$$y^2 = x^3 + ax + b, \quad (\text{where } a = -27c_4 \text{ and } b = -54c_6)$$

and in this situation Tate's algorithm reduces to considering the multiplicities  $v_C(c_4)$ ,  $v_C(c_6)$ ,  $v_C(\Delta)$  over the component  $C$ , reading downwards along the respective columns in Table 2.22 [KMSS11].

### 3 Intersection theory and cohomology

In this section, we review the basics of intersection theory, and use the theory of Euler characteristics and Hirzebruch-Riemann-Roch to bridge algebraic geometry and algebraic topology in the complex case.

#### 3.1 Chern classes and adjunction

We first introduce the theory of Chern classes from the perspective of algebraic geometry. One might be familiar with Chern classes from the perspective of algebraic topology or Chern-Weil theory, whereby complex vector bundles of complex rank  $n$  are associated with elements in the integral cohomology of the base manifold.

In algebraic geometry, Grothendieck introduced an axiomatic development of Chern classes valid for arbitrary varieties and fields. We will use this as our path to the adjunction formula and modern intersection theory. We begin with the following setting:

**Definition 3.1** (Chow groups). Let  $X$  be an  $n$ -dimensional variety.

- The group of  $k$ -cycles  $Z_k(X)$  is the free abelian group over the  $k$ -dimensional subvarieties of  $X$ . Taking the direct sum, we get the graded **group of cycles**  $Z(X) = \bigoplus_{k \geq 0} Z_k(X)$ .
- If  $V$  a  $(k+1)$ -dimensional subvariety of  $X$ , there is a **divisor map** on  $V$

$$\operatorname{div}_V : K(V)^\times \rightarrow Z_k(X), \quad f \mapsto \sum_{W \in Z_k(V)} \nu_W(f)W,$$

using the natural inclusion  $Z_k(V) \subseteq Z_k(X)$ . A  $k$ -cycle  $D \in Z_k(X)$  is **rationally equivalent to zero** if  $D = \sum_{i=1}^m \operatorname{div}_{V_i}(f_i)$  for some  $(k+1)$ -dimensional subvarieties  $V_1, \dots, V_m$  with associated  $f_i \in K(V_i)^\times$ . These  $k$ -cycles form a subgroup  $\operatorname{Rat}_k(X)$ .

- The  $k$ -th Chow group  $A_k(X)$  is the quotient  $Z_k(X)/\operatorname{Rat}_k(X)$ , and their direct sum gives the graded **Chow group**  $A(X) = \bigoplus_{k \geq 0} A_k(X)$ . We write  $[D] \in A(X)$  for the equivalence class of a cycle  $D \in Z(X)$ .

**Example 3.2.** These notions generalize the group of **Weil divisors**  $\operatorname{Div}(X) = Z_{n-1}(X)$ , the group of **principal divisors**  $P(X) = \operatorname{Rat}_{n-1}(X) = \operatorname{im}(\operatorname{div}_X)$ , and the **divisor class group**  $\operatorname{Cl}(X) = \operatorname{Div}(X)/P(X) = A_{n-1}(X)$  respectively (see [Sha13]).

These constructions begin to algebraize the geometric notion of subvarieties inside varieties. Continuing the theme of making manifold-like properties algebraic, we define transversality in our language:

**Definition 3.3.** Two subvarieties  $V, W \subseteq X$  are **transverse at  $p$**  if  $V, W, X$  are smooth at  $p$  and their Zariski tangent spaces satisfy

$$T_p(V) + T_p(W) = T_p(X).$$

They are **generically transverse** if they are transverse at the generic point of every component of  $V \cap W$ .

Intersection theory generalizes Bézout's theorem to arbitrary pairs of codimension via the setting of a graded ring. This is nicest when working in smooth varieties over algebraically closed fields:

**Theorem 3.4.** *Let  $X$  be a smooth  $n$ -dimensional quasiprojective variety over an algebraically closed field. There is a unique and well-defined **intersection product***

$$A_k(X) \times A_\ell(X) \rightarrow A_{k+\ell-n}(X)$$

such that

$$[V][W] = [V \cap W]$$

when  $V, W$  are generically transverse subvarieties of  $X$ . This turns the dimension-graded Chow group into the codimension-graded **Chow ring**, also written  $A(X)$ . In particular, if  $k + \ell = n$ , one gets a 0-cycle  $\alpha = \sum n_P [P]$  summed over closed points. The sum of its coefficients gives the **degree map**, written

$$\deg(\alpha) = \sum n_P,$$

and when  $\alpha = [V][W]$ , we call  $\deg(\alpha)$  the **intersection number** of  $V$  and  $W$ .

*Remark.* This is a difficult theorem in its full generality. Its central idea involves the cap product from algebraic topology. We do not prove this theorem here; see [EH16, §1] for a discussion and [Ful98, §8] for an explicit construction.

The Chow group/ring thus encodes the rich structure of subvarieties and their intersections. This structure is preserved by sufficiently nice morphisms:

**Proposition 3.5.** *If  $\varphi : X \rightarrow Y$  is a proper morphism, there is a well-defined **pushforward**  $\varphi_* : A_k(X) \rightarrow A_k(Y)$ . If  $\varphi : X \rightarrow Y$  is a flat morphism of relative dimension  $n$ , there is a well-defined **pullback**  $\varphi^* : A_k(Y) \rightarrow A_{k+n}(X)$ . These extend to group homomorphisms (but not ring homomorphisms!) between  $A(X), A(Y)$ . The **pullback Chow group**  $\varphi^* A(Y)$  is also well defined.*

We defer the details to [EH16, §1]. One should think of *proper* as analogous to differential topology, mapping (quasi-)compact sets (e.g., subvarieties) to (quasi-)compact. For our use, we can interpret *flat of relative dimension  $n$*  as a type of projection with continuous fibers to a variety  $\dim Y = \dim X - n$ , such as the ones in our projective bundles. In a sense, the pushforward takes a subvariety to its image of the same dimension (with multiplicity in the degree of  $\varphi$  at the subvariety), and the pullback takes a subvariety to the algebraic subset “projecting” down to it (though  $\varphi$  does not have to be surjective; for example, inclusion maps). We will clarify these intuitions through computation.

The most important subvarieties of  $X$  are those which are vanishing loci of sections of vector bundles over  $X$ . One may be familiar with the first Chern class of a line bundle over (possibly singular) varieties. The following generalizes this construction:

**Definition 3.6.** For each  $k$ , the  **$k$ -th Chern class** of a vector bundle  $\mathcal{F}$  over a variety  $X$  is an operation which for every  $\ell$  gives a homomorphism  $c_k(\mathcal{F}) \frown - : A_\ell(X) \rightarrow A_{\ell-k}(X)$ , and the **(total) Chern class** is a homomorphism

$$c(\mathcal{F}) \frown - : A(X) \rightarrow A(X), \quad c(\mathcal{F}) \frown - = \sum_{k=0}^{\infty} (c_k(\mathcal{F}) \frown -),$$

which collectively satisfy the following properties:

- **Normalization:**  $c_0(\mathcal{F})$  is always the identity, which we write  $c_0(\mathcal{F}) = 1$ . More generally, if  $\mathcal{L}$  a line bundle with  $\mathcal{O}(D) \cong \mathcal{L}$  for some Cartier divisor  $D$ , then:

$$c(\mathcal{L}) \frown [X] = (1 + c_1(\mathcal{L})) \frown [X] = (1 \frown [X]) + (c_1(\mathcal{L}) \frown [X]) = [X] + [D].$$

- **Projection:** If  $\varphi : X' \rightarrow X$  a proper morphism, then the following are equal as maps  $A(X') \rightarrow A(X)$ :

$$\varphi_*(c(\varphi^*\mathcal{F}) \frown -) = c(\mathcal{F}) \frown \varphi_*(-)$$

- **Functoriality:** If  $\varphi : X' \rightarrow X$  a flat morphism, then the following are equal as maps  $A(X) \rightarrow A(X')$ :

$$\varphi^*(c(\mathcal{F}) \frown -) = c(\varphi^*\mathcal{F}) \frown \varphi^*(-)$$

- **Whitney sum:** For an exact sequence  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  of vector bundles over  $X$ , we have:

$$c(\mathcal{F}) \frown - = c(\mathcal{F}') \frown (c(\mathcal{F}'') \frown -)$$

We now have this interpretation of Chern classes as maps between gradings of the Chow group/ring. However, as before, when working with smooth varieties in algebraically closed fields, there is an intuitive viewpoint that reconciles this modern perspective with the classical notion of a section's zero-set:

**Theorem 3.7.** *For any vector bundle  $\mathcal{F}$ , the total Chern class  $c(\mathcal{F})$  exists and is uniquely determined (hence we can talk about “the” Chern class). Furthermore, if the base variety  $X$  is quasiprojective, smooth, and over an algebraically closed field, then there is a unique  $[V] \in A_{n-i}(X)$  such that for all classes  $[W] \in A(X)$ ,  $c_i(\mathcal{F}) \frown [W] = [V][W]$  in the sense of intersection product.*

The first part was shown by Grothendieck; the explicit construction can be found in [Ful98, §3]. The intuition here is equivalent to that of Poincaré duality (Theorem 3.31).

**Notation 3.8.** Moving forward, when possible we omit  $\frown$  when writing products and sums of Chern classes. For example, the Whitney sum property can be written as  $c(\mathcal{F}) = c(\mathcal{F}')c(\mathcal{F}'')$ , where composition is understood. This has the nice consequence that, when possible (Theorem 3.7), we can interpret these as intersection products of elements in  $A(X)$ . Likewise, abusing notation we write  $c_k(\mathcal{F}) \in A_{n-k}(X)$  and  $c(\mathcal{F}) \in A(X)$  when possible.

We are ready for some computations. For example, taking the Whitney sum and normalization properties together gives the immediate corollary:

**Corollary 3.9.** *Let  $\mathcal{L}_i$  be line bundles. Then the vector bundle  $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{L}_i$  of rank  $r$  has total Chern class*

$$c(\mathcal{F}) = \prod_{i=1}^r (1 + c_1(\mathcal{L}_i)) = 1 + \sum_{i=1}^r c_1(\mathcal{L}_i) + \sum_{1 \leq i < j \leq r} c_1(\mathcal{L}_i)c_1(\mathcal{L}_j) + \cdots + \prod_{i=1}^r c_1(\mathcal{L}_i)$$

*In particular,  $c_k(\mathcal{F}) = 0$  for all  $k > r$ .*

However, not all vector bundles (e.g., tangent and normal bundles) are direct sums of line bundles! We might also want to consider, e.g., tensor products of bundles. The properties of the Chern class do not seem amenable to this; however, we are saved by the following construction and principle:

**Proposition 3.10** (Splitting construction). *For any vector bundle  $\mathcal{F}$  over a smooth variety  $X$ , there exists a flat morphism  $\varphi : Y \rightarrow X$  such that the pullback  $\varphi^* : A(X) \rightarrow A(Y)$  is injective and  $\varphi^*\mathcal{F} = \bigoplus_{i=1}^{\text{rk}\mathcal{F}} \mathcal{L}_i$ , where  $\mathcal{L}_i$  are line bundles over  $Y$ .*

*Proof sketch.* Consider the projective bundle  $\varphi_1 : Y_1 = \mathbb{P}(\mathcal{F}) \rightarrow X$ . Then the pullback bundle  $\varphi_1^*\mathcal{F}$  has a tautological subbundle  $\mathcal{S} \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)$  (Definition 1.4). One then takes the quotient bundle  $\mathcal{F}_1 = \mathcal{Q} \cong \varphi_1^*\mathcal{F}/\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)$  and projectivizes it to get  $\varphi_2 : Y_2 = \mathbb{P}(\mathcal{F}_1) \rightarrow Y_1$ . Repeating this process gives a direct sum decomposition of  $\varphi^*\mathcal{F} = \varphi_{\text{rk}\mathcal{F}}^* \circ \cdots \circ \varphi_1^*\mathcal{F}$  over  $Y = Y_{\text{rk}\mathcal{F}}$ . Finally, the pullback  $\varphi^* : A(X) \rightarrow A(Y)$  is injective due to the existence of global coordinates.  $\square$

We will see a similar construction of iterated projective bundles when we perform blowups of the Weierstrass total space. We give the classes of these latent elemental line bundles a name, as we will use them to define the Chern character, Todd class, and thus the Hirzebruch-Riemann-Roch theorem:

**Definition 3.11.** The **Chern roots**  $\alpha_i$  of a vector bundle  $\mathcal{F}$  are the first Chern classes  $\alpha_i = c_1(\mathcal{L}_i)$ , where  $\mathcal{L}_i$  are the  $\text{rk}\mathcal{F}$  line bundles given by the splitting construction.

**Corollary 3.12** (Splitting principle). *Any formal power series on Chern classes that holds for direct sums of line bundles holds in general.*

*Proof.* From the splitting construction we have  $c(\varphi^*\mathcal{F}) = \prod_{i=1}^{\text{rk}\mathcal{F}} (1 + \alpha_i)$  for an appropriate  $\varphi : Y \rightarrow X$ . The injectivity of  $\varphi^*$  and the functoriality property show that the Chern roots determine  $c(\mathcal{F})$ . We appeal to Theorem 3.7 so that this determination is unique and well-defined.  $\square$

**Corollary 3.13** (Chern class identities). *Let  $\mathcal{F}, \mathcal{G}$  be vector bundles over a variety  $X$ . The following hold:*

- **Vanishing:**  $c_k(\mathcal{F}) = 0$  for  $k > \text{rk}\mathcal{F}$ .
- **Dual bundle:**  $c_k(\mathcal{F}) = (-1)^k c_k(\mathcal{F}^\vee)$ .
- **Tensor product:**  $c_1(\mathcal{F} \otimes \mathcal{G}) = (\text{rk}\mathcal{G})c_1(\mathcal{F}) + (\text{rk}\mathcal{F})c_1(\mathcal{G})$ .

*Proof sketch.* One proves these using the Chern classes of an appropriate sum of line bundles, and then invoking the splitting principle. In this way, for example, the vanishing property follows from Corollary 3.9.  $\square$

Tying things together, we demonstrate the “big idea” of passing from explicit vanishing loci of sections to their classes via Chern classes. We do so using our main object of study, the Weierstrass model:

**Notation 3.14.** Let  $\psi : \mathcal{W} \xrightarrow{\iota} \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \xrightarrow{\pi} B$  be a Weierstrass model. We will always write:

$$X_0 = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}), \quad L = c_1(\mathcal{L}), \quad H = c_1(\mathcal{O}_{X_0}(1)),$$

though  $H$  will also be used in general to refer to the **hyperplane class**  $c_1(\mathcal{O}_P(1))$  for a projective space  $P$ .

**Example 3.15** (The total space  $X_0$ ). Since  $X_0 = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ , then by tensoring the universal sequence of  $X_0$  (Definition 1.4) with  $\mathcal{S}^\vee \cong \mathcal{O}_{X_0}(1)$  we get:

$$0 \rightarrow \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_0}(1) \otimes \pi^*(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \rightarrow \mathcal{T}_{X_0/B} \rightarrow 0.$$

Since  $c(\mathcal{O}_{X_0}) = 1$ , by the Whitney sum property and Corollary 3.9:

$$\begin{aligned} c(\mathcal{T}_{X_0/B}) &= c(\mathcal{O}_{X_0}(1) \otimes \pi^*(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)) = c(\mathcal{O}_{X_0}(1))c(\mathcal{O}_{X_0}(1) \otimes \pi^*\mathcal{L}^2)c(\mathcal{O}_{X_0}(1) \otimes \pi^*\mathcal{L}^3) \\ &= (1+H)(1+H+2\pi^*L)(1+H+3\pi^*L). \end{aligned}$$

Finally, we have the relative tangent exact sequence

$$0 \rightarrow \mathcal{T}_{X_0/B} \rightarrow \mathcal{T}_{X_0} \rightarrow \pi^*\mathcal{T}_B \rightarrow 0,$$

and so the Whitney sum property gives

$$c(\mathcal{T}_{X_0}) = (1+H)(1+H+2\pi^*L)(1+H+3\pi^*L)c(\pi^*\mathcal{T}_B).$$

This is a precursor to the pullback/pushforward formulas we will discuss in Section 5.3.

The conventional means by which one computes Chern classes of intersections is via adjunction, which can be expressed in terms of the canonical class:

**Definition 3.16.** The **canonical bundle** is  $\omega_X = \wedge^n \Omega_X$  (where  $\Omega_X$  is the cotangent bundle), which is the line bundle of  $n$ -forms. The **canonical class** is  $K_X = c_1(\omega_X)$ .

**Example 3.17.** By working with differential forms on two affine coordinate patches of  $\mathbb{P}^n$ , one concludes:

$$K_{\mathbb{P}^n} = -(n+1)c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = -(n+1)H.$$

**Proposition 3.18** (Adjunction). *Let  $\iota : V \rightarrow X$  be a  $(n-1)$ -dimensional subvariety, where  $X$  is smooth and  $V$  is the vanishing locus of a section of a line bundle  $\mathcal{L} \cong \mathcal{O}_X(V)$ . Then:*

$$\mathcal{N}_{V/X} \cong \iota^*\mathcal{L}$$

Furthermore, if  $V$  smooth we have the more familiar forms:

$$\omega_V = \iota^*\omega_X \otimes \iota^*\mathcal{O}_X(V) \quad (\implies \quad K_V = \iota^*(K_X + [V])).$$

Ultimately, we will use adjunction along with the pushforward formulas (Section 5.3) to compute Chern classes of smooth varieties birational to our elliptic fibrations of interest. For example, we will need the following result later:

**Example 3.19** (Normal bundle of Weierstrass). Let  $\iota : \mathcal{W} \rightarrow X_0$  be the inclusion of a Weierstrass model into its total space. In particular, recall that  $\mathcal{W}$  is the vanishing locus of a section of  $\mathcal{O}_{X_0}(3) \otimes \pi^*\mathcal{L}^{\otimes 6}$ . Then by adjunction and the properties of Chern classes:

$$\begin{aligned} \iota_*c(\mathcal{N}_{\mathcal{W}/X_0}) &= \iota_*c(\iota^*(\mathcal{O}_{X_0}(3) \otimes \pi^*\mathcal{L}^{\otimes 6})) = 1 + c_1(\mathcal{O}_{X_0}(3) \otimes \pi^*\mathcal{L}^{\otimes 6}) = 1 + c_1(\mathcal{O}_{X_0}(1)^{\otimes 3}) + \pi^*c_1(\mathcal{L}^{\otimes 6}) \\ &= 1 + 3H + 6\pi^*L. \end{aligned}$$

We used the following to show that Weierstrass equations are elliptic curves:

**Proposition 3.20** (Genus-degree formula). *If  $C$  is an irreducible smooth curve in  $\mathbb{P}^2$  of degree  $d$ , then its arithmetic genus (Definition 3.41) is:*

$$p_a(C) = \frac{1}{2}(d-1)(d-2).$$

*Proof.* Since we are in  $\mathbb{P}^2$ , we have  $[C] = dH$  (being rationally equivalent divisors), and  $\deg H^2 = 1$  (a single point). We saw that  $K_{\mathbb{P}^2} = -3H$  (Example 3.17), and we will prove later that  $\deg K_C = 2p_a(C) - 2$  (Corollary 3.50). Using the first Chern class formulation (and manifesting the pullback as intersection), we conclude:

$$\deg K_C = \deg((K_{\mathbb{P}^2} + [C])[C]) \implies 2g - 2 = \deg((-3H + dH)(dH)).$$

Rearranging using  $\deg H^2 = 1$  gives the result.  $\square$

### 3.2 Euler characteristics

*Remark.* This section is not intended as a self-contained introduction, but rather a (motivated) list of relevant statements and constructions. References are provided.

In algebraic topology, *homology* and *cohomology* are the means by which we assign modules to topological spaces (see Definition 3.26). However, homology and cohomology are “axiomatic,” in the sense that they can be constructed for any *chain complex*. From these we get an abstract numerical invariant known as the Euler characteristic, of which one type is the *topological Euler characteristic* which we seek to compute for our Weierstrass models. Here,  $R$  always denotes a commutative, unital ring:

**Definition 3.21.** A **chain complex**  $(C_\bullet, \partial_\bullet)$  of  $R$ -modules is a  $\mathbb{Z}$ -indexed sequence of  $R$ -modules  $C_i$  and homomorphisms  $\partial_i$  (called **boundary maps** or **differentials**)

$$\cdots \xrightarrow{\partial_{i+2}} C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that  $\partial_i \circ \partial_{i+1} \equiv 0$  for all  $i \in \mathbb{Z}$ .

**Definition 3.22.** The  $n$ -th **homology** (with coefficients in  $R$ ) of a chain complex  $(C_\bullet, \partial_\bullet)$  of  $R$ -modules is defined by

$$H_n(C_\bullet, \partial_\bullet) = \ker \partial_n / \text{im } \partial_{n+1}.$$

The intuition is as follows: for  $R$ -modules, we see that the chain complex condition requires  $\text{im } \partial_{i+1} \subseteq \ker \partial_i$ . This generalizes the notion of an *exact sequence*, which imposes  $\text{im } \partial_{i+1} = \ker \partial_i$ . Hence, the  $n$ -th homology can be viewed as measuring the extent to which a chain complex fails to be exact at  $C_n$ .

