Set-theoretic Geology, the Ultimate Inner Model, and New Axioms

Justin William Henry Cavitt

(860) 949-5686

cavittjustin@gmail.com

Advisor: W. Hugh Woodin

Harvard University

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1 Introduction

The Zermelo-Fraenkel axioms for set theory with the Axiom of Choice (ZFC) are central to mathematics.[1] Set theory is foundational in that all mathematical objects can be modeled as sets, and all theorems and proofs trace back to the principles of set theory. For much of mathematics, the ZFC axioms suffice.

However, the independence phenomenon complicates this picture. There are mathematical statements that can neither be proven nor refuted on the basis of ZFC alone. Among these are examples that arose naturally from mathematical inquiry, such as the Continuum Hypothesis (CH), the assertion that all sets of reals

[1] See the Appendix for a list of axioms.

are either in bijection with the natural numbers or with the set of all reals. The independence of these statements presents us with several questions. Do such sentences have truth values? If so, what are they?

To make progress on these questions, we embark on the project of finding new axioms. If we can successfully justify the extension of ZFC by new axioms that resolve the problem posed by independence, then the answer to the first question is positive, and we will be able to find the answers to the second. If the project fails, then one has a case that independent questions have no answers.

In this thesis, we will discuss a current proposal for a new axiom that, under certain conjectures, would wipe away the independence problem. This axiom candidate, $V =$ Ultimate L, asserts that the universe of sets is equal to what is hoped to become a well-understood "sub-universe," named Ultimate L. We will present some of the main ideas in the search for new axioms that lead us to consider Ultimate L, and discuss the central ideas around the axiom. We will then add to the case for the axiom by considering some recent results in set-theoretic geology due to Toshimichi Usuba that parallel Ultimate L. Throughout, we will not only present the mathematical facts, but also the philosophical ideas that inspire and are inspired by them.

In Chapter 2 we will develop in more detail the independence problem. In Chapter 3, we will develop some of the mathematical ideas that lead to new axioms. In Chapter 4 we will sketch the basic facts about the new axiom candidate, and discuss some of the philosophical issues surrounding it. In Chapter 5 we will prove in detail Usuba's main result, the Strong Downward Directed Grounds Hypothesis, and its philosophically interesting consequences.

1.1 Author's Note

This thesis is meant to bring to the fore not only mathematical ideas, but also philosophical ones. We hope to give the reader a sense of the story that leads one to investigate Ultimate L (Chapter 4), or to emphasize what were originally just corollaries at the end of a paper (Section 5.3). While the earlier chapters philosophically build up to the later ones, they also do so mathematically. We will spend much of the earlier chapters laying down the necessary groundwork in order to approach the later chapters, giving definitions and at least stating, if not proving, results. We hope to make accessible the results of the final chapter to a reader with only a basic knowledge of set theory and mathematical logic, both in terms of learning the results and understanding why they matter.

We will presume a basic familiarity with set theory, including topics such as the ordinals, the class of sets, and basic constructions of mathematical objects within set theory. We will also presume general mathematical background, and at least a little familiarity with standard structures such as partial orders. We will sketch some of these ideas in the Appendix.

1.2 Acknowledgements

First, I would like to thank my advisor, Professor Hugh Woodin, for all of his help as I worked through this material. I would also like to thank Professor Warren Goldfarb for sparking my interest in mathematical logic and Professor Peter Koellner for introducing me to set theory. More generally, I am grateful to all the teachers I have had at Harvard, who have helped me to learn and thrive these past years. Finally, I would like to thank all of my friends for their continued support as I worked on this project.

2 The Independence Problem

In this chapter, we will further discuss the independence problem. We will discuss, in particular, the ideas of *consistency strength*, *natural independence*, and *forcing*.

2.1 Gödelian Independence and Consistency Strength

Let us further discuss the problem of independence. We will not dive into the formalities of syntax, however everything that we do can be formalized in the language of set theory, with classical first-order logic in the background.

We call a set of sentences a *theory* (usually thought of as the collection of axioms); examples of theories include the axioms for group theory, or the ZFC axioms themselves. We will only consider theories that extend ZF.[2] We write "$T \vdash \varphi$" to mean that φ is provable from T. We say that φ is *independent* of T iff $T \nvdash \varphi$ and $T \nvdash \neg\varphi$. Equivalently, φ is independent of T iff both $T + \varphi$ and $T + \neg\varphi$ are consistent. A fortiori, if there is φ independent of T, then T is consistent.[3] Formally, the independence problem is this: We usually establish sentences φ as mathematical truths by showing that ZFC $\vdash \varphi$, but there are sentences that are independent of ZFC.

Gödel's Incompleteness Theorems show the existence of independent sentences. We will need the following form of the Second Incompleteness Theorem:

Theorem 2.1 (Second Incompleteness Theorem)**.** *Let T be a theory. Let $\mathrm{Con}(T)$ formalize "T is consistent". If T is consistent, then $T \nvdash \mathrm{Con}(T)$. If T is Σ^0_1-Sound,*[4] *then $T \nvdash \neg\,\mathrm{Con}(T)$.*

[2] We will also require that all of our theories be recursively enumerable, which means (somewhat informally) that a computer program can list the axioms.

[3] Due to the Second Incompleteness Theorem below, in order to show independence results for T we thus need to assume the consistency of T.

[4] A formula is Σ^0_1 iff it is of the form "$(\exists v \in \omega)\varphi$", where φ has only bounded quantifiers that range over the natural numbers. A theory is Σ^0_1-Sound iff all of its Σ^0_1 consequences are true.

One may be tempted to just dismiss the problem. The only example of a sentence we have given that is independent is $\mathrm