In fact, chain complexes and homology can be constructed over any *abelian category*. However, the following invariants require the notion of module rank (which is well-defined under our assumptions on  $R$ ):

**Definition 3.23.** The  $n$ -th **Betti number** of a chain complex  $(C_\bullet, \partial_\bullet)$  of  $R$ -modules is

$$h_n(C_\bullet, \partial_\bullet) = \text{rk}_R H_n(C_\bullet, \partial_\bullet).$$

**Definition 3.24.** The **Euler characteristic of a chain complex**  $(C_\bullet, \partial_\bullet)$  of  $R$ -modules, when the sum is unconditionally convergent, is

$$\chi(C_\bullet, \partial_\bullet) = \sum_{k \in \mathbb{Z}} (-1)^k h_k(C_\bullet, \partial_\bullet).$$

**Definition 3.25.** A **cochain complex**  $(C^\bullet, \partial^\bullet)$  of  $R$ -modules is a  $\mathbb{Z}$ -indexed sequence of  $R$ -modules  $C^i$  and homomorphisms  $\partial^i$  (**coboundary maps**) such that



$$\dots \xrightarrow{\partial^{i-2}} C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \xrightarrow{\partial^{i+1}} \dots$$

such that  $\partial^{i+1} \circ \partial^i \equiv 0$  for all  $i \in \mathbb{Z}$ . Taking quotients, one retrieves equivalent notions of **cohomology**, as well as **Betti numbers** and **Euler characteristic**. (Due to Poincaré duality (Theorem 3.31), this will not be too confusing.)

We now rapidly recall the three types of (co)homology relevant to this thesis, with appropriate references and relevant examples:

### Singular homology (topology)

**Definition 3.26.** The **singular homology over  $R$**  of a topological space  $X$  is given by:

- **Chain objects:** Let  $C_n(R)$  be the free  $R$ -module over the **singular  $n$ -simplices**, i.e., all continuous maps  $\sigma_n : \Delta^n \rightarrow X$ , where  $\Delta^n$  is the standard  $n$ -simplex.
- **Boundary map:** Let  $\partial_n : C_n \rightarrow C_{n-1}$  be the formal sum of singular  $(n-1)$ -simplices produced by restricting  $\sigma_n \in C_n$  to the faces of  $\Delta^n$ , alternating the sign to account for orientation.
- **Notation:**  $H_n(X; R) = H_n(C_\bullet(R), \partial_\bullet)$  using the chain complex defined above.

A standard and accessible reference for this (and algebraic topology in general) is [Hat02].

**Definition 3.27.** The **(topological) Euler characteristic**  $\chi_{\text{top}}(X)$  of a variety/space/manifold  $X$  is the Euler characteristic of the singular chain complex (Definition 3.26), with coefficients in  $\mathbb{Z}$ , of its underlying topological space. By construction, it is invariant under homeomorphism of  $X$ . *This is what is meant when we solve for the Euler characteristics of our resolutions of Weierstrass models.*

**Proposition 3.28.** *The following properties hold.*

- **Homotopy-invariance:** *Singular homology is invariant for homotopy-equivalent spaces  $X$ . In particular,  $\chi_{\text{top}}$  is invariant.*
- **Coefficient independence:** *If  $X$  is a compact, topological manifold, then its Euler characteristic is independent of the coefficients of singular homology.*
- **Betti numbers and characteristic:** *The Betti numbers  $h_n(C_\bullet, \partial_\bullet)$  for a chain complex over a field  $K$  depends only on  $\text{char}(K)$ . In particular,  $h_n(X; \mathbb{Z}) = h_n(X; \mathbb{R}) = h_n(X; \mathbb{C})$ , and so we just write  $h_n(X)$ .*
- **Normalization:**  $\chi_{\text{top}}(pt) = 1$ .
- **Product:**  $\chi_{\text{top}}(X \times Y) = \chi_{\text{top}}(X)\chi_{\text{top}}(Y)$ .
- **Excision, inclusion-exclusion:** *If  $X$  is a complex or compact real variety and  $V_1, V_2$  are (closed) algebraic subsets that cover  $X$ , then*

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(V_1) + \chi_{\text{top}}(X - V_1), \quad \chi_{\text{top}}(X) = \chi_{\text{top}}(V_1) + \chi_{\text{top}}(V_2) - \chi_{\text{top}}(V_1 \cap V_2).$$

*Proof sketch.* Homotopy-invariance is a standard result for singular homology, and intuitively follows from the chain complex of modules of continuous maps. Coefficient independence follows since compact, topological manifolds are homotopy-equivalent to a finite CW-complex, which implies finite integral homology. Then for this, and for Betti numbers, one can invoke the universal coefficient theorem. See [Hat02] for the relevant statements. The remaining properties follow with some work; see the notes of [Fu193].  $\square$

**Example 3.29** (Invariants of curves). We use the properties (Proposition 3.28) of  $\chi_{\text{top}}$  and make reference to Figure 2.6 to deduce, at least in case of complex curves:

- **Circle:** A circle  $S^1$  is a union of two closed intervals  $[0, 1]$  joined at the ends. Since  $[0, 1]$  is homotopic to pt, then by inclusion-exclusion:

$$\chi_{\text{top}}(S^1) = 2\chi_{\text{top}}([0, 1]) - 2\chi_{\text{top}}(\text{pt}) = 0.$$

- **Elliptic curve:** An elliptic curve  $(E, O)$  over  $\mathbb{C}$  is topologically a torus. By the product property:

$$\chi_{\text{top}}(E) = \chi_{\text{top}}(S^1 \times S^1) = \chi_{\text{top}}(S^1)\chi_{\text{top}}(S^1) = 0.$$

- **Nodal curve:** Topologically, this is a torus pinched to a point. Removing the node leaves an open-ended cylinder, which is homotopic to  $S^1$ . By excision:

$$\chi_{\text{top}}(\text{nodal}) = \chi_{\text{top}}(S^1) + \chi_{\text{top}}(\text{pt}) = 0 + 1 = 1$$

The point here may seem belabored or elementary, but it is by a complicated combinatorial argument due to excision through which Grassi and Morrison [GM03] first computed the Euler characteristics of particular Weierstrass models with singular fibers. The intuition is that the Euler characteristic of an elliptic curve fiber is 0. Hence, the Euler characteristic of a model is determined by its singular fibers (like nodal curves), and the singular loci. In particular, inclusion-exclusion and this observation gives

$$\chi_{\text{top}}(\mathcal{E}) = \sum (-1)^k \chi_{\text{top}}(\Delta'_i) \chi_{\text{top}}(\varphi^{-1}(\Delta'_i)),$$

where  $\Delta'_i$  iterates over irreducible components of  $\Delta$  (codimension 1), along with these components' intersections (codimension 2), and so forth (to arbitrary codimension), where  $k$  depends on properties such as codimension, and where  $\varphi^{-1}(\Delta'_i)$  denotes the corresponding fiber over the generic point of the variety (e.g., a nodal curve, a cuspidal curve, etc.).

*Remark.* We do not pursue this approach further, but we hope this gives some geometric intuition to the Euler characteristic as an invariant, while highlighting the difference from our own strategy of taking pull-backs/pushforwards.

## De Rham cohomology and Hodge decomposition (differential topology)

**Definition 3.30.** The **de Rham cohomology** of a smooth manifold  $X$  is given by

- **Cochain objects:** Let  $C^n = \wedge^n \Omega$  be the  $C^\infty(X)$  vector space of differential  $n$ -forms on  $X$ .
- **Coboundary map:** Let  $\partial^n : C^n \rightarrow C^{n-1}$  be the exterior derivative.
- **Notation:**  $H_{\text{dR}}^n(X) = H^n(C_\bullet, \partial_\bullet)$  using the cochain complex defined above.

A standard and accessible reference for this (and differential geometry in general) is [Lee13].

This cohomology is notable for the following classical result (see [GH78]):

**Theorem 3.31** (De Rham's theorem, Poincaré duality). *For a smooth, oriented, closed  $n$ -dimensional manifold  $X$  (e.g., when  $X$  a complex projective variety with its analytic topology), we have:*

$$H_{\text{dR}}^k(X) \cong H_{n-k}(X; \mathbb{R}).$$

Furthermore, when  $X$  is a complex manifold, the space of  $n$ -forms is in fact the direct sum of the spaces of  $(p, q)$ -forms ( $p$  variables,  $q$  complex-conjugate variables) which satisfy  $p + q = n$ . That is:

**Theorem 3.32** (Hodge decomposition). *For a compact Kähler manifold, we have the decomposition:*

$$H_{dR}^k(X) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

where  $H^{p,q}(X)$  is the vector subspace of classes of forms of type  $(p, q)$ .

Similarly, in practical terms we care most about the dimensions of these vector subspaces. Motivated by Betti numbers, we have:

**Definition 3.33.** The  $(p, q)$ -th **Hodge number** is given by  $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ .

The Hodge numbers of a manifold are of interest to both mathematicians and physicists. However, in our case we work from the perspective of elliptic fibrations as (possibly singular) complex projective varieties. Happily, we will pass to their (smooth) resolution, so once one had a smooth complex projective variety  $X$ , one gets an embedding  $X \subseteq \mathbb{P}_{\mathbb{C}}^N$  and the Riemannian metric on  $\mathbb{P}_{\mathbb{C}}^N$  induces  $X$  to be a compact Kähler manifold and thus have Hodge numbers. (See also: Chow's theorem.)

Before we complete this discussion, we recall the following basic identities on Hodge numbers; we will use them to compute Hodge numbers in the case of Calabi-Yau elliptic threefolds in Section 9.3.

**Proposition 3.34.** *The following identities hold for all Hodge numbers of a smooth complex projective variety:*

$$h_{n-r}(X; \mathbb{R}) = h_{dR}^r(X) = \sum_{p+q=r} h^{p,q}(X), \quad h^{p,q}(X) = h^{q,p}(X), \quad h^{p,q}(X) = h^{n-p, n-q}(X)$$

*Proof.* The first identity follows from Hodge decomposition. The second identity follows as every  $(p, q)$ -form is the complex conjugate of a respective  $(q, p)$ -form, giving a vector space isomorphism  $H^{p,q}(X) \cong H^{q,p}(X)$ . The last property is a consequence of Serre duality and the relationship with sheaf cohomology; see the next part.  $\square$

In Hodge theory, there is an eponymous *Hodge conjecture* which relates  $H^{k,k}(X)$  to the rational cohomology of the complex subvarieties of  $X$ . The following is the only case proven; we will use it later to compute Hodge numbers:

**Definition 3.35.** The **fundamental class** of a divisor (element of  $A_{n-1}(X)$ ) for smooth complex projective varieties  $X$  is its image in  $H^2(X; \mathbb{Z}) \cong H_{2n-2}(X; \mathbb{Z})$  (by Poincaré duality).

**Theorem 3.36** (Lefschetz (1,1)-theorem). *Let  $X$  be a smooth, complex  $n$ -dimensional variety. If  $\alpha \in H^2(X; \mathbb{Z})$  has an image in  $H^2(X; \mathbb{C})$  which lies in  $H^{1,1}(X)$ , then  $\alpha$  is the the fundamental class of an element of  $A_{n-1}(X)$*

### Sheaf cohomology (algebraic geometry)

**Definition 3.37.** The **coherent sheaf cohomology** of a coherent sheaf  $\mathcal{F}$  on a space  $X$  is given by

- **Cochain objects:** There exists an injective resolution  $0 \rightarrow \mathcal{F} \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$ . Let  $C^n = I_n$ .
- **Coboundary map:** Let  $\partial^n$  be the maps of the injective resolution.

- **Notation:**  $H^n(X, \mathcal{F}) = H^n(C_\bullet, \partial_\bullet)$  using the cochain complex defined above. One can prove that this construction is independent of the choice of resolution.

The full theory is developed in [Har77].

The following result, due to Serre [Ser56], is the essential bridge between algebraic geometry and complex differential geometry:

**Theorem 3.38** (GAGA principle). *Let  $X$  be a projective complex variety. The natural map  $\varphi : X^{\text{an}} \rightarrow X$ , where  $X^{\text{an}}$  is the analytification (intuitively,  $X$  as a complex analytic space), induces an equivalence of categories between the coherent (algebraic) sheaves of  $X$  and the coherent (analytic) sheaves of  $X^{\text{an}}$ . In particular,  $H^k(X, \mathcal{F}) \cong H^k(X^{\text{an}}, \varphi^* \mathcal{F})$ , although we write the former under both interpretations.*

*Remark.* The principle ultimately validates our overloading of expressions such as *tangent bundle*, *complex space*, etc., and our implicit passage from  $X$  as a smooth projective complex variety to  $X$  as complex manifold, such as when we use algebraic interpretations of Calabi-Yau varieties (which passes via GAGA to analytic interpretations of Calabi-Yau manifolds).

We are only interested in the case where  $\mathcal{F}$  is a locally free coherent sheaf, i.e., a vector bundle. Here are some observations to build intuition:

**Example 3.39.** Let  $\mathcal{F}$  be a vector bundle over  $X$ . Then  $H^0(X, \mathcal{F})$  is the  $\mathcal{O}_X$ -module of global sections of  $\mathcal{F}$ . In the case of  $\mathcal{F} = \mathcal{O}_X(D)$ , this reconciles with our earlier notation of  $\mathcal{L}([D])$ .

**Definition 3.40.** The **holomorphic Euler characteristic**  $\chi(X, \mathcal{F})$  of a vector bundle  $\mathcal{F}$  over  $X$  is the Euler characteristic of its sheaf cohomology.

**Definition 3.41.** The **arithmetic genus** of a variety  $X$  is  $p_a(X) = (-1)^{\dim X} (\chi(X, \mathcal{O}_X) - 1)$ . When  $X$  is a smooth complex curve, this coincides with the classic geometric genus of a Riemann surface (Corollary 3.52).

Proofs of the next two assertions can be found in [GH78, §0]. The following is an important duality of sheaf cohomology, highlighting the privileged role of the canonical bundle:

**Proposition 3.42** (Serre duality). *For a vector bundle  $\mathcal{F}$  over smooth projective varieties, we have:*

$$H^k(X, \mathcal{F}) \cong H^{n-k}(X, \omega_X \otimes \mathcal{F}^\vee)^\vee.$$

It goes hand-in-hand with an important relationship between Hodge decomposition and coherent sheaf cohomology that we will need (by GAGA, we can interpret the right side analytically or algebraically):

**Theorem 3.43** (Dolbeault theorem (+ GAGA)). *For  $X$  a smooth complex projective variety, we have:*

$$H^{p,q}(X) \cong H^q(X, \Omega^p).$$

### 3.3 The Hirzebruch-Riemann-Roch theorem

We have now established two seemingly unrelated sets of notions: The first set of concepts were the Chow ring, Chern class, and adjunction, by which one has an algebraic setting to consider subvarieties and their intersections within a variety (e.g., a projective bundle). The second set of concepts were the various homology and cohomology theories, and their associated invariants. We have seen glimpses of a bridge (e.g., via the fundamental classes of divisors).

We relate the two views with the Hirzebruch-Riemann-Roch theorem. Its statement requires the definition of two “analytic” expressions of Chern classes, previewing the philosophy that leads to the analytic pushforward formulas of Section 5.3. Despite being formal power series, these expressions are all well-defined since the Chern classes past the rank of  $\mathcal{F}$  are zero; i.e.,  $c_k(\mathcal{F}) = 0$  by the vanishing property:

**Definition 3.44.** We define the following polynomials of Chern classes. Let  $\mathcal{F}$  be a vector bundle of rank  $r$ , and let  $\alpha_i$  be its Chern roots (Definition 3.11). Then, writing  $c_i = c_i(\mathcal{F})$ :

- The **Chern character**  $\text{ch}(\mathcal{F})$  is given by

$$\text{ch}(\mathcal{F}) = \sum_{i=1}^r \exp(\alpha_i) = \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha_1^k + \cdots + \alpha_r^k) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots.$$

- The **Todd class**  $\text{td}(\mathcal{F})$  is given by

$$\text{td}(\mathcal{F}) = \prod_{i=1}^r \frac{\alpha_i}{1 - \exp(-\alpha_i)} = \prod_{i=1}^r \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} \alpha_i^k = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots.$$

The power series arise as we get elementary symmetric polynomials in the  $\alpha_i$ , making them expressible in terms of  $c_i(\mathcal{F})$ . (Also, the rational coefficients will not pose a problem when we take degree maps.)

These two expressions are related by the following identity:

**Proposition 3.45** (Borel-Serre lemma). *Let  $\mathcal{F}$  be a vector bundle of rank  $r$ . Then:*

$$\sum_{i=1}^r (-1)^i \text{ch}(\wedge^i \mathcal{F}^\vee) = c_r(\mathcal{F}) \text{td}(\mathcal{F})^{-1}$$

*Proof.* Let  $\alpha_i$  be the Chern roots of  $\mathcal{F}$ . Expanding the left side gives

$$\sum_{i=1}^r (-1)^i \sum_{k_1 < \cdots < k_i} \exp(-\alpha_{k_1} - \cdots - \alpha_{k_i}) = \prod_{i=1}^r (1 - \exp(-\alpha_i)) = \left( \prod_{i=1}^r \alpha_i \right) \cdot \left( \prod_{i=1}^r \frac{1 - \exp(-\alpha_i)}{\alpha_i} \right),$$

which is the right side. □

**Notation 3.46.** We introduce the “integral notation”  $\int_X \alpha$  for the degree of the 0-th component  $\text{deg } \alpha_0$  of a cycle  $\alpha \in A(X)$ . This gives the following suggestive relation when  $\varphi : X \rightarrow Y$ :

$$\int_X \alpha = \int_Y \varphi_*(\alpha).$$

**Theorem 3.47** (Hirzebruch-Riemann-Roch). *Let  $\mathcal{F}$  be a vector bundle over a smooth projective variety  $X$ . Then*

$$\chi(X, \mathcal{F}) = \int_X \text{ch}(\mathcal{F}) \text{td}(\mathcal{T}_X).$$

*Remark.* This is proven in [Ful98, §15] as a corollary to the even more general Grothendieck-Riemann-Roch theorem. However, before then it was proven by Hirzebruch for complex varieties. The key idea is that the Chern character is a nice ring homomorphism from the Grothendieck ring of vector bundles, the Todd class has a similarly nice defining property, and these induce the relationship to sheaf cohomology.

In particular, the Hirzebruch-Riemann-Roch theorem via GAGA implies the “algebraic version” of Poincaré-Hopf, an *index theorem* which enables our later computations of Euler characteristics:

**Theorem 3.48** (Poincaré-Hopf). *Let  $X$  be a smooth,  $n$ -dimensional, complex projective variety. Then:*

$$\chi_{\text{top}}(X) = \int_X c(\mathcal{T}_X) \frown [X]$$

*Proof.* We take  $\mathcal{F} = \mathcal{T}_X$  to be the rank  $r = n$  bundle in the Borel-Serre lemma (Proposition 3.45). Then

$$\int_X c(\mathcal{T}_X) \frown [X] = \int_X \left( \sum_{i=1}^n (-1)^i \text{ch}(\wedge^i \mathcal{T}_X^\vee) \right) \text{td}(\mathcal{T}_X) = \sum_{i=1}^n (-1)^i \int_X \text{ch}(\wedge^i \Omega_X) \text{td}(\mathcal{T}_X) = \sum_{i=1}^n (-1)^i \chi(X, \wedge^i \Omega_X)$$

by Hirzebruch-Riemann-Roch. By the definition of Euler characteristic (Definition 3.24), the Dolbeault theorem + GAGA (Theorem 3.43), and Hodge decomposition (Proposition 3.34), this equals:

$$\sum_{i=1}^n (-1)^i \sum_{j=1}^{\infty} (-1)^j h^j(X, \wedge^i \Omega_X) = \sum_{i,j=1}^{\infty} (-1)^{i+j} h^{i,j}(X) = \sum_{r=1}^{2n} (-1)^r h^r(X),$$

which is the definition of  $\chi_{\text{top}}(X)$ . □

Ultimately, we will need this statement in full generality (that is, for arbitrary dimensions  $n$ ). However, from these two theorems we also retrieve a number of relevant, classical algebro-geometric results:

- In constructing Weierstrass models we used the 1-dimensional case of Hirzebruch-Riemann-Roch, which with Serre duality implies the Riemann-Roch theorem for curves.
- Also, from  $\chi_{\text{top}}(C) = \deg c_1(\mathcal{T}_C)$  for curves  $C$ , one gets the classical relationship between the genus of a Riemann surface and its Euler characteristic.
- For the Hodge numbers of Calabi-Yau elliptic threefolds, we will use Noether’s formula, which follows from the 2-dimensional case of Hirzebruch-Riemann-Roch along with the equivalence  $\chi_{\text{top}}(B) = \deg c_2(\mathcal{T}_B)$  for surfaces  $B$ .

We now formally state these corollaries, along with some related results:

**Corollary 3.49** (Riemann-Roch for curves). *Let  $C$  be an irreducible smooth projective curve. For a divisor  $D$ :*

$$\ell(D) - \ell(K_C - D) = \deg D + 1 - p_a(C).$$

where  $\ell(D) = h^0(C, \mathcal{O}_C(D))$ .

*Proof.* Using Theorem 3.47 for the line bundle  $\mathcal{L} = \mathcal{O}_C(D)$ :

$$\chi(C, \mathcal{O}_C(D)) = \int_C \text{ch}(\mathcal{O}_C(D)) \text{td}(\mathcal{T}_C) = \int_C (1 + c_1(\mathcal{O}_C(D)) + \cdots) \left( 1 + \frac{1}{2} c_1(\mathcal{T}_C) + \cdots \right)$$

Since  $C$  is dimension 1, the top Chern class is  $c_1(\mathcal{O}_C(D)) + \frac{1}{2} c_1(\mathcal{T}_C)$  and we get:

$$h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C) = \deg D + \frac{1}{2} \deg c_1(\mathcal{T}_C)$$

Now observe that if  $D = 0$  (i.e., we take  $\mathcal{L} = \mathcal{O}_C$ ), we get  $\chi(C, \mathcal{O}_C) = \frac{1}{2} \deg c_1(\mathcal{T}_C)$ , which gives  $p_a(C) = 1 - \frac{1}{2} \deg c_1(\mathcal{T}_C)$ , using the definition of  $p_a(C)$  (Definition 3.41). Finally, Serre duality gives us

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \omega_C \otimes \mathcal{O}_C(D)^\vee) = h^0(C, \mathcal{O}_C(K_C - D)).$$

Placing these last two results into the first and adopting the  $\ell(D)$  notation gives the result.  $\square$

We used the following assertion for the genus-degree formula:

**Corollary 3.50.** *For any irreducible curve  $C$ , we have  $\deg K_C = 2p_a(C) - 2$ .*

*Proof.* In the Riemann-Roch theorem for curves, taking  $D = 0$  gives  $\ell(0) - \ell(K_C) = 1 - p_a(C)$ . Taking  $D = K_C$  gives  $\ell(K_C) - \ell(0) = \deg K_C + 1 - p_a(C)$ . Adding the two relations gives the result.  $\square$

We used the following assertion to construct the Weierstrass equation:

**Corollary 3.51.** *Let  $E$  an elliptic curve. Then for any point  $P \in E$  and  $n > 0$ , we have  $\ell(n[P]) = n$ .*

*Proof.* An elliptic curve  $(E, \mathcal{O})$  is an irreducible smooth projective curve of genus  $p_a(E) = 1$ . In the Riemann-Roch theorem for curves, taking this and  $D = n[P]$  gives  $\ell(n[P]) - \ell(K_C - n[P]) = n$ . But  $\ell(K_C - n[P]) = 0$  and we are done.  $\square$

Here is the well-known relationship between the Euler characteristic of a complex curve (Riemann surface) and its genus:

**Corollary 3.52.** *For  $C$  an irreducible smooth projective curve, we have  $\chi_{\text{top}}(C) = 2 - 2p_a(C)$ .*

*Proof.* In the proof of Corollary 3.49, we saw  $p_a(C) = 1 - \frac{1}{2} \deg c_1(\mathcal{T}_C)$ . From Theorem 3.48 we have  $\deg c_1(\mathcal{T}_C) = \chi_{\text{top}}(C)$ , which gives the result.  $\square$

We will later use the following 2-dimensional version, whose corollary due to Noether lets us compute Hodge numbers for Calabi-Yau threefolds:

**Corollary 3.53** (Riemann-Roch for surfaces). *Let  $S$  be an irreducible smooth projective surface. Then:*

$$\chi(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S) + \frac{1}{2} \deg c_1(\mathcal{O}_S(D))(c_1(\mathcal{O}_S(D)) - c_1(\mathcal{T}_S)).$$

*Proof.* As in Corollary 3.49, using Theorem 3.47 for the line bundle  $\mathcal{L} = \mathcal{O}_S(D)$  and that  $S$  is 2-dimensional:

$$\chi(S, \mathcal{O}_S(D)) = \frac{1}{2} \deg c_1(\mathcal{O}(D))^2 + \frac{1}{2} \deg c_1(\mathcal{O}_S(D))c_1(\mathcal{T}_S) + \frac{1}{12} (\deg c_1(\mathcal{T}_S)^2 - \deg c_2(\mathcal{T}_S)).$$

If  $D = 0$ , we get  $\chi(S, \mathcal{O}_S) = \frac{1}{12} (\deg c_1(\mathcal{T}_S)^2 - \deg c_2(\mathcal{T}_S))$ , giving the result.  $\square$

**Corollary 3.54** (Noether's formula). *Let  $S$  be an irreducible smooth projective surface over an algebraically closed field. Then:*

$$\chi(S, \mathcal{O}_S) = \frac{1}{12} (\deg K_S^2 + \chi_{\text{top}}(S))$$

*Proof.* We observe that  $c_1(\mathcal{T}_S) = -K_S$  by the dual bundle property of Chern classes. We can then pass to intersection products due to our imposed conditions. From Theorem 3.48 we have  $\deg c_2(\mathcal{T}_C) = \chi_{\text{top}}(C)$ , which gives the result.  $\square$

## 4 Torsion and non-Kodaira fibers

In this section, we consider the arithmetic structure of elliptic fibrations and construct Weierstrass models which capture this structure. We also discuss the state-of-the-art regarding singular fibers in higher codimension.

### 4.1 The Mordell-Weil group of a fibration

Thus far we have only discussed the “geometric” properties of elliptic fibrations. Because of this, we wryly observe that all of our examples and proofs so far could have been equally performed using genus-one fibrations. The choice of a distinguished point on an elliptic curve is related to properties of interest to arithmetic geometry and number theory, via the group law:

**Proposition 4.1.** *An elliptic curve  $(E, O)$  over  $K$  has a **group law**. Namely, there is a unique abelian group operation which makes  $E$  into a group with identity  $O$ . (Formally, we say that an elliptic curve is an abelian variety, or more generally a group scheme.) We call this group the **Mordell-Weil group**  $MW(E, O)$ .*

One could prove this abstractly and then pass from the abstract concept of the group to its concrete realization in terms of the Weierstrass equation. However, let us finally reap some rewards from our development of the Weierstrass equation by describing the group law directly (a proof by example, if you will). Our construction would be preserved under isomorphism, proving the existence of a group law for any elliptic curve (at least in the cases  $\text{char}(K) \neq 2, 3$ ). We will use this explicit description soon to derive torsion parameterizations of Weierstrass models.

**Example 4.2.** Consider the simplified Weierstrass equation  $y^2 = x^3 + ax + b$ , and call the elliptic curve it defines  $(E, O)$ , where  $O = [0, 1, 0]$ . Working in  $z \neq 0$  (i.e., in the affine plane), the group law is given as follows (refer to Figure 4.3 for clarity):

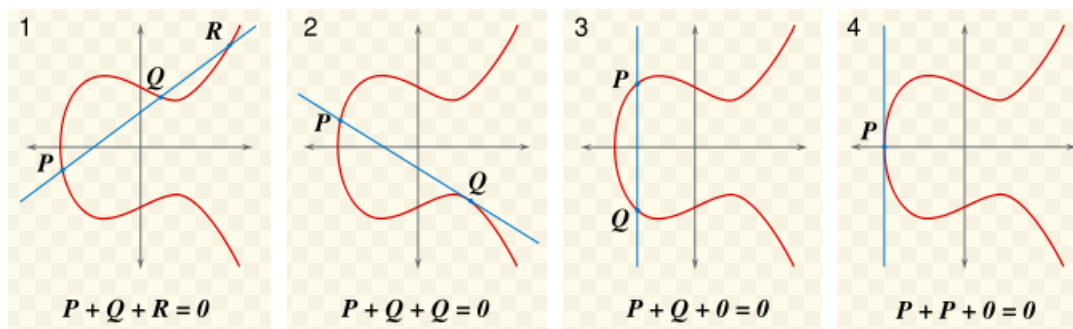
- First, if  $P = (x, y)$  is on  $E$ , we write  $-P = (x, -y)$ , which is necessarily also on  $E$ .
- Let  $P, Q \in E$ . Constructing the line through  $P$  and  $Q$ , when  $Q \neq -P$ , gives a third point of intersection  $R$  by Bézout’s theorem. We then define  $P + Q = -R$ . If two of  $P, Q, R$  coincide, then we appeal to tangents in the obvious way.
- When  $P + (-P)$ , we see that the line goes to the point of infinity, which is  $O$ ; this constructs the inverse.
- Treating  $O$  as the point at infinity, one sees that  $O$  acts as the identity element under these constructions.

Validating this rigorously (in particular associativity, which is not geometrically obvious) can be done using tedious algebra on the Weierstrass equation (see [Sil09, §III]).

**Definition 4.4.** The **Mordell-Weil group**  $MW(\varphi)$  of an elliptic fibration  $\varphi : \mathcal{E} \rightarrow B$  is the group of its rational sections  $B \rightarrow \mathcal{E}$ , where the identity element is  $\sigma$  and the group operation is the fiberwise group operation on the fiberwise Mordell-Weil group  $MW(\varphi^{-1}(p), \sigma(p))$ . The **Mordell-Weil torsion**  $MW_{\text{tors}}(\varphi)$  is the subgroup of torsion elements.

In particular, observe that the Mordell-Weil group is, by construction, invariant under birational morphisms of our elliptic fibration over  $B$ . A major motivation of this thesis is that the Mordell-Weil group of a fibration is related to the gauge group in F-theory that the space describes (Section 9.2). Here is a relevant result:





**Figure 4.3:** Examples of the group law on an elliptic curve from a simplified Weierstrass equation. Graphic by [Emmanuel Boutet](#), Wikimedia Commons.

**Theorem 4.5** (Mordell-Weil theorem). *The Mordell-Weil group of an elliptic curve over a number field is finitely generated. More generally, an elliptic fibration over a function field is finitely generated unless  $\mathcal{E}$  is birational to  $E \times B$  for some elliptic curve  $E$ .*

*Remark.* For the first result, see the author’s undergraduate junior paper (“Descent by 2-isogeny on elliptic curves: from Fermat to Weil”, supervised by Noam Elkies) for a friendly introduction, or [Sil09, §VIII] for an exhaustive one. For the (relevant) function field case, see [LN59].

Furthermore, there are partial results which state that the torsion part of the Mordell-Weil group can take only one of certain forms, given certain constraints. This program was initiated by Mazur’s famous result for elliptic curves over  $\mathbb{Q}$ . Relevant to us, however, is that if one’s elliptic fibration is an elliptic K3 surface, then the Mordell-Weil group is one of only 15 possibilities (not the same as Mazur’s!). There are results for other special cases of elliptic fibrations. These considerations are overviewed in [MMTW14].

*Remark.* In particular, this means that extending our thesis’ strategy to find all crepant resolutions of Weierstrass models for all torsion groups is not impossible, if one restricts to fibrations of certain natural types.

Remarkably, there is an important correspondence between the algebraic structure of  $\varphi : \mathcal{E} \rightarrow B$  as a variety, and the group of its rational sections as a fibration. It was first proven by Shioda and Tate for elliptic surfaces, and then extended to threefolds by Wazir [Waz04]:

**Theorem 4.6** (Shioda-Tate-Wazir). *If  $\varphi : \mathcal{E} \rightarrow B$  is a smooth elliptic threefold, then*

$$\mathrm{rk}(\mathrm{NS}(\mathcal{E})) = \mathrm{rk}(\mathrm{NS}(B)) + f + \mathrm{rk}(\mathrm{MW}(\varphi)) + 1,$$

where  $f$  is the number of geometrically irreducible fibral divisors away from the zero section and  $\mathrm{NS}(X)$  is the Néron-Severi group (the subgroup in  $H^2(X; \mathbb{Z})$  of fundamental classes of Cartier divisors).

The proof of this is beyond the scope of this thesis, and lies closer to the domain of arithmetic geometry. However, the formula has found use in recent F-theory literature (see [MMTW14], [KMSS11]), as it does once more here.

## 4.2 Weierstrass models with given torsion

The families of Weierstrass models we will use are already well-known. The arguments that were used to derive parameterizations in the original elliptic curve context generalize to Weierstrass models via the

local passage to function fields. For example, [Kub76, Table 3] is a large list of parameterizations one might wish to refer to when encoding various torsion groups in generality. We now derive the general Weierstrass model families we require to exactly enforce our desired torsion groups  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , and  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . Here, points  $P$  are coordinates on a general fiber:

**Example 4.7** ( $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  family). Let  $P \in \mathcal{W}(K(B))$ . We can always take  $x \mapsto x - x(P)$ ,  $y \mapsto y - y(P)$  to place  $P$  at  $(0, 0)$ ; this automorphism occurs over  $K$ . Suppose  $P \in \mathcal{W}(K(B))[2] \neq \emptyset$  (the 2-torsion component); then we can assume under a  $K(B)$ -automorphism that  $P = (0, 0)$ . Via the simplified Weierstrass form, we can assume our Weierstrass model has the form

$$\mathcal{W}/K(B) : y^2 z = c(x, z), \quad x \mid c(x, z),$$

where  $c(x, z)$  is a homogeneous cubic with  $\Delta \neq 0$ . Over  $\overline{K(B)}$ , the equation  $c(x, z)$  splits into:

$$\mathcal{W}/\overline{K(B)} : y^2 z = x(x - \beta_2 z)(x - \gamma_2 z), \quad \beta_2 \neq \gamma_2 \neq 0.$$

By the group law on the Weierstrass form, since  $y = 0$ , these points plus the section-at-infinity  $O$  exhaust  $\mathcal{W}(\overline{K(B)})[2] \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .

In preparation for Tate's algorithm, note that we can take  $x \mapsto x + \frac{1}{3}(\beta_2 + \gamma_2)$  to get

$$\mathcal{W}/K(B) : y^2 z = x^3 + f x z^2 + g z^3,$$

where we have

$$f = \frac{1}{3}(\beta_2 \gamma_2 - \beta_2^2 - \gamma_2^2), \quad g = \frac{1}{27}(2\beta_2 - \gamma_2)(2\gamma_2 - \beta_2)(\beta_2 + \gamma_2), \quad \Delta = 4f^3 + 27g^2 = -\beta_2^2 \gamma_2^2 (\beta_2 - \gamma_2)^2.$$

**Example 4.8** ( $\mathbb{Z}/3\mathbb{Z}$  family). The following derivation is based on [Hus04]. Suppose  $P \in \mathcal{W}/K(B)[3] \neq \emptyset$  (the 3-torsion component). Then we can assume under a linear  $K(B)$ -automorphism that  $P = (0, 0)$  and our Weierstrass equation is

$$y^2 z + a_1 x y z + a_3 x z = x^3 + a_2 x^2 z + a_4 x z^2.$$

Summarizing the argument: taking  $y'$  shows that the slope at  $(0, 0)$  is  $a_4/a_3$ . To prevent both a singularity and to enforce  $P$  having order  $\geq 2$ , we must have  $a_3 \neq 0$ . The linear  $K(B)$ -automorphism  $y \mapsto y + (a_2/a_3)x$  allows us to assume the form

$$y^2 + a_1 x y z + a_3 y z^2 = x^3 + a_2 x^2 z$$

where  $(0, 0)$  has tangent slope 0. For  $y = 0$  to give an intersection of multiplicity 3 with  $\mathcal{W}/K(B)$  at  $(0, 0)$ , we must have  $a_2 = 0$ . Hence, a general Weierstrass form with  $\mathcal{W}(K(B))[3] \neq \emptyset$  is:

$$\mathcal{W}/K(B) : y^2 z + a_1 x y z + a_3 y z^2 = x^3, \quad \Delta \neq 0.$$

In preparation for Tate's algorithm, note that we can take  $y \mapsto y - \frac{1}{2}(a_1 x + a_3 z)$  to get

$$y^2 z = x^3 + \frac{1}{4}a_1^2 x^2 z + \frac{1}{2}a_1 a_3 x z^2 + \frac{1}{4}a_3^2 z^3.$$

Take  $x \mapsto x - \frac{1}{12}a_1^2$  to get

$$\mathscr{W}/K(B) : y^2z = x^3 + f xz^2 + gz^3,$$

where we have

$$f = -\frac{a_1}{48}(a_1^3 - 24a_3), \quad g = \frac{1}{864}(a_1^6 - 36a_1^3a_3 + 216a_3^2), \quad \Delta = 4f^3 + 27g^2 = -\frac{a_3^3}{16}(a_1^3 - 27a_3).$$

**Example 4.9** ( $\mathbb{Z}/2\mathbb{Z}$  family). This follows from the  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  family, but we now disallow one of the automorphisms. The easiest one is to not require  $c(x, z)$  to factor further than  $xq(x, z)$ . This gives the family that we will denote

$$y^2z = x(x^2 + a_2xz + a_4z^2),$$

which corresponds to taking  $a_1 = a_3 = 0$  in the general Weierstrass equation. However, we must also enforce  $a_2^2 - 4a_4 \neq 0$  and  $a_4 \neq 0$  to ensure that this is not a cuspidal curve.

In preparation for Tate's algorithm, note that we can take  $x \mapsto x - \frac{1}{3}a_2$  to get

$$\mathscr{W}/K(B) : y^2z = x^3 + f xz^2 + gz^3, \quad a_2^2 - 4a_4 \notin K(B)^{\times 2}, \quad a_4 \neq 0$$

with

$$f = -\frac{1}{3}(a_2^2 - 3a_4), \quad g = \frac{a_2}{27}(2a_2^2 - 9a_4), \quad \Delta = -a_4^2(a_2^2 - 4a_4).$$

*Remark.* Note that these torsion families of Weierstrass models match the classical Legendre family  $y^2 = x(x-1)(x-\lambda)$  and Hesse families of elliptic curves with particular torsion.

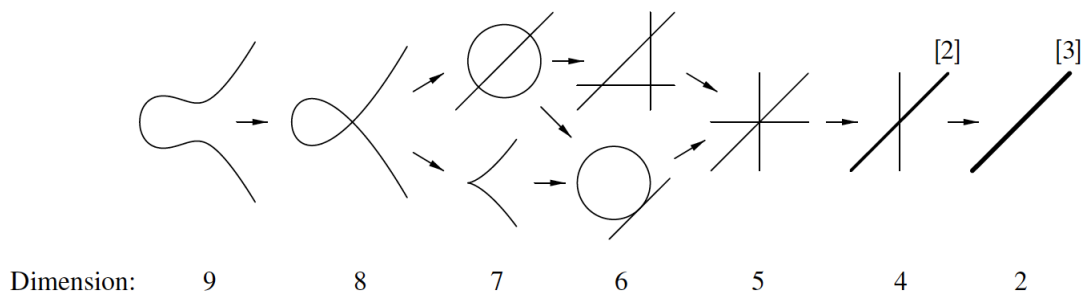
### 4.3 Collisions and degenerations of fibers

As noted before, Kodaira and Néron were interested in classifying the singular fibers in elliptic surfaces. In particular, their fibers occurred over 0-dimensional points, giving these points no opportunity to intersect. However, in the case of elliptic threefolds and higher, it is inevitable that two components of the discriminant locus are distinct and thus intersect.

**Example 4.10** (Plane cubics). We revisit the parameter space  $\mathbb{P}^9$  of plane cubics (Example 2.8) over an algebraically closed field. As noted before, we can view the plane cubics as a genus-one fibration. Explicitly, we have the hypersurface:

$$\mathscr{E} = \{([a_{3,0}, \dots, a_{0,0}], [x, y, z]) \mid \sum_{0 \leq i+j \leq 3} a_{i,j} x^i y^j z^{3-i-j}\} \subseteq \mathbb{P}^9 \times \mathbb{P}^2$$

If  $\Delta$  is the cubic discriminant, we see that  $\Delta = 0$  determines a hypersurface in  $\mathbb{P}^9$  over which we no longer have a smooth (genus 1) cubic. It can be shown that generically, one gets nodal fibers over  $\{\Delta = 0\}$ ; however, over codimension 2 our fibers factor or give a double point. In general, we see that the cubic fibers *degenerate* further and further over higher codimension loci; see Figure 4.11.



**Figure 4.11:** In the genus-one fibration of all plane cubics, the generic smooth fiber degenerates over loci of higher codimension. Original image from D. Eisenbud and J. Harris, *3264 and All That: A Second Course in Algebraic Geometry* (2016).

Let us try to match the singular plane cubics to Kodaira’s classification (Table 2.22):

Dimension:	9	8	7	6	5	4	2
Codimension:	0	1	2	3	4	5	7
Fiber type:		$I_1$	$I_2$ $II$	$I_3$ $III$	$IV$	?	?

Here, Kodaira’s classification is surprisingly tenacious, but it is not comprehensive in very degenerate situations (e.g., the first instance of multiplicity-2 in Kodaira’s classification, requiring five irreducible components, is  $I_0^*$ ; compare this with the three co-incident lines we see in codimension 7). We see that in codimension 2 there are two types of collisions:

$$I_1 + I_1 = I_2, \quad \text{or} \quad I_1 + I_1 = II.$$

This is the kind of analysis we are interested in. However, as Kodaira did by requiring smooth minimality, we also aim to resolve our total space first; however, we work in arbitrary dimension, we want to preserve our base, and we only take (crepant) resolutions. It is these resolutions which makes large numbers of rational curves possible, as exhibited by the infinite  $I_n$  family of singular fibers in Kodaira’s classification. In the case of plane cubics, our intuition is informed by the factorizations of a cubic polynomial; such intuition is lost when new curves are created by blowups.

Miranda, extending Kodaira’s work, was able to classify the geometric singular fibers over codimension-2 in  $\text{char}(K) = 0$  [Mir83]; however, he only did so upon further restriction to  $\Delta$  being a simple normal crossings divisor and preserving  $j$ -invariant at the intersections. In his Harvard PhD thesis [Szy99], Szydło extended Miranda’s results to arbitrary  $n$ -folds, non-perfect fields, and higher codimension; however, similar constraints remained.

**Definition 4.12.** A divisor  $\Delta = \sum_i \Delta_i$  of irreducible components on a smooth variety  $X$  of dimension  $n$  is a **simple normal crossing divisor** if any  $\Delta_i$  is smooth and for all  $p \in X$ ,  $\Delta$  locally looks like  $\prod_{i=1}^r x_i = 0$  for  $r \leq n$  independent local parameters  $x_i$  (i.e., it looks locally like  $r$  mutually transversal hyperplanes everywhere).

**Example 4.13.** In the plane cubic case, the first collision type is on Miranda’s list [Mir83]; the second type is not. The latter type’s occurrence implies that one of the normal or  $j$ -invariant constraints is unmet.

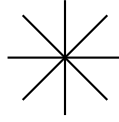
Simple normal crossings with preserved  $j$  invariant are not particularly natural; especially in physics, F-theory finds stranger collisions more non-trivial. In fact, in our own models:

**Proposition 4.14.** *The Weierstrass models with torsion listed in Section 4.2 do not have simple normal crossings.*

*Proof sketch.* This is visually immediate; noted the depicted of tangency in the discriminant loci for  $\mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}/3\mathbb{Z}$ , as shown in Figure 6.1 and Figure 7.1. In the case of  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  (Figure 8.1), each pair of components is transversal, but not all three together (at their shared point of intersection) in the sense of simple normal crossings; they are mutually dependent. (Despite this, the first two give Kodaira fibers in codimension 2; the third does not.)  $\square$

As a preview of our later work, we consider an example of (non-Weierstrass), non-Kodaira degeneration in codimension 2. In Section 8 we will also encounter it in codimension 2, but furthermore for a crepant resolution of a Weierstrass model. This is because we do not attempt to satisfy simple normal crossings, as produce these generically would alter the Chern class of the base, changing the topology and nature of our question entirely.

**Example 4.15** (The  $I_0^{*-}$  fiber). We construct an instance of what we shall call the **blown-down**  $I_0^*$ . The long name comes from being the product of taking an  $I_0^*$  and contracting the central component. The short name comes from a notable sighting of this fiber over a codimension-4 locus in a smooth  $D_5$  (non-Weierstrass) elliptic fibration [EFY15]. This fiber looks like four rational curves intersecting at the same point:



The following construction is based on the observation that the intersection of two quadrics (degree 2 surfaces)  $Q, Q'$  is generically an elliptic curve (after imposing a zero section); see the Remark for details. Now, consider the following two families of quadrics in  $\mathbb{P}^3$  with coordinates  $[w, x, y, z]$ :

$$Q_\alpha : w^2 - x^2 + \alpha z^2 = 0, \quad Q'_\beta : w^2 - y^2 + \beta z^2 = 0.$$

We consider their intersection  $Q_\alpha \cap Q'_\beta$ , which is hence a two-parameter family of elliptic curves (an elliptic threefold) parameterized by  $\alpha$  and  $\beta$ . Then on the locus  $\alpha = \beta = 0$ , we get

$$w^2 - x^2 = w^2 - y^2 = 0 \iff w^2 = x^2 = y^2$$

giving four lines  $w = x = y, w = x = -y, w = -x = y, w = -x = -y$ , all pairwise transverse at  $[0, 0, 0, 1]$ .

*Remark.* To confirm that our intersection is generically an elliptic curve, one can use the formula for Chern classes of complete intersections such as  $\iota : X = Q \cap Q' \subseteq \mathbb{P}^3$ . See [EH16, §5]; this can be proven by adjunction, the Whitney sum property, and  $c(\mathcal{T}_{\mathbb{P}^n}) = (1 + H)^{n+1}$  (which follows from the Euler sequence as in Example 3.15):

$$\iota_* c(\mathcal{T}_X) = \frac{\iota_* c(\iota^* \mathcal{T}_{\mathbb{P}^3})}{\iota_* c(\mathcal{N}_{X/\mathbb{P}^3})} = \frac{(1 + H)^{3+1}}{(1 + 2H)(1 + 2H)} = 1 + 2H^2 - 4H^3 + \dots = 1,$$

where division will be justified by Proposition 5.10. This means it has trivial canonical class, implying it has genus 1 by Corollary 3.49 and is thus an elliptic curve.

*Remark.* A preprint of Cattaneo claims that codimension-2 fibers for crepant resolutions of Weierstrass threefolds are necessarily Kodaira or contractions of Kodaira fibers, consistent with our encounter in Section 8. The proof is not yet fully complete [Cat13].

## 5 Pushforwards along crepant resolutions

### 5.1 Resolution of singularities

In Section 2.2 we saw how elliptic fibrations are birational over their base to respective Weierstrass models. Such models are potentially singular, which is problematic if one wanted to compute “the” topological invariants of a Weierstrass model (in particular, there are inequivalent characterizations of Euler characteristic for singular varieties). Furthermore, many results in algebraic geometry assume a smooth variety, and topological invariants are not generally preserved under birational maps.

However, the birational map from an elliptic fibration to a Weierstrass model preserves the Mordell-Weil group, which is a primary motivation for studying torsional Weierstrass models; they represent torsion elliptic fibrations in generality while giving a constrained, familiar form through the Weierstrass equation. Furthermore, singularities are often quite local; thus, it is useful to consider smooth varieties that are isomorphic almost everywhere to a singular Weierstrass model of interest. This leads to the following concept:

**Definition 5.1.** A **resolution of singularities** of  $X$  is a proper, birational morphism of varieties  $\varphi : X' \rightarrow X$  such that  $X'$  is smooth and  $\varphi$  is an isomorphism away from the singular points  $\text{Sing}(X)$  of  $X$ .

*Remark.* This definition is stricter than what others might require, and is what some authors call a **strong desingularization**.

The classic means by which one attempts to construct a resolution of a singular variety is via iterating the blowup construction, which replaces a subvariety of singular points with its projective tangent space. The exposition here is synthesized from [Har92, §7] and [Hau06].

There is a universal property that characterizes the blowup of a subvariety; however, we specialize to its definition for projective varieties as this gives the explicit construction we will use repeatedly later on:

**Definition 5.2.** The **blowup**  $\text{Bl}_Z(X)$  of a projective variety  $X \subseteq \mathbb{P}^n$  along a subvariety  $Z$  is constructed as follows: take homogeneous polynomials  $F_0, \dots, F_n$  of equal degree generating an ideal with saturation  $I(Z)$ . Then

$$\text{Bl}_Z(X) = \text{graph}([F_0(X), \dots, F_n(X)])$$

We also associate with a blowup  $\pi : \text{Bl}_Z(X) \rightarrow X$  are a few privileged subvarieties, in both  $\text{Bl}_Z(X)$  and  $X$ :

**Definition 5.3.** The **exceptional divisor**  $E$  of a blowup  $\pi : \text{Bl}_Z(X) \rightarrow X$  is given by  $E = \pi^{-1}(Z) \subseteq \text{Bl}_Z(X)$ .

**Definition 5.4.** Let  $V \subseteq X$  be a subvariety, and consider the blowup  $\pi : \text{Bl}_Z(X) \rightarrow X$ . The **proper transform** of  $V$  is the closure of  $\pi^{-1}(V \setminus Z)$ .

**Example 5.5.** The situation most to us are the repeated blowups of  $X_0 = \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3})$ . We consider the blowup of  $X_0$  along  $Z_0 = \langle r_0, \dots, r_m \mid e_1 \rangle$ , where  $r_i$  are the intersecting divisors the give our center (this notation lists the ideal and also names the exceptional divisor as  $e_1 = 0$ ). We get:

$$X_1 = \text{Bl}_{Z_0}(X_0) = \{[z, x, y], [\bar{r}_0, \dots, \bar{r}_m] \mid r_0 = e_1 \bar{r}_0, \dots, r_m = e_1 \bar{r}_m\} \subseteq X_0 \times \mathbb{P}^m$$

where  $e_1$  is the section whose zero locus is the exceptional divisor  $E_1 \subseteq X_1 = \text{Bl}_Z(X_0)$ . This construction iterates; if we take  $X_1$ 's blowup along  $Z_1 = \langle s_0, \dots, s_n \mid e_2 \rangle$ , we get

$$X_2 = \text{Bl}_{Z_1}(X_1) = \{r_0 = e_1 \bar{r}_0, \dots, r_m = e_1 \bar{r}_m, s_0 = e_2 \bar{s}_0, \dots, s_n = e_2 \bar{s}_n\} \subseteq X_0 \times \mathbb{P}^m \times \mathbb{P}^n,$$

where  $[z, x, y], [\bar{r}_0, \dots, \bar{r}_m], [\bar{s}_0, \dots, \bar{s}_n]$  are the coordinates of  $X_0 \times \mathbb{P}^m \times \mathbb{P}^n$ , and  $e_2$  is the section whose zero locus is the exceptional divisor  $E_2$  of  $X_2 = \text{Bl}_{Z_1}(X_1)$ .

It turns out that this construction suffices to resolve the models we consider in this thesis. More generally, Hironaka [Hir64, Main Thm.] famously showed that:

**Theorem 5.6.** *For a variety  $X$  over a field of characteristic zero, the singular locus is exactly a subvariety, and there exists a resolution of singularities of  $X$  constructed by finitely many repeated blowups.*

The proof is a very lengthy, case-by-case argument by induction on the dimension of the singular locus. However, it is a very important result that led to the plausibility of, and ongoing research in, the *minimal model program*, which seeks to classify all complex projective varieties up to birationality by finding particularly nice smooth models within the birational class.

## 5.2 Crepant blowups and Betti invariance

However, Hironaka’s result would still not provide the preservation of topological invariants we desire. As noted before, in general one does not expect birational maps to preserve topological invariants (consider for example, the Riemann sphere to the complex plane). However, for certain types of varieties and certain birational morphisms, we can ensure good results.

**Definition 5.7.** A morphism  $\varphi : X' \rightarrow X$  is **crepant** if it preserves the canonical class.

In general, canonical class invariance is a desirable property; for example, the *Calabi-Yau condition* is expressible as having trivial canonical class, and string theorists often restrict themselves to such varieties as this condition implies essential physical properties such as Ricci-flatness. There are important interpretations of crepant resolutions in particular, relating to the “network” they form for a given singular variety; see [ESY15].

There are purely mathematical outcomes as well. The one relevant to us is due to Batyrev [Bat99, Thm. 4.2], who was inspired by string theoretic computations (cf., *stringy invariants*) that for “physical” reasons suggested that two crepant birational spaces should have the same topological invariant. Remarkably:

**Theorem 5.8.** *If  $\varphi : X \rightarrow Y$  is a birational map of smooth projective (irreducible) varieties over  $\mathbb{C}$  that preserves the canonical class ( $K_{X'} = \varphi^*K_X$ ), then  $X, Y$  have the same topological Betti numbers (over  $\mathbb{C}$ ).*

This result is beyond the scope of this thesis and is particularly dependent on taking projective varieties over  $\mathbb{C}$ , as it invokes  $p$ -adic integration and the Weil conjectures. However, this result is powerful enough for our practical needs. Combining definitions, a **crepant resolution** is a birational map between a singular variety and a smooth “resolved” one. However, this means two such resolved varieties are related by crepant birational maps, and hence:

**Corollary 5.9.** *All crepant resolutions of the same singular projective irreducible variety over  $\mathbb{C}$  have the same topological Betti numbers.*

In summary: each Weierstrass model (over a projective base in  $\mathbb{C}$ ) lives in a birational class of varieties. In some subclass of that, there is a class of varieties that furthermore have the same Betti numbers, and in particular have the same Euler characteristic. Thus, Batyrev’s result shows that the crepant resolutions of the Weierstrass model live in this “homology equivalent” class. In particular, it thus suffices to compute the Euler characteristic of any smooth representative, namely any single crepant resolution, of the Weierstrass model—as this is well-defined across the class!

### 5.3 Pushforward formulas for Chern classes

Adjunction lets us to compute Chern classes for subvarieties like  $\mathcal{W} \subseteq X_0$ . We now also have the *crepant resolution*, a conceptual bridge that relates our singular Weierstrass models to smooth elliptic fibrations resolving them, ones with Euler characteristics that are appropriately independent of the choice of crepant resolution. However, having an expression in terms of Chern classes in the resolved total space is still detached from an expression in terms of Chern classes in the base, which is what we have and what one modifies.

In particular, suppose we had a formal power series  $f(H)$  where  $H$  is a first Chern class (such as the hyperplane class) in  $A(X)$ . By linearity, its pushforward is determined by the pushforwards  $\pi_* H^i$ . Is it possible to compute these all at once? In response, we suggestively write

$$\frac{1}{1-H} = 1 + H + H^2 + H^3 + \dots$$

In particular, the right side is always a finite sum by the vanishing property. Hence, these inverse (total) Chern classes are always well-defined; we call them **Segre classes**. We will not work with these explicitly, but we do appeal to their existence when we “divide” total Chern classes. In particular:

**Proposition 5.10** (Chern class division). *For any Chern class  $c(\mathcal{F}) \in A(X)$ , there is a well-defined inverse endomorphism  $\frac{1}{c(\mathcal{F})} \frown - : A(X) \rightarrow A(X)$  such that*

$$\frac{1}{c(\mathcal{F})} \frown (c(\mathcal{F}) \frown -) = -$$

Hence, when the conditions of Theorem 3.7 are met (i.e., Chern classes as Chow ring elements), then for any total Chern class  $c(\mathcal{F}) \in A(X)$  we have an inverse element  $\frac{1}{c(\mathcal{F})} \in A(X)$ .

In practical terms, this justifies division by Chern classes. Let us now realize our objective: pushforward formulas necessary for our computations. The above idea works particularly nicely for projective bundles:

**Proposition 5.11.** *Consider the projective bundle  $\pi : \mathbb{P}(\mathcal{F}) \rightarrow B$ . Then*

$$\pi_* 1 + \pi_* H + \pi_* H^2 + \pi_* H^3 + \dots = \pi_* \left( \frac{1}{1-H} \right) = \frac{1}{c(\mathcal{F})}.$$

*Proof.* Applying the Whitney sum property to the universal sequence of  $\mathbb{P}(\mathcal{F})$  (Definition 1.4), where  $\mathcal{S} \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)$  is the tautological subbundle, we get:

$$c(\mathcal{Q}) = \frac{c(\pi^* \mathcal{F})}{c(\mathcal{S})} = \frac{c(\pi^* \mathcal{F})}{c(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1))} = \frac{c(\pi^* \mathcal{F})}{1-H} = c(\pi^* \mathcal{F})(1 + H + H^2 + \dots),$$

where the last step follows by normalization. Then the projection property gives

$$\pi_* c(\mathcal{Q}) = \pi_* (c(\pi^* \mathcal{F})(1 + H + H^2 + \dots)) = c(\mathcal{F}) \pi_* (1 + H + H^2 + \dots) = c(\mathcal{F})(\pi_* 1 + \pi_* H + \pi_* H^2 + \pi_* H^3 + \dots)$$

Comparing dimensions shows that the left-hand side is a constant, and by the right side, necessarily 1. Dividing both sides by  $c(\mathcal{F})$  gives the result.  $\square$



In some sense this is backwards; [Ful98]’s treatment gives the Segre classes primacy, from which the Chern classes are derived. This is arguably more natural, as Segre classes can be extended to *cones*, while Chern classes are confined to the less-general vector bundles. However, Chern classes have the advantage of the vanishing property (Corollary 3.13). Regardless, the above result allows us to perform “Fulton-style” intersection theory, and is what motivates the rest of the identities in this section. One reason is that by taking blowups of projective bundles, we continue to work in projective bundles. We continue to see that Segre classes play nice with these varieties:

**Corollary 5.12.** *Let  $\varphi : X' \rightarrow X$  be a blowup of a complex projective variety along a complete intersection of  $n$  transversal divisors  $\{U_i\}$  giving an exceptional divisor  $E$ . Then:*

$$\varphi_* \left( \frac{E}{1+E} \right) = \prod_{i=1}^n \frac{U_i}{1+U_i}.$$

*Proof.* By construction (see Example 5.5), restriction to the exceptional locus gives  $\mathbb{P}(\mathcal{N}_X(Z))$ . Then

$$\varphi_* \left( \frac{E}{1+E} \right) = \varphi_* \left( \frac{1}{1+E} \right) \frown [E] = \frac{1}{c(\mathcal{N}_X(Z))} \frown [Z] = \prod_{i=1}^n \frac{U_i}{1+U_i} \in A(X),$$

where we used functoriality, Proposition 5.11, and adjunction. Note that blowups  $\varphi$  are an isomorphism away from the center (by definition), so we take  $\varphi_* 1 = 1$ . Comparing the series expansions gives results for different numbers of transversally intersecting divisors listed in Table 5.13 (in fact, one can prove the parenthesized expressions are the complete homogeneous symmetric polynomials in  $U_i$ ; see [EJK17]).  $\square$

In the same vein, this gives us an expression for the pushforwards  $\varphi_* E^i$ :

	2 divisors	3 divisors
$\varphi_* 1$	1	1
$\varphi_* E$	0	0
$\varphi_* E^2$	$-U_1 U_2$	0
$\varphi_* E^3$	$-U_1 U_2 (U_1 + U_2)$	$U_1 U_2 U_3$
$\varphi_* E^4$	$-U_1 U_2 (U_1^2 + U_1 U_2 + U_2^2)$	$U_1 U_2 U_3 (U_1 + U_2 + U_3)$
$\varphi_* E^5$	$-U_1 U_2 (U_1^3 + U_1^2 U_2 + U_1 U_2^2 + U_2^3)$	$U_1 U_2 U_3 (U_1^2 + U_1 U_2 + U_1 U_3 + U_2^2 + U_2 U_3 + U_3^2)$

**Table 5.13:** The pushforward of the exceptional divisor in terms of the intersecting divisors at the center

At this point we now have the tools to, term by term, push forward powers of  $E$  down through blowups, so that we are only left with  $H$  and  $\pi^* L$  once back in  $A(X_0)$ . For the final projection  $\pi : X_0 \rightarrow B$ , we saw that the pushforwards of  $H^k$  are determined by the Chern class of the originating vector bundle  $\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}$ .

This motivates the next result, which is due to Aluffi but was popularized by Fullwood in his Sethi-Vafa-Witten generalizations [Ful11]:

**Theorem 5.14** (Pushforward from  $X_0$ ). *Let  $\pi : X_0 \rightarrow B$  be the Weierstrass projective bundle with  $H, L$  as in Notation 3.14. For a general formal power series  $f(H) = \sum_{k=0}^{\infty} \pi^* a_k H^k$  with  $a_k \in A(B)$ , we have:*

$$\pi_* f(H) = 3 \left( \frac{f(H) - a_0 - a_1 H}{H^2} \right) \Big|_{H=-3L} - 2 \left( \frac{f(H) - a_0 - a_1 H}{H^2} \right) \Big|_{H=-2L}.$$

*Proof.* We have  $\pi : \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \rightarrow B$ . Using Proposition 5.11 then Corollary 3.9, we get

$$\pi_* \left( \frac{1}{1-H} \right) = \frac{1}{c(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)} = \frac{1}{(1)(1+2L)(1+3L)} = \frac{3}{1+3L} - \frac{2}{1+2L},$$

Recalling that  $\dim X_0 = \dim B + 2$ , we compare series terms of the same dimension to get

$$\pi_* H^0 = \pi_* H^1 = 0, \quad \pi_* H^{2+k} = 3(-3L)^k - 2(-2L)^k \text{ for } k \geq 0.$$

In general, by the projection property:

$$\pi_* f(H) = \sum_{k=0}^{\infty} \pi_*(\pi^* a_k H^k) = \sum_{k=0}^{\infty} a_{2+k} \pi_* H^{2+k} = 3 \sum_{k=0}^{\infty} a_{k+2} (-3L)^k - 2 \sum_{k=0}^{\infty} a_{k+2} (-2L)^k,$$

which gives the result.  $\square$

For reference:

$\pi_* 1$	$\pi_* H$	$\pi_* H^2$	$\pi_* H^3$	$\pi_* H^4$	$\pi_* H^5$	$\pi_* H^6$	$\pi_* H^7$	$\dots$
0	0	1	$-5L$	$19L^2$	$-65L^3$	$211L^4$	$-665L^5$	$\dots$

We now deduce a classic (and surprisingly recent) result due to Aluffi and Esole [AE09]:

**Theorem 5.15** ( $\chi_{\text{top}}$  for smooth Weierstrass). *If  $\varphi : \mathcal{E} \rightarrow B$  is a smooth Weierstrass model, then*

$$\chi_t(\mathcal{E}) = \frac{12Lt^2}{1+6Lt} c_t(\mathcal{T}_B),$$

is the generating function for  $\chi_{\text{top}}(\mathcal{E})$  in the dimension  $n = \dim \mathcal{E} > 0$ , where  $c_t$  denotes the **Chern polynomial**  $c_t = 1 + c_1 t + c_2 t^2 + \dots$ .

*Proof.* We write  $\varphi : \mathcal{E} \xrightarrow{!} X_0 \xrightarrow{\pi} B$  where  $X_0 = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$  is the Weierstrass total space. To solve for  $c(\mathcal{T}_{\mathcal{E}}) \frown [\mathcal{E}]$ , we consider the normal exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{E}} \rightarrow i^* \mathcal{T}_{X_0} \rightarrow \mathcal{N}_{\mathcal{E}/X_0} \rightarrow 0.$$

Recall that we solved for  $c(\mathcal{T}_{X_0})$  in Example 3.15 and  $i_* c(\mathcal{N}_{\mathcal{E}/X_0})$  in Example 3.19. Using those results with the Whitney sum property, then applying projection and identifying with the Chow ring (Theorem 3.4):

$$i_*(c(\mathcal{T}_{\mathcal{E}}) \frown [\mathcal{E}]) = \frac{i_* c(i^* \mathcal{T}_{X_0})}{i_* c(\mathcal{N}_{\mathcal{E}/X_0})} \frown [\mathcal{E}] = \left( \frac{(1+H)(1+H+2\pi^*L)(1+H+3\pi^*L)c(\pi_* \mathcal{T}_B)}{1+3H+6\pi^*L} \right) (3H+6\pi^*L) \in A(X_0).$$

Factoring out  $c(\pi^* \mathcal{T}_B)$ , we can view the remaining expression as a formal power series  $f(H)$  with coefficients in  $\pi^* A(B)$ :

$$i_*(c(\mathcal{T}_{\mathcal{E}}) \frown [\mathcal{E}]) = f(H) c(\pi^* \mathcal{T}_B), \quad f(H) = \frac{(1+H)(1+H+2\pi^*L)(1+H+3\pi^*L)(3H+6\pi^*L)}{1+3H+6\pi^*L}.$$

Using the result of Theorem 5.14, one gets

$$\pi_* i_*(c(\mathcal{T}_{\mathcal{E}}) \frown [\mathcal{E}]) = \frac{12L}{1+6L} c(\mathcal{T}_B).$$

By Theorem 3.48, we have

$$\chi_{\text{top}}(\mathcal{E}) = \int_{\mathcal{E}} c(\mathcal{T}_{\mathcal{E}}) \frown [\mathcal{E}] = \int_B \pi_* \iota_* (c(\mathcal{T}_{\mathcal{E}}) \frown [\mathcal{E}]) = \int_B \frac{12L}{1+6L} c(\mathcal{T}_B),$$

so that for  $\dim \mathcal{E} = n$ , we take degree of the  $\dim B = (n-1)$ -th Chern class (the 0-cycle). Passing to Chern polynomials and writing  $c_t(\mathcal{T}_B) = 1 + c_1 t + c_2 t^2 + \dots$  gives the generating function

$$\chi_t(\mathcal{E}) = \frac{12Lt^2}{1+6Lt} c_t(\mathcal{T}_B),$$

where we introduce an extra factor  $t$  since  $\dim \mathcal{E} = \dim B + 1$ . □

Here is our table of Euler characteristics of  $\mathcal{E}$  in terms of the Chern class  $c(\mathcal{T}_B) = 1 + c_1 + c_2 + \dots$

$n = \dim \mathcal{E}$	$\chi_{\text{top}}(\mathcal{E})$
2	$12L$
3	$12c_1 L - 72L^2$
4	$12c_2 L - 72c_1 L^2 + 432L^3$
5	$12c_3 L - 72c_2 L^2 + 432c_1 L^3 - 2592L^4$
6	$12c_4 L - 72c_3 L^2 + 432c_2 L^3 - 2592c_1 L^4 + 15552L^5$

This is the computation we wish we could perform for every Weierstrass model; however, our computation (and in fact, the notion of Euler characteristic itself) is only valid in the smooth case! This is why we need to *resolve* our singular elliptic fibrations. The final key will be the following identity, noted in [Alu10] as a special case of a classic formula for the Chern classes of a blowup [Ful98, §15], which itself follows from Riemann-Roch. This case will suffice for our work:

**Proposition 5.16.** *Let  $\pi : X' \rightarrow X$  be a blowup along a complete intersection of  $n$  transversal divisors  $\{U_i\}$  giving an exceptional divisor  $E$ . Then:*

$$c(\mathcal{T}_{X'}) = \frac{(1+E) \prod_{i=1}^n (1 + \pi^* U_i - E)}{\prod_{i=1}^n (1 + \pi^* U_i)} \pi^* c(\mathcal{T}_X).$$

As a consequence, we get an easy criterion to check that our blowups are crepant morphisms, and thus will compose to give a crepant resolution:

**Proposition 5.17.** *Let  $\pi : X' \rightarrow X$  be a blowup along a complete intersection of  $n$  divisors  $\{U_i\}$  with exceptional divisor  $E$ . Let  $\mathcal{W}' \subseteq X'$  be the proper transform of  $\mathcal{W} \subseteq X$ . Then  $\pi$  is crepant if and only if*

$$e = n - 1, \quad (\text{where } e \cdot E \text{ removed to give the proper transform } \mathcal{W}')$$

*Proof.* From the identities on pushforwards along blowups (Corollary 5.12, Proposition 5.16), we get the following additive relation on the first Chern classes:

$$c_1(\mathcal{T}_{X'}) = \pi^* c_1(\mathcal{T}_X) + \left( E + \sum_{i=1}^n (\pi^* U_i - E) \right) - \left( \sum_{i=1}^n \pi^* U_i \right) = \pi^* c_1(\mathcal{T}_X) - (n-1)E.$$

We can use this to justify that each blowup is crepant with respect to each  $\mathcal{T}_{\mathcal{E}}$ . One can pass by adjunction to the tangent bundles of  $\mathcal{W}$  and  $\mathcal{W}'$ :

$$\begin{aligned} c_1(\mathcal{T}_{\mathcal{W}'}) &= c_1(\mathcal{T}_X) - [\mathcal{W}'] = (\varphi^* c_1(\mathcal{T}_X) - (n-1)E) - (\varphi^*[\mathcal{W}] - eE) \\ &= \varphi^*(\mathcal{T}_{\mathcal{W}}) - (n-1-e)E, \end{aligned}$$

where  $e$  indicates the number of copies of the exceptional divisor  $E$  we had to remove to give the proper transform  $\mathcal{W}'$ . Hence, if our partial resolution is given by a blowup along a complete intersection of  $n$  divisors, it is crepant exactly when  $e = n - 1$ .  $\square$

In fact, the above results suffice to compute the Euler characteristics to arbitrary degree, from which one could impute the generating function. This was the author's original approach. However, the contemporaneous work of [EJK17] presented "ultimate" analytic formula that we used to verify our identities without passing to series:

**Theorem 5.18** (Analytic pushforwards for blowup). *Let  $\pi : X' \rightarrow X$  be a blowup along a smooth complete intersection  $Z$  of  $n$  smooth transversal divisors  $\{U_i\}$  giving an exceptional divisor  $E$ . If  $Q(t)$  is a formal power series*

$$Q(t) = \sum_a \pi^* Q_a t^a, \quad Q_a \in A(X),$$

then

$$\pi_* Q(E) = \sum_{i=1}^d Q(U_i) M_i, \quad M_i = \prod_{j \neq i} \frac{U_j}{U_j - U_i}.$$

The procedure implicit in this section is generalized to other singular fibrations of projective hypersurfaces within Section 9.1, and so we refer the reader there for the general construction. More practically, refer to our constructions of each resolution in Section 6, Section 7, etc. for a flavor of this strategy in action. We implemented the procedure in code on a symbolic mathematical system (*Mathematica*) to derive the closed-forms listed in Table 1.2, which are "general" due to Batyrev's result in Theorem 5.8.

## 6 The case $\mathbb{Z}/2\mathbb{Z}$

In Example 4.9 we derived a general Weierstrass parameterization for any elliptic fibration with  $\mathbb{Z}/2\mathbb{Z}$  torsion. This was

$$\mathscr{W}_0 : y^2 z = x(x^2 + a_2 x z + a_4 z^2).$$

Passing to normal form  $y^2 z = x^3 + f x z^2 + g z^3$  we computed:

$$f = -\frac{1}{3}(a_2^2 - 3a_4), \quad g = \frac{a_2}{27}(2a_2^2 - 9a_4), \quad \Delta = -a_4^2(a_2^2 - 4a_4).$$

Hence, by applying Tate's algorithm (Theorem 2.23) to the irreducible components  $a_4 = 0$ ,  $a_2^2 - 4a_4 = 0$  in  $\Delta$ , we get:

$\Delta$ -component	$v_C(f)$	$v_C(g)$	$v_C(\Delta)$	Kodaira type	Diagram
$a_4 = 0$	0	0	2	$I_2$	$\bar{A}_1$
$a_2^2 - 4a_4 = 0$	0	0	1	$I_1$	-

Rearranging and working in  $z \neq 0$  gives:

$$\mathscr{W}_0 : y^2 - x(x^2 + a_2 x + a_4) = 0 \quad \text{in} \quad X_0 : [x, y, 1]$$

which (by taking partials) has  $\text{Sing}(\mathscr{W}_0) = \mathscr{W}_0 \cap \{x = y = a_4 = 0\}$ . However, Tate's algorithm only shows the outcome after resolution; in  $\mathscr{W}_0$  we only see Figure 6.1.

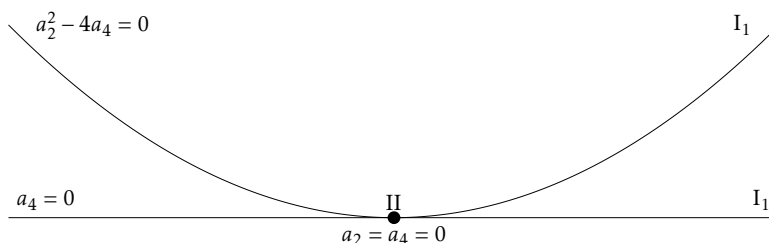


Figure 6.1: Discriminant locus in (unresolved)  $\mathscr{W}_0$ .

### 6.1 Blowup along $(x, y \mid e_1)$

In the notation of Example 5.5, we blow up  $X_0$  along  $\langle x, y \mid e_1 \rangle$  to get  $\pi_1 : X_1 \rightarrow X_0$ . Then  $\mathscr{W}_0$  has the proper transform

$$\mathscr{W}_1 : e_1 y_1^2 - x_1(e_1^2 x_1^2 + a_2 e_1 x_1 + a_4) = 0 \quad \text{in} \quad X_1 : [e_1 x_1, e_1 y_1, 1], [x_1, y_1],$$

where 1 copy of the exceptional divisor  $e_1 = 0$  was removed, making  $\pi_1$  a crepant blowup as our center was the intersection of 2 hyperplanes (Proposition 5.17). One checks that  $\mathscr{W}_1$  is smooth and is thus a crepant resolution. We now study the singular fibers:

- **Codimension 1 (Kodaira):**

- $I_2^{\text{ns}}$  **fiber over**  $a_4 = 0$ : Working in  $\mathscr{W}_1$  and taking  $a_4 = 0$ , we get

$$e_1(y_1^2 - e_1x_1^3 - a_2x_1^2) = 0.$$

This gives two irreducible components:

$$\begin{cases} C_0 : a_4 = y_1^2 - e_1x_1^3 - a_2x_1^2 = 0, \\ C_1 : a_4 = e_1 = 0 \end{cases} \implies C_0 \cap C_1 : a_4 = e_1 = y_1^2 - a_2x_1^2 = 0,$$

which is a variety of degree 2. This confirms that we have an  $I_2$  fiber over  $a_4 = 0$ . Furthermore, this fiber is non-split unless  $a_2$  is a perfect square, which occurs generically only when  $\dim B = 1$  (to give  $(y_1 - \sqrt{a_2}x_1)(y_1 + \sqrt{a_2}x_1) = 0$ ).

- $I_1$  **fiber over**  $a_2^2 - 4a_4 = 0$ : Working in  $\mathscr{W}_1$  and taking  $a_2^2 - 4a_4 = 0$ , we can write  $a_4 = \frac{1}{4}a_2^2$ , which allows the quadratic to factor and gives the single irreducible component

$$C'_0 : a_2^2 - 4a_4 = e_1y_1^2 - x_1 \left( e_1x_1 - \frac{a_2}{2} \right)^2 = 0,$$

which corresponds to the nodal fiber on the original variety  $\mathscr{W}_0$ .

- **Codimension 2:**

- **III fiber over**  $a_2 = a_4 = 0$ : Our two  $\Delta$  components intersect when  $a_2^2 - 4a_4 = a_4 = 0$ , i.e.,  $a_2 = a_4 = 0$ . Viewing this as a degeneration of the  $I_2^{\text{ns}}$  fiber, we get the components:

$$\begin{cases} D_0 : a_2 = a_4 = y_1^2 - e_1x_1^3 = 0, \\ D_1 : a_2 = a_4 = e_1 = 0. \end{cases} \implies D_0 \cap D_1 : a_2 = a_4 = e_1 = y_1^2 = 0,$$

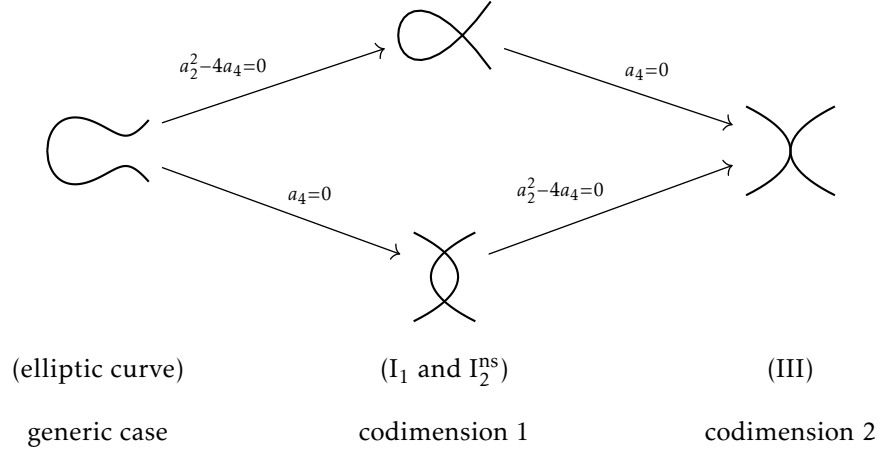
which is a double point intersection  $2([0, 0, 1], [x_1, 0])$  of two rational curves; one is a line and the other is the normalization of a cusp. This is a fiber of type III.

## 6.2 Degeneration of fibers

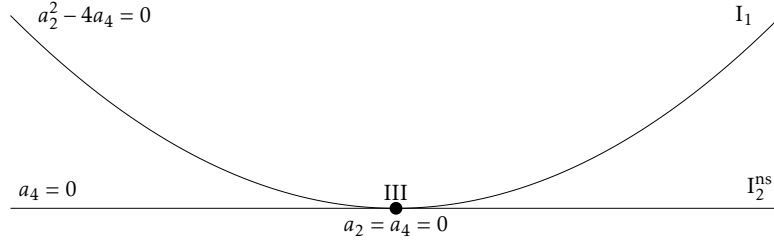
Coming from  $a_4 = 0$ , this corresponds to the  $I_2 \rightarrow \text{III}$  situation where the two intersection points coincide (as two copies of  $y_1 = 0$ ). Coming from  $a_2^2 - 4a_4 = 0$ , this is a blown up version of the original degeneration  $I_1 \rightarrow \text{II}$  where the nodal singularity coincided with the third root to give a cusp, which was a singular point in  $\mathscr{W}_0$ . Upon blow up, this gave the above III fiber. In summary:

$$C'_0 \mapsto D_0 + D_1, \quad \begin{cases} C_0 \mapsto D_0 \\ C_1 \mapsto D_1 \end{cases}$$

The network of degenerations is shown in Figure 6.2; they are shown over the base in Figure 6.3.



**Figure 6.2:** Fiber degenerations in (resolved)  $\mathscr{W}_1$ .



**Figure 6.3:** Discriminant locus in (resolved)  $\mathscr{W}_1$ .

### 6.3 Pushforward of the Chern class

We performed one blowup  $\pi_1$ , whose center was the complete intersection of two transversal divisors with classes

$$U_1 = [x = 0] = H + 2\pi^*L, \quad U_2 = [y = 0] = H + 3\pi^*L$$

in  $A(X_0)$ . By Aluffi's formula, we get

$$c(\mathcal{T}_{X_1}) = \frac{(1 + E_1)(1 + \pi_1^*(H + 2\pi^*L) - E_1)(1 + \pi_1^*(H + 3\pi^*L) - E_1)}{(1 + \pi_1^*(H + 2\pi^*L))(1 + \pi_1^*(H + 3\pi^*L))} \pi_1^*c(\mathcal{T}_{X_0}).$$

To solve for  $c(\mathcal{T}_{\mathscr{W}_1})$ , we consider the normal exact sequence on  $\iota: \mathscr{W}_1 \subseteq X_1$ :

$$0 \rightarrow \mathcal{T}_{\mathscr{W}_1} \rightarrow \iota^*\mathcal{T}_{X_1} \rightarrow \mathcal{N}_{\mathscr{W}_1/X_1} \rightarrow 0,$$

to which we can apply the Whitney sum property. By normalization, and where we also subtract  $E$  to account for the removed exceptional divisor (being a proper transform), we get:

$$\iota_*c(\mathcal{N}_{\mathscr{W}_1/X_1}) = (1 + 3\pi_1^*H + 6\pi_1^*\pi^*L - E)$$

Altogether,

$$\begin{aligned} \iota_* c(\mathcal{T}_{\mathcal{W}_1} \frown [\mathcal{W}_1]) &= \frac{\iota_* c(t^* \mathcal{T}_{X_1})}{\iota_* c(\mathcal{N}_{\mathcal{W}_1/X_1})} \frown [\mathcal{W}_1] \\ &= \frac{(1 + E_1)(1 + \pi_1^*(H + 2\pi^*L) - E_1)(1 + \pi_1^*(H + 3\pi^*L) - E_1) \cdot \pi_1^* c(\mathcal{T}_{X_0})}{(1 + \pi_1^*(H + 2\pi^*L))(1 + \pi_1^*(H + 3\pi^*L)) \cdot (1 + 3\pi_1^*H + 6\pi_1^*\pi^*L - E)} \\ &\quad \cdot (3\pi_1^*H + 6\pi_1^*\pi^*L - E) \in A(X_1). \end{aligned}$$

Since we have expressions for  $U_1 = H + 2\pi^*L$  and  $U_2 = H + 3\pi^*L$ , we can solve for the corresponding  $M_1$  and  $M_2$  (Theorem 5.18) for  $\pi_1$ , and then apply our procedure (see Theorem 5.15, Theorem 9.3) for solving the generating function (this corresponds to one application of the analytic blowup formula and then the hyperplane pushforward formula) in the manner of Theorem 5.15, to get:

$$\frac{12Lt^2}{1 + 4Lt} c_t(\mathcal{T}_B)$$

as the generating function  $\chi_t(\mathcal{W}_1)$ . This was done in *Mathematica*. We extract the coefficients for various degrees of  $t$  to get the explicit dimension cases of the Euler characteristics listed in Table 1.2 for  $\mathbb{Z}/2\mathbb{Z}$ .



## 7 The case $\mathbb{Z}/3\mathbb{Z}$

In Example 4.8 we derived a general Weierstrass parameterization for any elliptic fibration with  $\mathbb{Z}/3\mathbb{Z}$  torsion. This was

$$\mathscr{W}_0 : y^2z + a_1xyz + a_3yz^2 - x^3 = 0.$$

Passing to normal form  $y^2z = x^3 + fxz^2 + gz^3$  we computed:

$$f = -\frac{a_1}{48}(a_1^3 - 24a_3), \quad g = \frac{1}{864}(a_1^6 - 36a_1^3a_3 + 216a_3^2), \quad \Delta = 4f^3 + 27g^2 = -\frac{a_3^3}{16}(a_1^3 - 27a_3).$$

Hence, by applying Tate's algorithm (Theorem 2.23) to the irreducible components  $a_3 = 0$ ,  $a_1^3 - 27a_3 = 0$  in  $\Delta$ , we get:

$\Delta$ -component	$v_C(f)$	$v_C(g)$	$v_C(\Delta)$	Kodaira type	Diagram
$a_3 = 0$	0	0	3	$I_3$	$\bar{A}_2$
$a_1^3 - 27a_3 = 0$	0	0	1	$I_1$	-

Rearranging and working in  $z \neq 0$  gives:

$$\mathscr{W}_0 : y^2 + a_1xy + a_3y - x^3 = 0 \quad \text{in} \quad X_0 : [x, y, 1]$$

which (by taking partials) has  $\text{Sing}(\mathscr{W}_0) = \mathscr{W}_0 \cap \{x = y = a_3 = 0\}$ . However, Tate's algorithm only shows the outcome after resolution; in  $\mathscr{W}_0$  we only see Figure 7.1.

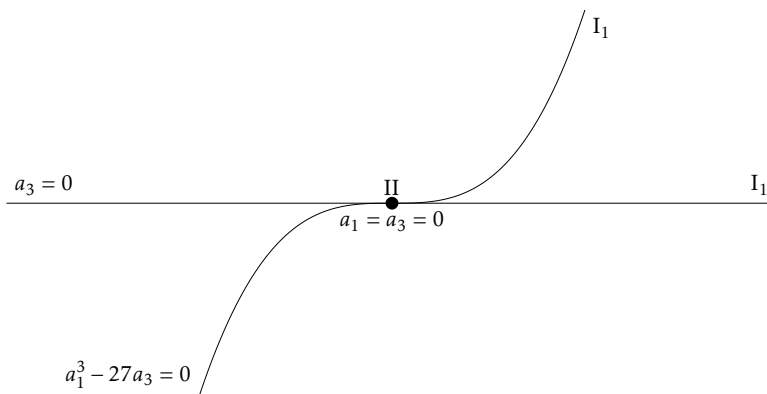


Figure 7.1: Discriminant locus in (unresolved)  $\mathscr{W}_0$ .

### 7.1 Blowup along $(x, y, a_3 \mid e_1)$

In the notation of Example 5.5, we blow up  $X_0$  along  $\langle x, y, a_3 \mid e_1 \rangle$  to get  $\pi_1 : X_1 \rightarrow X_0$ . Then  $\mathscr{W}_0$  has the proper transform

$$\mathscr{W}_1 : y_1^2 + a_1x_1y_1 + a_{31}y_1 - e_1x_1^3 = 0 \quad \text{in} \quad X_1 : [e_1x_1, e_1y_1, 1], [x_1, y_1, a_{31}],$$

where 2 copies of the exceptional divisor  $e_1 = 0$  were removed, making  $\pi_1$  a crepant blowup as our center was the intersection of 3 hyperplanes (Proposition 5.17). One checks that  $\mathscr{W}_1$  is smooth and is thus a crepant resolution. We now study the singular fibers:

• **Codimension 1 (Kodaira):**

- $I_3^s$  **fiber over**  $a_3 = 0$ : Working in  $\mathscr{W}_1$  and taking  $a_3 = 0$  is equivalent to taking either  $a_{31} = 0$  and  $e_1 = 0$ . Substitution shows these give three irreducible components:

$$\begin{cases} C_0 : a_{31} = y_1^2 + a_1 x_1 y_1 - e_1 x_1^3 = 0, \\ C_{1a} : e_1 = y_1 = 0, \\ C_{1b} : e_1 = y_1 + a_1 x_1 + a_{31} = 0, \end{cases} \implies \begin{cases} C_0 \cap C_{1a} : a_{31} = e_1 = y_1 = 0, & [0, 0, 1], [1, 0, 0] \\ C_0 \cap C_{1b} : a_{31} = e_1 = y_1 + a_1 x_1 = 0, & [0, 0, 1], [1, -a_1, 0] \\ C_{1a} \cap C_{1b} : e_1 = y_1 = a_1 x_1 + a_{31}, & [0, 0, 1], [1, 0, -a_1] \end{cases}$$

which are three distinct points generically (away from  $a_1 = 0$ ). This confirms that we have a split  $I_3$  fiber over  $a_3 = 0$ .

- $I_1$  **fiber over**  $a_1^3 - 27a_3 = 0$ : Working in  $\mathscr{W}_1$  and taking  $a_1^3 - 27a_3 = 0$ , we can write  $a_3 = e_1 a_{31} = \frac{1}{27} a_1^3$ . We know *a priori* by Tate's algorithm that this remains an irreducible rational curve  $C'_0$  away from  $a_1 = a_3 = 0$ , so we will not write things out here. If one wished to convince themselves that this is still the nodal fiber on the original variety  $\mathscr{W}_0$ , complete the square as in Example 4.8 with  $y_1 \mapsto y_1 - \frac{1}{2}(a_1 x_1 + \frac{1}{27} a_1^3 z)$  and deduce that the resulting cubic in  $x_1$  gives a node away from  $a_1 = 0$ .

• **Codimension 2:**

- **IV fiber over**  $a_1 = a_3 = 0$ : Our two  $\Delta$  components intersect when  $a_1^3 - 27a_3 = a_3 = 0$ , i.e.,  $a_1 = e_1 a_{31} = 0$ . Viewing this as a degeneration of the  $I_3$  fiber, we get the components:

$$\begin{cases} D_0 : a_1 = a_{31} = y_1^2 - e_1 x_1^3 = 0, \\ D_{1a} : a_1 = e_1 = y_1 = 0, \\ D_{1b} : a_1 = e_1 = y_1 + a_{31} = 0, \end{cases} \implies \begin{cases} D_0 \cap D_{1a} : a_1 = a_{31} = e_1 = y_1 = 0, & [0, 0, 1], [1, 0, 0] \\ D_0 \cap D_{1b} : a_1 = a_{31} = e_1 = y_1 = 0, & [0, 0, 1], [1, 0, 0] \\ D_{1a} \cap D_{1b} : a_1 = e_1 = y_1 = a_{31} = 0, & [0, 0, 1], [1, 0, 0] \end{cases}$$

This shows our curves are still distinct, but the additional restriction  $a_1 = 0$  makes the three individual intersection points coincide. Two are lines and one is the normalization of a cusp (and is thus rational), so we have a fiber of type IV.

## 7.2 Degeneration of fibers

Coming from  $a_3 = 0$ , this corresponds to the  $I_3 \rightarrow IV$  situation where the three intersection points coincide (as  $3([0, 0, 1], [1, 0, 0])$ ). Coming from  $a_1^3 - 27a_3 = 0$ , this is a blown up version of the degeneration  $I_1 \rightarrow II$  where the nodal singularity coincides with the third root to give a cusp, which was a singular point in  $\mathscr{W}_0$ . Upon blow up, this gives the above IV fiber. In summary:

$$C'_0 \mapsto D_0 + D_{1a} + D_{1b}, \quad \begin{cases} C_0 \mapsto D_0 \\ C_{1a} \mapsto D_{1a} \\ C_{1b} \mapsto D_{1b} \end{cases}$$

The network of degenerations is shown in Figure 7.2; they are shown over the base in Figure 7.3.

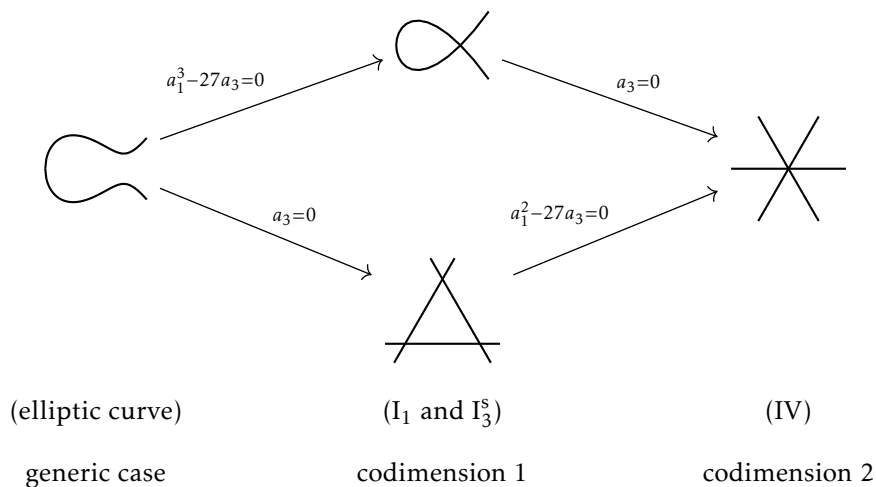


Figure 7.2: Fiber degenerations in (resolved)  $\mathcal{W}_1$ .

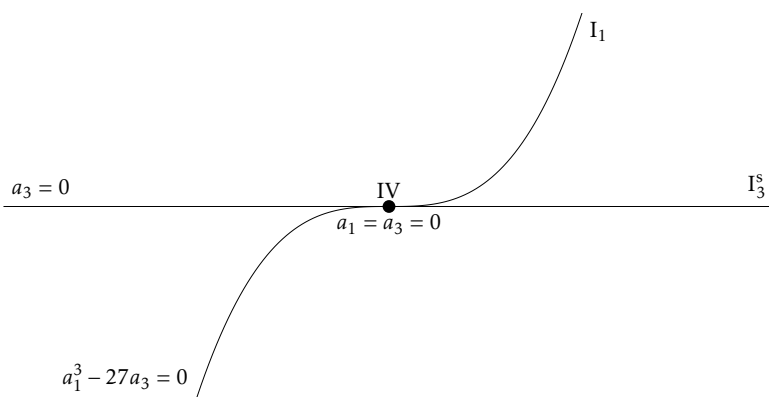


Figure 7.3: Discriminant locus in (resolved)  $\mathcal{W}_1$ .

### 7.3 Pushforward of the Chern class

We performed one blowup  $\pi_1$ , whose center was the complete intersection of three transversal divisors with classes

$$U_1 = [x = 0] = H + 2\pi^*L, \quad U_2 = [y = 0] = H + 3\pi^*L, \quad U_3 = [a_3 = 0] = 3\pi^*L$$

in  $A(X_0)$ . By Aluffi's formula, we get

$$c(\mathcal{T}_{X_1}) = \frac{(1 + E_1)(1 + \pi_1^*(H + 2\pi^*L) - E_1)(1 + \pi_1^*(H + 3\pi^*L) - E_1)(1 + 3\pi_1^*\pi^*L - E_1)}{(1 + \pi_1^*(H + 2\pi^*L))(1 + \pi_1^*(H + 3\pi^*L))(1 + 3\pi_1^*\pi^*L)} \pi_1^*c(\mathcal{T}_{X_0}).$$

To solve for  $c(\mathcal{T}_{\mathcal{W}_1})$ , we consider the normal exact sequence on  $\iota: \mathcal{W}_1 \subseteq X_1$ :

$$0 \rightarrow \mathcal{T}_{\mathcal{W}_1} \rightarrow \iota^* \mathcal{T}_{X_1} \rightarrow \mathcal{N}_{\mathcal{W}_1/X_1} \rightarrow 0,$$

to which we can apply the Whitney sum property. By normalization, and where we also subtract  $2E$  to account for the removed exceptional divisors (being a proper transform), we get:

$$\iota_* c(\mathcal{N}_{\mathcal{W}_1/X_1}) = (1 + 3\pi_1^* H + 6\pi_1^* \pi^* L - 2E)$$

Altogether,

$$\begin{aligned} \iota_* c(\mathcal{T}_{\mathcal{W}_1} \frown [\mathcal{W}_1]) &= \frac{\iota_* c(\iota^* \mathcal{T}_{X_1})}{\iota_* c(\mathcal{N}_{\mathcal{W}_1/X_1})} \frown [\mathcal{W}_1] \\ &= \frac{(1 + E_1)(1 + \pi_1^*(H + 2\pi^* L) - E_1)(1 + \pi_1^*(H + 3\pi^* L) - E_1)(1 + 3\pi_1^* \pi^* L - E_1) \cdot \pi_1^* c(\mathcal{T}_{X_0})}{(1 + \pi_1^*(H + 2\pi^* L))(1 + \pi_1^*(H + 3\pi^* L))(1 + 3\pi_1^* \pi^* L) \cdot (1 + 3\pi_1^* H + 6\pi_1^* \pi^* L - 2E)} \\ &\quad \cdot (3\pi_1^* H + 6\pi_1^* \pi^* L - 2E) \in A(X_1). \end{aligned}$$

Since we have expressions for  $U_1, U_2, U_3$ , we can solve for the corresponding  $M_1, M_2, M_3$  (Theorem 5.18) for  $\pi_1$ , and then apply our procedure (see Theorem 5.15, Theorem 9.3) for solving the generating function (this corresponds to one application of the analytic blowup formula and then the hyperplane pushforward formula) in the manner of Theorem 5.15, to get:

$$\frac{12Lt^2}{1 + 3Lt} c_t(\mathcal{T}_B)$$

as the generating function for  $\chi_t(\mathcal{W}_1)$ . This was done in *Mathematica*. We evaluate the coefficients for various degrees of  $t$  to get the explicit dimension cases of the Euler characteristics listed in Table 1.2 for  $\mathbb{Z}/3\mathbb{Z}$ .

## 8 The case $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

In Example 4.7 we derived a general Weierstrass parameterization for any elliptic fibration with  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  torsion. This was

$$\mathscr{W}_0 : y^2 z = x(x - \beta_2 z)(x - \gamma_2 z).$$

Passing to normal form  $y^2 z = x^3 + f x z^2 + g z^3$  we computed:

$$f = \frac{1}{3}(\beta_2 \gamma_2 - \beta_2^2 - \gamma_2^2), \quad g = \frac{1}{27}(2\beta_2 - \gamma_2)(2\gamma_2 - \beta_2)(\beta_2 + \gamma_2), \quad \Delta = 4f^3 + 27g^2 = -\beta_2^2 \gamma_2^2 (\beta_2 - \gamma_2)^2.$$

Hence, by applying Tate's algorithm (Theorem 2.23) to the irreducible components  $\beta_2 = 0$ ,  $\gamma_2 = 0$ , and  $\beta_2 - \gamma_2 = 0$  in  $\Delta$ , we get:

$\Delta$ -component	$v_C(f)$	$v_C(g)$	$v_C(\Delta)$	Kodaira type	Diagram
$\beta_2 = 0$	0	0	2	$I_2$	$\tilde{A}_1$
$\gamma_2 = 0$	0	0	2	$I_2$	$\tilde{A}_1$
$\beta_2 - \gamma_2 = 0$	0	0	2	$I_2$	$\tilde{A}_1$

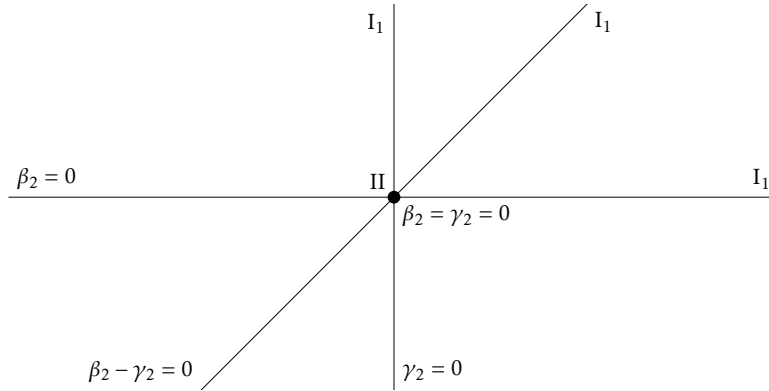
Rearranging and working in  $z \neq 0$  gives:

$$\mathscr{W}_0 : y^2 - x(x - \beta_2)(x - \gamma_2) = 0 \quad \text{in} \quad X_0 : [x, y, 1]$$

which (by taking partials) has

$$\text{Sing}(\mathscr{W}_0) = \mathscr{W}_0 \cap (\{x = x - \beta_2 = y = 0\} \cup \{x = x - \gamma_2 = y = 0\} \cup \{x - \beta_2 = x - \gamma_2 = y = 0\})$$

However, Tate's algorithm only shows the outcome after resolution; in  $\mathscr{W}_0$  we only see Figure 8.1.



**Figure 8.1:** Discriminant locus in (unresolved)  $\mathscr{W}_0$ .

### 8.1 Blowing up in patches

For our analysis of this torsion, we take the convention of writing our model as

$$y^2 = xuv, \quad (u = x - \beta_2, v = x - \gamma_2).$$

This emphasizes the inherent symmetry we see in this fibration (and visually in Figure 8.1). This also reveals why  $y = x = u = 0$ ,  $y = x = v = 0$ , and  $y = u = v = 0$  are the loci of singular points.

We do this since we already saw from  $\text{Sing}(\mathcal{W}_0)$  that we might need blowups with respect to (intersections of) hypersurfaces such as  $x - \beta_2$ . This makes testing for smoothness less straightforward, as we would get multiple defining equations (and thus may require the Jacobian). Instead, we give a more intuitive argument to justify that the sequence of blowups with centers

$$X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0$$

will fully resolve our singular model.

Let us blow up along  $\langle x, y \mid e_1 \rangle$ . Then  $x = e_1 x_1$  and  $y = e_1 y_1$ . Working in patches means either taking  $x_1 \neq 0$  or taking  $y_1 \neq 0$  on the added  $\mathbb{P}^1$  component. Doing the former lets us take  $e_1 = \frac{x}{x_1}$  and hence  $y = (\frac{x}{x_1})y_1$ . For those familiar, this is often accompanied by the following abuse of notation:

$$(x, y) \rightarrow \begin{cases} (\frac{x}{x_1}, \frac{x}{x_1}y_1) & \text{on } x_1 \neq 0 \\ (\frac{y}{y_1}x_1, y) & \text{on } y_1 \neq 0 \end{cases} \quad \text{“=”} \quad \begin{cases} (x, xy) & \text{on } x \neq 0 \\ (xy, y) & \text{on } y \neq 0 \end{cases}$$

Recall we have  $y^2 = xuv$ . We proceed as follows:

- We blow up along  $(x, y)$  and work in  $y \neq 0$ . Writing  $x \rightarrow xy$  and taking the proper transform gives  $y = uv$ , which is smooth.
- Thus our singularity is in  $x \neq 0$ . Write  $y \rightarrow xy$  and taking the proper transform gives  $xy^2 = uv$ , which is singular. We start from this form (i.e., we restrict ourselves to this patch moving forwards).
- We blow up along  $(y, u, v)$  and work in  $y \neq 0$ . Writing  $u \rightarrow uy$ ,  $v \rightarrow vy$  and taking the proper transform gives  $x = uv$ , which is smooth.
- Any remaining singularity must further be in  $u \neq 0$  (or  $w \neq 0$ , by symmetry). Writing  $y \rightarrow yu$ ,  $v \rightarrow uv$  and taking the proper transform gives  $xy^2 = v$ , which is smooth.

Thus, our two blowups lead to smoothness in all patches, and we are done.

*Remark.* Due to the simplicity of this local view of blowups, one might wonder why we do not use this method for our computations here (where we instead view blowups as intersections within projective bundles). There are two issues:

- Studying intersections of curves locally is hard, as one must check adjacent patches to verify if two curves that do not intersect in a patch do not intersect at any point globally. The number of combinations can explode if one had to do three or more blowups.
- When we take the centers of blowups for our pushforward formula, this is a global expression (along with the Euler characteristic itself). It may be difficult to propagate from a local expression for a divisor to the global one we need for our formulas.

## 8.2 First blowup along $(x, y \mid e_1)$

In the notation of Example 5.5, we blow up  $X_0$  along  $\langle x, y \mid e_1 \rangle$  to get  $\pi_1 : X_1 \rightarrow X_0$ . Then  $\mathscr{W}_0$  has the proper transform

$$\mathscr{W}_1 : e_1 y_1^2 - x_1(e_1 x_1 - \beta_2)(e_1 x_1 - \gamma_2) = 0 \quad \text{in} \quad X_1 : [e_1 x_1, e_1 y_1, 1], [x_1, y_1],$$

where 1 copy of the exceptional divisor  $e_1 = 0$  was removed, making  $\pi_1$  a crepant blowup as our center was the intersection of 2 hyperplanes (Proposition 5.17). By our preliminary work, we know  $\mathscr{W}_1$  is not yet smooth. We now study the singular fibers:

- **Codimension 1 (Kodaira):**

- $I_2^{\text{ns}}$  **fiber over  $\beta_2 = 0$  (or  $\gamma_2 = 0$ ):** It suffices to consider  $\beta_2 = 0$  by symmetry. Working in  $\mathscr{W}_1$  over the locus  $\beta_2 = 0$ , our fiber looks like

$$e_1(y_1^2 - x_1^2(e_1 x_1 - \gamma_2)) = 0,$$

so we have two irreducible components:

$$\begin{cases} C_0 : \beta_2 = y_1^2 - x_1^2(e_1 x_1 - \gamma_2) = 0, \\ C_1 : \beta_2 = e_1 = 0 \end{cases} \quad \implies \quad C_0 \cap C_1 : \beta_2 = e_1 = y_1^2 - \gamma_2 x_1^2 = 0, \quad [0, 0, 1], [x_1, y_1]$$

where  $y_1^2 - \gamma_2 x_1^2 = 0$  is a variety of degree 2. We conclude that our fiber is a type  $I_2$  that is non-split, unless  $\gamma_2$  is a perfect square. This also suggests a degeneration at the codimension-2 locus  $\beta_2 = \gamma_2 = 0$ .

(To see why  $C_0$  is a rational curve, working in the patch  $x_1 \neq 0$  to get  $C_0 : \left(\frac{y_1}{x_1}\right)^2 - (e_1 x_1 - \gamma_2) = 0$ . Taking  $t = \frac{y_1}{x_1}$ , then  $e_1 x_1 = t^2 + \gamma_2$  and  $e_1 y_1 = e_1 x_1 \cdot \frac{y_1}{x_1} = t(t^2 + \gamma_2)$ . Hence  $C_1$ 's coordinates are parametrizable as  $[t^2 + \gamma_2, t(t^2 + \gamma_2), 1], [1, t]$ .)

- **Unresolved  $I_1$  fiber over  $\beta_2 - \gamma_2 = 0$ :** Working in  $\mathscr{W}_1$  and taking  $\beta_2 - \gamma_2 = 0$  gives the fiber

$$C'_0 : e_1 y_1^2 - x_1(e_1 x_1 - \beta_2)^2 = 0.$$

To see why this is a rational curve, first note that  $e_1 = 0$  gives  $\beta_2^2 x_1 = 0$ . When away from the codimension-2 locus  $\beta_2 = \gamma_2 = 0$ , this implies  $x_1 = 0$ , which means in the exceptional divisor  $e_1 = 0$  we still only get the one point  $[0, 0, 1], [0, 1]$  (intuitively, we are away from the blowup center, where we took  $x = 0$ ). Hence, our fiber is isomorphic to its image in  $\mathscr{W}_0$  (the  $I_1$  nodal curve  $y^2 - x(x - \beta_2)^2 = 0$ ) when away from  $\beta_2 = \gamma_2 = 0$ . It is unresolved as it contains the singular  $e_1 x_1 - \beta_2 = e_1 x_1 - \gamma_2 = y = 0$ .

- **Codimension 2:**

- **Unresolved III fiber over  $\beta_2 = \gamma_2 = 0$ :** Our three  $\Delta$  components intersect when  $\beta_2 = \gamma_2 = 0$ . Viewing this as a degeneration of the  $I_2^{\text{ns}}$  fiber from  $\beta_2 = 0$ , we get the components:

$$\begin{cases} D_0 : \beta_2 = \gamma_2 = y_1^2 - e_1 x_1^3 = 0, \\ D_1 : \beta_2 = \gamma_2 = e_1 = 0, \end{cases} \quad \implies \quad D_0 \cap D_1 : \beta_2 = e_1 = y_1^2 = 0$$

The intersection  $D_0 \cap D_1$  is now a double point  $2([0, 0, 1], [x_1, 0])$ . Note that as before with  $t = \frac{y_1}{x_1}$ , for  $D_0$  we get the rational parameterization  $[t^2, t^3, 1], [1, t]$ . We conclude that our fiber thus far is a III. It is unresolved since it contains the point  $x - \beta_2 = x - \gamma_2 = y = 0$  (albeit with  $\beta_2 = \gamma_2 = 0$ ).

Figure 8.2 depicts the “intermediate” network of degenerations. Observe that it is identical (minus either one of the the  $I_2^{\text{ns}}$   $\Delta$ -components) to the final network for  $\mathbb{Z}/2\mathbb{Z}$ -torsion (Figure 6.2)! This is certainly related to  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -torsion being a specialization of  $\mathbb{Z}/2\mathbb{Z}$ -torsion, as  $a_4, a_2^2 - 4a_4$  can be thought of as a transform from  $\beta_2, \beta_2 - \gamma_2$ .

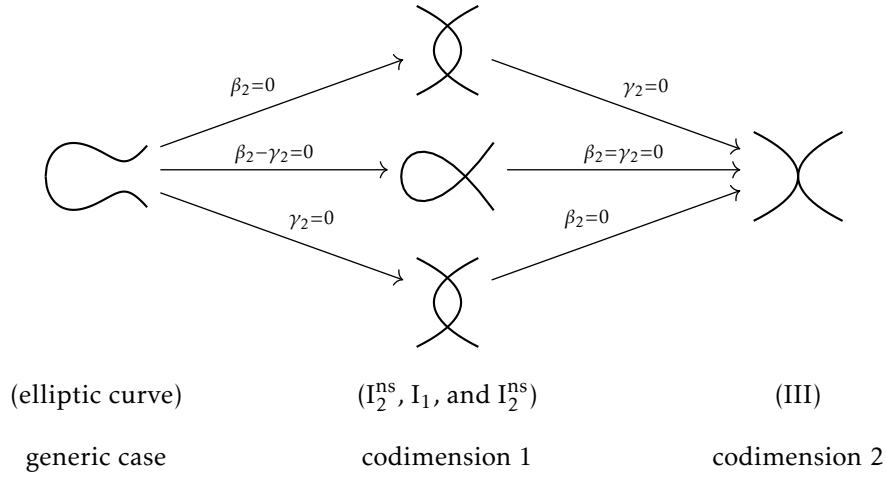


Figure 8.2: Fiber degenerations in (unresolved)  $\mathscr{W}_1$ .

### 8.3 Second blowup along $(y_1, u, v \mid e_2)$

In the notation of Example 5.5 and our preliminary work, we blow up  $X_1$  along  $\langle y_1, u, v \mid e_2 \rangle$  to get  $\pi_2 : X_2 \rightarrow X_1$ . Then  $\mathscr{W}_1$  has the proper transform

$$\mathscr{W}_2 : e_1 y_2^2 - x_1 u_2 v_2 = 0, \quad (e_2 u_2 = e_1 x_1 - \beta_2, \quad e_2 v_2 = e_1 x_1 - \gamma_2) \quad \text{in} \quad X_2 : [e_1 x_1, e_2 e_1 y_2, 1], [x_1, e_2 y_2], [y_2, u_2, v_2]$$

where 2 copies of the exceptional divisor  $e_2 = 0$  were removed, making  $\pi_2$  a crepant blowup as our center was the intersection of 3 hypersurfaces (Proposition 5.17). By our preliminary work, we know  $\mathscr{W}_2$  is smooth, and since via the composition  $\pi_1 \circ \pi_2$  it is crepant birational to  $\mathscr{W}_0$ , we have constructed a crepant resolution. We now study the singular fibers:

• **Codimension 1 (Kodaira):**

- $I_2^{\text{ns}}$  **fiber over**  $\beta_2 = 0$  (or  $\gamma_2 = 0$ ): It suffices to consider  $\beta_2 = 0$  by symmetry. Since our blowup was away from this locus, the curves from the first blowup will remain the same away from  $\gamma_2 = 0$ . The fiber is

$$e_1 y_2^2 - x_1 u_2 v_2 = 0, \quad (\text{with } e_2 u_2 = e_1 x_1 = e_2 v_2 + \gamma_2).$$

In particular, note that  $e_2 = 0$  implies  $\gamma_2 = 0$ , so we can assume  $e_2 \neq 0$  generically. For when we



degenerate we write:

$$\begin{cases} C_0^\beta : \beta_2 = y_1^2 - x_1^2(e_1 x_1 - \gamma_2) = 0, & [e_1 x_1, e_1 y_1, 1], [x_1, y_1], \left[ \frac{y_1}{e_2}, \frac{e_1 x_1}{e_2}, \frac{e_1 x_1 - \gamma_2}{e_2} \right] \\ C_1^\beta : \beta_2 = e_1 = 0, & [0, 0, 1], [x_1, y_1], \left[ \frac{y_1}{e_2}, 0, -\frac{\gamma_2}{e_2} \right]. \end{cases}$$

$$\implies C_0^\beta \cap C_1^\beta : \beta_2 = e_1 = y_1^2 - \gamma_2 x_1^2 = 0, \quad [0, 0, 1], [x_1, y_1], [1, 0, 0],$$

where  $y_1^2 - \gamma_2 x_1^2 = 0$  is a variety of degree 2 that is non-split as before. One can retrieve the rational parameterization  $[t^2 + \gamma_2, t(t^2 + \gamma_2), 1], [1, t], [t(t^2 + \gamma_2), e_1(t^2 + \gamma_2), e_1 t^2]$  as before.

–  $\mathbb{I}_2^{\text{ns}}$  **fiber over**  $\beta_2 - \gamma_2 = 0$ : Working in  $\mathscr{W}_2$  over the locus  $\beta_2 - \gamma_2 = 0$ , our variety becomes

$$e_1 y_2^2 - x_1 u_2 v_2 = 0, \quad (\text{with } e_2 u_2 = e_1 x_1 - \beta_2 = e_2 v_2).$$

Observe that the  $\beta_2 = \gamma_2$  condition is exactly expressed by the condition  $u = v$ , that is,  $e_2(u_2 - v_2) = 0$ . Thus, we have a net decomposition of two curves:

$$\begin{aligned} C_0^{\beta-\gamma} : u_2 - v_2 = e_1 y_2^2 - x_1 u_2^2 = 0, & \quad [e_1 x_1, e_2 e_1 y_2, 1], [x_1, e_2 y_2], [y_2, u_2, u_2] \\ C_1^{\beta-\gamma} : e_2 = e_1 y_2^2 - x_1 u_2 v_2 = 0, & \quad [\beta_2, 0, 1], [1, 0], [y_2, u_2, v_2]. \end{aligned}$$

To see why  $C_1^{\beta-\gamma}$  is a rational curve, note that the first set of coordinates are  $[\beta_2, 0, 1]$  since  $e_2 = 0$ , which implies  $e_1 x_1 - \beta_2 = 0$ . The second set of coordinates are  $[1, 0]$  because  $y_1 = e_2 y_2 = 0$ . Finally, the equation  $e_1 y_2^2 - x_1 u_2 v_2 = 0$  is a conic on the last set of coordinates, corresponding to a  $\mathbb{P}^2$  ( $[y_2, u_2, v_2]$ ). This means it is isomorphic to  $\mathbb{P}^1$ . Making this explicit, note that we have  $x_1 \neq 0$  and we can write

$$y_2 = st, \quad u_2 = \frac{e_1}{x_1} s^2, \quad v_2 = t^2,$$

which describes an isomorphism between  $C_1^{\beta-\gamma}$  and a  $\mathbb{P}^1$  with coordinates  $[s, t]$  (locally, the map looks like, e.g.,  $[y_2, u_2, v_2] \mapsto [y_2, v_2]$  on  $v_2 \neq 0$  to  $t \neq 0$ ).

To see why  $C_0^{\beta-\gamma}$  is a rational curve, note that away from  $e_2 = 0$ , our curve must be isomorphic to the  $I_1$  component we saw in  $\mathscr{W}_1$ , suggesting that it is irreducible and parametrizable as a (normalized) nodal curve. For an explicit parametrization, we can for example work in the patch  $u_2 \neq 0$  to get  $x_1 = \frac{e_1 y_2^2}{u_2^2}$ . Then letting  $t = \frac{y_2}{u_2}$ , we have

$$x_1 = e_1 t^2, \quad y_1 = e_2 y_2 = \frac{u y_2}{u_2} = t(e_1 x_1 - \beta_2) = t(e_1^2 t^2 - \beta_2)$$

which gives the parameterization  $[e_1^2 t^2, e_1 t(e_1^2 t^2 - \beta_2), 1], [e_1 t^2, t(e_1^2 t^2 - \beta_2)], [t, 1, 1]$ . The intersection  $C_0^{\beta-\gamma} \cap C_1^{\beta-\gamma}$  is given by

$$C_0^{\beta-\gamma} \cap C_1^{\beta-\gamma} : e_2 = u_2 - v_2 = e_1 y_2^2 - x_1 u_2^2 = 0, \quad [\beta_2, 0, 1], [1, 0], [y_2, u_2, u_2],$$

where  $e_1 y_2^2 - x_1 u_2^2 = 0$  is a variety of degree 2. We conclude that our fiber (away from the  $\beta_2 = \gamma_2 = 0$  locus) is a type  $I_2$  that is non-split.

• **Codimension 2:**

–  $I_0^*$  (**blown-down**  $I_0^*$ ) **over**  $\beta_2 = \gamma_2 = 0$ : Viewing this as a degeneration of the  $I_2^{\text{ns}}$  fiber from  $\beta_2 - \gamma_2 = 0$ , we examine how  $C_0^{\beta-\gamma}$  and  $C_1^{\beta-\gamma}$  each degenerate:

\* Under the condition  $u_2 - v_2 = 0$ , the change from the codimension-1 case can be seen by working in the patch  $y_2 \neq 0$  as before. In fact, suppose  $y_2 = 0$ . Then our equation gives  $x_1 u_2^2 = 0$ . But  $x_1 \neq 0$  since  $[x_1, e_2 y_2]$  are coordinates. However,  $u_2 \neq 0$  as well, since  $[y_2, u_2, u_2]$  are coordinates.

In conclusion,  $C_0^{\beta-\gamma}$  is entirely contained in  $y_2 \neq 0$ . The model equation lets us write  $e_1 = \frac{x_1 u_2^2}{y_2^2}$ , and substituting this into the only remaining constraint  $e_1 x_1 = e_2 u_2$ , we see that we get the factorization

$$u_2 \left( \frac{x_1^2 u_2}{y_2^2} - e_2 \right) = 0.$$

Note that  $u_2 = 0 \implies e_1 = 0$ . Hence we get two irreducible components

$$D_0 : \beta_2 = \gamma_2 = u_2 - v_2 = \frac{x_1^2 u_2}{y_2^2} - e_2 = e_1 - \frac{x_1 u_2^2}{y_2^2} = 0 \quad [e_1 x_1, e_2 e_1 y_2, 1], [x_1, e_2 y_2], [y_2, u_2, u_2],$$

$$D_1 : \beta_2 = \gamma_2 = u_2 - v_2 = u_2 = e_1 = 0 \quad [0, 0, 1], [x_1, y_1], [1, 0, 0]$$

with the corresponding degeneration

$$C_0^{\beta-\gamma} \mapsto D_0 + D_1.$$

To see why  $C_0^{\beta,\gamma}$  is a rational curve, first note that  $x_1 = 0 \implies e_2 = 0$ , which is impossible since  $[x_1, e_2 y_2]$  are coordinates. That is, this component is contained in  $x_1 \neq 0$ . Thus, rescaling coordinates such that  $x_1 = 1$ , we can exhibit an explicit parametrization in  $t = \frac{u_2}{y_2}$ . Using the component equations, we get:

$$y_1 = e_2 y_2 = \frac{x_1^2 u_2}{y_2} = x_1^2 t = t, \quad e_1 = \frac{x_1 u_2^2}{y_2^2} = x_1 t^2 = t^2$$

which gives the parametrization  $[t^2, t^3, 1], [1, t], [1, t, t]$ .

\* Under the condition  $e_2 = 0$ , the change from the codimension-1 case is that we now have the equality

$$e_2 u_2 = e_1 x_1 - \beta_2 = e_1 x_1.$$

as  $\beta_2 = 0$ . Here  $e_2 = 0 \implies e_1 x_1 = 0 \implies e_1 = 0$ , where the last implication follows as  $[x_1, e_2 y_2]$  are coordinates. Thus the  $C_1^{\beta-\gamma}$  component becomes  $e_2 = x_1 u_2 v_2 = 0$  with  $x_1 \neq 0$ . Hence we get two irreducible components

$$D_{2u} : \beta_2 = \gamma_2 = e_2 = u_2 = 0 \quad [0, 0, 1], [1, 0], [y_2, 0, v_2]$$

$$D_{2v} : \beta_2 = \gamma_2 = e_2 = v_2 = 0 \quad [0, 0, 1], [1, 0], [y_2, u_2, 0].$$

with the corresponding degeneration

$$C_1^{\beta-\gamma} \mapsto D_{2u} + D_{2v}.$$

To summarize, we have four components:

$$\begin{aligned} D_0 : \beta_2 = \gamma_2 = u_2 - v_2 = e_2 - \frac{x_1^2 u_2}{y_2^2} = 0 & \quad [e_1 x_1, e_2 e_1 y_2, 1], [x_1, e_2 y_2], [y_2, u_2, u_2], \\ D_1 : \beta_2 = \gamma_2 = u_2 = v_2 = e_1 = 0 & \quad [0, 0, 1], [x_1, y_1], [1, 0, 0], \\ D_{2u} : \beta_2 = \gamma_2 = e_2 = u_2 = 0 & \quad [0, 0, 1], [1, 0], [y_2, 0, v_2], \\ D_{2v} : \beta_2 = \gamma_2 = e_2 = v_2 = 0 & \quad [0, 0, 1], [1, 0], [y_2, u_2, 0]. \end{aligned}$$

One description would call  $D_0$  the normalization of the cuspidal curve,  $D_1$  the normalization of the first exceptional component, and  $D_{2u}$  and  $D_{2v}$  are the degeneration of the second exceptional component (the conic  $C_1^{\beta-\gamma}$ ) into lines.

We see that all the curves are rational, and intersect at one point  $[0, 0, 1], [1, 0], [1, 0, 0]$ . Certainly the lines  $D_1, D_{2u}, D_{2v}$  pairwise intersect transversally (with multiplicity one). The intersection of  $D_0$  with each curve is also transverse; the partial with respect to  $u_2$  (equivalently  $v_2$ , since  $u_2 - v_2 = 0$  along the curve) is  $\frac{x_1^2}{y_2^2}$ , where we saw that the curve is contained in  $x_1 \neq 0$ . We called this non-Kodaira fiber a “blown-down”  $I_0^*$  and write  $I_0^{*-}$ ; see Example 4.15 for details.

#### 8.4 Degeneration of fibers

The alternative to approaching the fiber from the  $I_2$  over  $\beta_2 - \gamma_2 = 0$ , is approaching it from the  $I_2$  over  $\beta_2 = 0$ . There, we saw two components

$$\begin{aligned} C_0^\beta : \beta_2 = y_1^2 - x_1^2(e_1 x_1 - \gamma_2) = 0, & \quad [e_1 x_1, e_1 y_1, 1], [x_1, y_1], \left[ \frac{y_1}{e_2}, \frac{e_1 x_1}{e_2}, \frac{e_1 x_1 - \gamma_2}{e_2} \right] \\ C_1^\beta : \beta_2 = e_1 = 0, & \quad [0, 0, 1], [x_1, y_1], \left[ \frac{y_1}{e_2}, 0, -\frac{\gamma_2}{e_2} \right]. \end{aligned}$$

The second curve was determined by  $e_1 = 0$ , and is irreducible away from  $\beta_2 = \gamma_2 = 0$ . Recall that we now have the relation  $e_2 u_2 = e_1 x_1 = e_2 v_2$ , and so this second curve now factorizes as  $e_2 = 0$  and  $u_2 = v_2 = 0$ . We also saw that the case  $e_2 = 0$  gives two further components via the fiber equation:  $e_2 = u_2 = 0$  and  $e_2 = v_2 = 0$ .

To combine this with our earlier result, we see that:

$$\begin{cases} C_0^\beta, C_0^\gamma \mapsto D_0 \\ C_1^\beta, C_1^\gamma \mapsto D_1 + D_{2u} + D_{2v} \end{cases} \quad \begin{cases} C_0^{\beta-\gamma} \mapsto D_0 + D_1 \\ C_1^{\beta-\gamma} \mapsto D_{2u} + D_{2v} \end{cases}$$

The network of degenerations is shown in Figure 8.3; they are shown over the base in Figure 8.4.

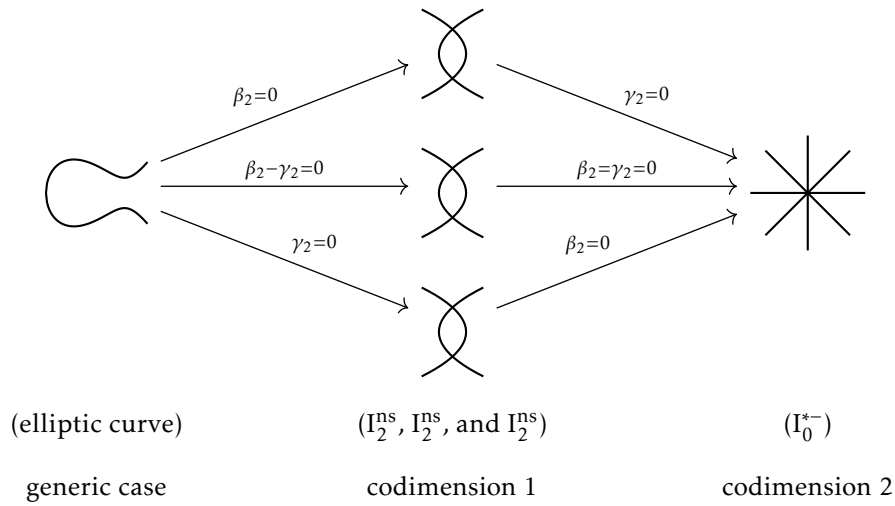


Figure 8.3: Fiber degenerations in (resolved)  $\mathcal{W}_2$ .

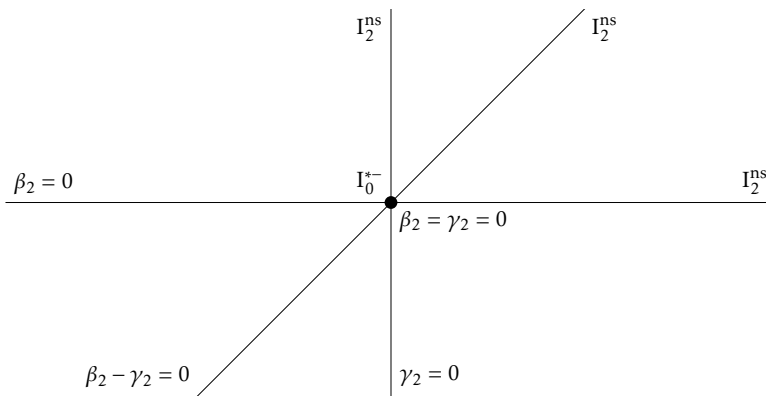


Figure 8.4: Discriminant locus in (resolved)  $\mathcal{W}_0$ .

### 8.5 Pushforward of the Chern class

*Remark.* To simplify notation, we keep the pullback and pushforward maps implicit (e.g.,  $L$  instead of  $\pi_2^* \pi_1^* \pi^* L$ ).

We first performed the blowup  $\pi_1$ , whose center was the complete intersection of three transversal divisors with classes

$$U_1 = [x = 0] = H + 2L, \quad U_2 = [y = 0] = H + 3L,$$

in  $A(X_0)$ . By Aluffi's formula, we get

$$c(\mathcal{T}_{X_1}) = \frac{(1 + E_1)(1 + H + 2L - E_1)(1 + H + 3L - E_1)}{(1 + H + 2L)(1 + H + 3L)} \pi_1^* c(\mathcal{T}_{X_0}).$$

We then performed the blowup  $\pi_2$ , whose center was the intersection of three divisors with classes

$$U_1 = [y_1 = 0] = H + 3L - E_1, \quad U_2 = [u = 0] = H + 2L, \quad U_3 = [v = 0] = H + 2L,$$

in  $A(X_1)$ . (Recall that e.g.,  $u = x - \beta_2 z$ , where we restore the  $z$  to be explicit.) By Aluffi's formula, we get

$$c(\mathcal{T}_{X_2}) = \frac{(1 + E_2)(1 + H + 3L - E_1 - E_2)(1 + H + 2L - E_2)^2}{(1 + H + 3L - E_1)(1 + H + 2L)^2} c(\mathcal{T}_{X_1}).$$

To solve for  $c(\mathcal{T}_{\mathcal{W}_2})$ , we consider the normal exact sequence on  $\iota: \mathcal{W}_2 \subseteq X_2$ :

$$0 \rightarrow \mathcal{T}_{\mathcal{W}_2} \rightarrow \iota^* \mathcal{T}_{X_2} \rightarrow \mathcal{N}_{\mathcal{W}_2/X_2} \rightarrow 0,$$

to which we can apply the Whitney sum property. By normalization, and where we subtract  $2E_2 + E_1$  to account for the removed exceptional divisors (being a proper transform), we get:

$$\iota_* c(\mathcal{N}_{\mathcal{W}_2/X_2}) = (1 + 3H + 6L - E_1 - 2E_2)$$

Altogether,

$$\begin{aligned} \iota_*(c(\mathcal{T}_{\mathcal{W}_2}) \frown [\mathcal{W}_2]) &= \frac{\iota_* c(\iota^* \mathcal{T}_{X_2})}{\iota_* c(\mathcal{N}_{\mathcal{W}_2/X_2})} \frown [\mathcal{W}_2] \\ &= \frac{(1 + E_2)(1 + H + 3L - E_1 - E_2)(1 + H + 2L - E_2)^2 \cdot \pi_2^* c(\mathcal{T}_{X_1})}{(1 + H + 3L - E_1)(1 + H + 2L)^2 \cdot (1 + 3H + 6L - E_1 - 2E_2)} \cdot (3H + 6L - E_1 - 2E_2) \in A(X_2). \end{aligned}$$

Since we have expressions for  $U_1, U_2, U_3$ , we can solve for the corresponding  $M_1, M_2, M_3$  (Theorem 5.18) for  $\pi_2$ , and then likewise again for  $\pi_1$ . Then, we apply our procedure (see Theorem 5.15, Theorem 9.3) for solving the generating function (this corresponds to two applications of the analytic blowup formula and then one of the hyperplane pushforward formula) in the manner of Theorem 5.15, to get:

$$\frac{4t^2(3 + 5Lt)}{(1 + 2Lt)^2} c_t(\mathcal{T}_B)$$

as the generating function  $\chi_t(\mathcal{W}_2)$ . This was done in *Mathematica*. We evaluate the coefficients for various degrees of  $t$  to get the explicit dimension cases of the Euler characteristics listed in Table 1.2 for  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .

## 9 Applications

### 9.1 $\chi_{\text{top}}$ for fibrations of projective hypersurfaces

To disambiguate the Euler characteristic strategy from its application to Weierstrass models (which have extra structure like torsion and Kodaira fibers), we state and prove a general feasibility theorem. It combines the Euler characteristic strategy via pushforwards expounded here and in [EJK17], with the following recent result due to Fullwood and Helmer [FH16] that concretely generalizes Theorem 5.14:

**Theorem 9.1** (Projective bundle pushforward). *Let  $\pi : \mathbb{P}(\mathcal{F}) \rightarrow X$  be a projective bundle over a smooth variety  $X$ . Then  $\pi_* : A(\mathbb{P}(\mathcal{F})) \rightarrow A(X)$  is defined by:*

$$\pi_*(\alpha) = \left( D \cdot \sum_{i=1}^m \frac{g_\alpha(x_i)}{\prod_{l=1, l \neq i}^m (x_i - x_l)} \right) \Big|_{x_1=-L_1, \dots, x_m=-L_m} \quad \text{with} \quad g_\alpha(x_i) = \frac{\alpha(x_i) - (\alpha_0 + \alpha_1 x_i + \dots + \alpha_{\text{rk}(\mathcal{F})-2} x_i^{\text{rk}(\mathcal{F})-2})}{x_i^{\text{rk}(\mathcal{F})-m} \prod_{l=1, l \neq i}^m (x_i - x_l)},$$

and  $D$  is the following differential operator on  $A(X)[x_1, \dots, x_m]$ :

$$D(g) = \frac{1}{k_1! \dots k_m!} \frac{\partial^{k_1 + \dots + k_m}}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} (x_1^{k_1} \dots x_m^{k_m} \cdot g).$$

*Proof sketch.* The proof is not particularly insightful; it generalizes the series manipulations we performed back with Theorem 5.14. There are additional subtleties they address in their proof, however:

- **Indecomposable vector bundles:** By the splitting principle (Corollary 3.12), it suffices to work with Chern roots and assumption of a direct sum of line bundles.
- **Repeated Chern roots:** Computationally, this is the classic situation from partial fraction decomposition where the various powers in the denominator must be taken in sum:

$$\frac{p(x)}{(1+ax)^k} = \sum_{i=0}^k \frac{q_i}{(1+ax)^i} \quad \text{for } \deg p < k.$$

This is addressed by representing Chern roots to be distinct with multiplicity and working accordingly, and some multilinear maneuvers give the result.  $\square$

The important point is that such a formula exists and is symmetric in the distinct Chern roots, and can thus be expressed in terms of  $c_i(\mathcal{F})$  and coefficients of the appropriate series. In particular (though they work with Chern-Schwartz-MacPherson classes of *morphisms* in their notation), they show this implies a nice product form when  $\pi : Y \rightarrow X$  a (possibly singular) fibration of hypersurfaces. We rephrase here:

**Corollary 9.2** (Analytic fibration pushforward). *For*

$$\psi : \mathcal{Y} \xrightarrow{\iota} \mathbb{P}(\mathcal{F}) \xrightarrow{\pi} B$$

*There is a rational expression  $Q_Y(L)$  (see [FH16]) defined in terms of  $d$ ,  $D$ , and the Chern roots  $L_i$  (where  $[\mathcal{Y}] = dH + \pi^*D$ ) such that*

$$\pi_*(c(\mathcal{T}_Y) \frown [Y]) = Q_Y(D, \{L_i\})c(\mathcal{T}_B) \frown [X].$$

This matches our experience with the smooth Weierstrass model giving the expression  $\frac{12}{1+6L}$  (Theorem 5.15). [FH16] demonstrates this effect using fibrations of K3 surfaces, del Pezzo surfaces, and so forth.

**Theorem 9.3** (Feasibility). *We work with complex projective varieties. Let  $\psi : \mathcal{Y} \rightarrow B$  be a possibly singular fibration of projective hypersurfaces, without loss of generality decomposed as:*

$$\psi : \mathcal{Y} \xrightarrow{I} \mathbb{P}(\mathcal{F}) \xrightarrow{\pi} B$$

where  $\mathcal{F}$  a known vector bundle. Suppose there exists a known crepant resolution with blowup centers  $\{I_k\}$  on  $\mathbb{P}(\mathcal{F})$ , where each  $I_k$  is along complete intersections of divisors that are polynomials in the exceptional divisors  $E_i$ , projective coordinates  $H$ , and distinct Chern roots  $L_i$  of  $\mathcal{F}$ . Then one can compute the generating function  $\chi_t(\mathcal{Y}')$  for the Euler characteristic of all crepant resolutions of  $\mathcal{Y}$ .

*Proof.* Here is a procedure that can be implemented in any symbolic mathematical program:

- By hypothesis, we are given a crepant resolution induced via  $\pi_1 \circ \dots \circ \pi_m : X_m \rightarrow X_0$ , such that by Aluffi's formula (Proposition 5.16), we can write  $c(\mathcal{T}_{X_m})$  in terms of  $E_i$ ,  $H$ , and  $L_i$  and  $c(\mathcal{T}_{X_{m-1}})$ . We also get  $\iota_{m*}c(\mathcal{N}_m)$  in terms of  $E_i$ ,  $H$ , and  $L_i$ .
- By Esole, Jefferson, and Kang (Theorem 5.18), we have  $A(X_m) \rightarrow A(X_0)$  as a pushforward on rational expressions (this eliminates the  $E_i$ ).
- By Fullwood and Helmer (Corollary 9.2), we have  $A(X_0) \rightarrow A(B)$  as a pushforward on rational expressions (this eliminates the  $H$ ).
- We then have a rational expression in terms of the distinct Chern roots  $L_i$ , and in  $c(\mathcal{T}_B)$ . Passing to Chern polynomials and applying Poincaré-Hopf as in Theorem 5.15 gives the generating function  $\chi_{\text{top}}(\mathcal{Y}_m)$ .
- By the result of Batyrev (Theorem 5.8), this generating function is valid for any crepant resolution of  $\mathcal{Y}$ .

These results are applicable under the hypotheses of the theorem statement.  $\square$

*Remark.* If one drops the desire for a rational expression (and a ready-made generating function), one might be able to use Theorem 9.1 in better generality.

## 9.2 Models in F-theory

In F-theory, elliptic fibrations are used to *geometrically engineer* desired gauge groups. This connection was succinctly expressed in [EJK17], but is of interest to all string theorists; see [ESY15], [MMTW14], [KLR98], etc.:

**Definition 9.4.** A *G-model* is an elliptic fibration  $\varphi : \mathcal{E} \rightarrow B$  where  $G$  is the associated Lie group given by

$$G(\varphi) = \frac{\exp(\mathfrak{g}^\vee)}{\text{MW}_{\text{tor}}(\varphi)} \times \text{U}(1)^{\text{rk MW}(\varphi)},$$

where  $\mathfrak{g}^\vee$  is the Langlands dual of  $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ , where  $\mathfrak{g}_i$  is the Lie algebra such that  $\tilde{\mathfrak{g}}_i^f$  is the dual graph of the  $i$ -th Kodaira codimension-1 fiber.

Thanks to Tate's algorithm, Weierstrass models are particularly amenable for constructing a desired  $G$ -model. In particular, we observe that the Weierstrass models we have studied are examples of  $G$ -models:

**Example 9.5.** Let  $\psi : \mathcal{W} \rightarrow B$  be our  $\mathbb{Z}/2\mathbb{Z}$ -model. By construction, we have:

$$MW_{\text{tor}}(\psi) \cong \mathbb{Z}/2\mathbb{Z}, \quad \text{rk} MW(\varphi) = 0.$$

Using Tate’s algorithm, we saw that  $\psi$ ’s generic singular fibers were  $I_2$  and  $I_1$ . These are associated to  $\tilde{A}_1$  and  $\tilde{A}_0$  respectively. [EJK17] notes that the dual graph of  $\tilde{\mathfrak{g}}_i^t$  after removing the extra node gives the dual graph of the Langlands dual of  $\mathfrak{g}$ . We avoid dual graphs by noting that  $A_k$  is simply laced, and so we get the regular projective  $A_1$  diagram, which is the Dynkin diagram for  $\mathfrak{su}(2)$ . We conclude that  $\exp(\mathfrak{g}^\vee) \cong \text{SU}(2)$ , and all together we get

$$G(\psi) = \frac{\text{SU}(2)}{\mathbb{Z}/2\mathbb{Z}} \times \{0\} \cong \text{SO}(3).$$

*Remark.* In particular, a similar argument shows that one requires  $\mathbb{Z}/2\mathbb{Z}$ -torsion in their elliptic fibration to construct any  $\text{SO}(n)$ -model (since it would arise from its double cover  $\text{SU}(n)$ , which has Dynkin diagram  $A_{n-1}$ ). This shows that the study of torsional models in F-theory is not only relevant, but unavoidable.

Proceeding similarly, the cases  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ ,  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  we have examined are  $G$ -models where  $G$  is  $\text{SO}(3)$ ,  $\text{PSU}(3)$ , and  $\text{Spin}(4)$  respectively, as we have listed in Table 1.3.

*Remark.* To further suggest connections between singular fibers, their degenerations, and physical interpretations in F-theory, we reproduce (in relative ignorance) the following correspondence noted in [EY13], in the hope that it piques the interest of an appropriate reader. We have only glimpsed the first row:

Singular locus	Physical interpretation
codimension 1	gauge group
codimension 2	matter fields
codimension 3	Yukawa couplings

We hope that those who are better informed can utilize the explicit fiber component degenerations in Section 6 to Section 8 to give physical interpretation to our models!

### 9.3 Hodge numbers in the Calabi-Yau case

For our purposes, we define the Calabi-Yau condition as follows:

**Definition 9.6.** A **Calabi-Yau variety**  $X$  is a smooth, compact, projective  $n$ -dimensional complex variety such that the canonical bundle is trivial ( $\omega_X \cong \mathcal{O}_X$ ) and  $H^p(X, \mathcal{O}_X) = 0$  for  $1 \leq p \leq n - 1$ .

Observe that birational Calabi-Yau varieties are crepant birational by the Calabi-Yau condition. In this situation, one can use the following specialization of Batyrev’s result due to Kontsevich:

**Theorem 9.7.** *If  $X, Y$  are birational (smooth, complex) Calabi-Yau varieties, then  $X, Y$  have the same Hodge numbers.*

The original proof was first presented at Kontsevich’s “String cohomology” talk (Orsay, 1995) but never published; however, stronger results have appeared hence, also using  $p$ -adic and motivic integration (e.g., [Ito03] gives a proof valid for all smooth minimal models). The Hodge numbers of a Calabi-Yau variety are relatively restricted:



**Proposition 9.8.** *If  $X$  is a Calabi-Yau variety, we have*

$$\begin{aligned} h^{0,0}(X) &= h^{n,0}(X) = h^{0,n}(X) = h^{n,n} = 1, \\ h^{p,0}(X) &= h^{0,p}(X) = h^{p,n}(X) = h^{n,p}(X) = 0 \quad (1 \leq p \leq n-1). \end{aligned}$$

*Proof.* We use Dolbeault's theorem (Theorem 3.43) and the definition of Calabi-Yau to get

$$h^{n,0}(X) = h^0(X, \omega_X) = h^0(X, \mathcal{O}_X) = h^{0,0}(X) = h^0(X) = 1,$$

being a connected component. We also have  $h^{p,0}(X) = h^p(X, \mathcal{O}_X) = 0$  for  $1 \leq p \leq n-1$  by hypothesis. Hodge symmetries (Proposition 3.34) give the rest of the equalities.  $\square$

**Proposition 9.9.** *A Calabi-Yau threefold's Hodge numbers are determined by  $h^{1,1}$  and  $h^{2,1}$ , which are related by  $\chi_{\text{top}} = 2(h^{1,1} - h^{2,1})$*

*Proof.* Applying the Calabi-Yau constraints of Proposition 9.8, we conclude that the Hodge diamond in the 3-dimensional case looks like:

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 0 & & 0 \\ & & & 0 & h^{1,1} & & 0 \\ 1 & & h^{2,1} & & h^{2,1} & & 1. \\ & & 0 & & h^{1,1} & & 0 \\ & & 0 & & 0 & & \\ & & & & 1 & & \end{array}$$

By Hodge decomposition and the definition of  $\chi_{\text{top}}$ :

$$\chi_{\text{top}} = h^0 - h^1 + h^2 - h^3 + h^4 - h^5 + h^6 = (1) - (0) + h^{1,1} - (2 + 2h^{2,1}) + (h^{1,1}) - (0) + (1),$$

which simplifies to the result.  $\square$

In particular, we are interested in when our resolved torsional Weierstrass models  $\psi : \mathcal{W} \rightarrow B$  are Calabi-Yau *elliptic* fibrations. It turns out this condition is also easily expressed in terms of the Chern class of the base:

**Proposition 9.10.** *If a smooth Weierstrass model  $\psi : \mathcal{W} \rightarrow B$  is Calabi-Yau, then  $c_1(\mathcal{T}_B) = L$ .*

*Proof.* We recall adjunction as applied to the Weierstrass model; in particular, by the tensor product property of Chern classes, we get:

$$c_1(\mathcal{T}_{\mathcal{W}}) = H + (H + 2\pi^*L) + (H + 3\pi^*L) + \pi^*c_1(\mathcal{T}_B) - (3H + 6\pi^*L) = \pi^*(c_1(\mathcal{T}_B) - L).$$

Since we require  $c_1(\mathcal{T}_{\mathcal{W}}) = 0$ , we must have  $c_1(\mathcal{T}_B) = L$ .  $\square$

Hence, substituting in  $c_1(\mathcal{T}_B) = L$  gives  $\chi_{\text{top}}$  in this special case; we have summarized the results for threefolds and fourfolds in Table 1.3. In the case of threefolds, our computation of the topological Euler characteristic  $\chi_{\text{top}}(\mathcal{W})$  completes the relationship between  $h^{1,1}(\mathcal{W})$  and  $h^{2,1}(\mathcal{W})$ . Finally, we can write  $h^{1,1}(\mathcal{W})$  in terms of  $h^{1,1}(B)$  using Hodge theory and Shioda-Tate-Wazir to get:

**Theorem 9.11.** *The Hodge numbers of a smooth, torsion-only, Calabi-Yau Weierstrass elliptic threefold  $\psi: \mathcal{W} \rightarrow B$  are:*

$$\begin{aligned} h^{1,1}(\mathcal{W}) &= 11 + f - \deg K_B^2 \\ h^{2,1}(\mathcal{W}) &= h^{1,1}(\mathcal{W}) - \frac{1}{2} \chi_{\text{top}}(\mathcal{W}) \end{aligned}$$

where  $f$  is the number of geometrically irreducible fibral divisors away from the zero section.

*Proof.* By hypothesis, we have  $\text{rk MW}(\psi) = 0$ . By the Lefschetz (1,1)-theorem (Theorem 3.36), we also have  $\text{rk NS}(X) = h^{1,1}(X)$  for  $X = B, \mathcal{E}$ . Hence Shioda-Tate-Wazir (Theorem 4.6) specializes to:

$$h^{1,1}(\mathcal{W}) = h^{1,1}(B) + f + 1.$$

Furthermore, the constraints on the Hodge diamond imply  $h^{0,1}(B) = h^{0,2}(B) = 0$ ,  $h^{0,0}(B) = 1$ . This gives

$$\begin{array}{ccccc} & & 1 & & \\ & & 0 & & 0 \\ & 0 & & h^{1,1} & & 0, \\ & & 0 & & 0 \\ & & & & 1 \end{array}$$

which gives  $\chi_{\text{top}}(B) = 1 - 0 + h^{1,1} - 0 + 1 = 2 + h^{1,1}$ . We then take Noether's formula (Corollary 3.54) on the base  $B$  to get:

$$1 - 0 + 0 = \chi(B, \mathcal{O}_B) = \frac{1}{12} (\deg K_B^2 + \chi_{\text{top}}(B)) = \frac{1}{12} (\deg K_B^2 + 2 + h^{1,1}(B)),$$

which simplifies to  $h^{1,1}(B) = 10 - \deg K_B^2$ . Substitution into Shioda-Tate-Wazir and into  $\chi_{\text{top}}(\mathcal{W})$  as given in Proposition 9.9 gives the result.  $\square$

We summarized the computational results in Table 1.3. We observe that our results match [EJK17] where they coincide, i.e., the Hodge number formulas (in some form), and their equation for the  $\mathbb{Z}/2\mathbb{Z}$ -model (their  $\text{SO}(3)$ -model).

## 9.4 Other connections and future work

The theory of elliptic fibrations, especially as it relates to string theory, is experiencing a flurry of exciting activity. Here are natural extensions of this project one could take:

- **Torsion sections:** How do the groups of our model's torsion elements act with respect to the fibers, in both the Kodaira and non-Kodaira cases? This is a straightforward exercise I did not have time to complete.
- **Enumerations of flop networks:** There is an important hypothesis/correspondence that relates the network of non-isomorphic crepant resolutions (and their relationships via flops) to Weyl chambers, physics, and representation theory. See [ESY15] for an introduction. In particular, what are the networks of crepant resolutions for our models? (This is related to the minimal model program. Toric varieties also seem to be involved.)

- **Other torsion groups:** Other than time and patience, nothing prevents other individuals (save, perhaps, a failure to find crepant resolutions) from performing the same analysis of other torsion groups (e.g.,  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ ). In particular, catalogs of parameterizations do exist (see the table of [Kub76], or [AM98] for a view towards gauge groups), and we have shown that parameterization seems to fully encode the effect of torsion (at least for Betti numbers). One still has to inspect fiber degenerations, however.
- **Fibers of crepant resolutions of Weierstrass models:** The work of [Cat13] shows one might still get classification results by imposing a model and restricted birational class. Is Cattaneo’s result “sufficient”; e.g., are all contracted Kodaira fibers realizable over codimension 2 under these constraints? In general: can we find more fibers, and/or disqualify others as Kodaira, Miranda, and Szydło did?
- **Hodge number computations:** In the case of Calabi-Yau elliptic fourfolds, one has four undetermined Hodge numbers  $h^{1,1}$ ,  $h^{2,1}$ ,  $h^{3,1}$ ,  $h^{2,2}$ . The topological Euler characteristic gives one relation. Are there Shioda-Tate-Wazir variants or generalizations available to give more? Can the Lefschetz hyperplane theorem be invoked? What about the more fine-grained arithmetic genera  $\chi_i$ ?
- **Fixed generating functions:** Since the generating function is fixed across the crepant birational class, this suggests that if one does not deduce the correct Euler characteristic generating function as seen from another crepant resolution, one can disqualify a sequence of crepant blowups as non-resolvent. How does one interpret this?
- **Code:** We could generalize our Mathematica code that symbolically computes generating functions of  $\chi_{\text{top}}$  for Weierstrass models to arbitrary hypersurfaces in arbitrary projective bundles, as per Theorem 9.3.
- **Generalizations:** For “safety” reasons, we still ultimately ended up with results on complex projective varieties. One could carefully relax the assumptions made in this thesis to maximize the generality of our results (quasiprojective varieties? algebraic schemes?). One could also vary  $\text{char}(K)$  and algebraic closure. Perhaps there are generalizations to arithmetic schemes, where families of elliptic curves occur naturally?

As the reader can see: I find the topic extremely fascinating, being a threefold intersection of my interests in mathematics, physics, and computation. I am grateful to my advisors, my peers, the math department, and Harvard, for giving me the support and education I needed to produce this independent, capstone work during my senior year.  $\square$

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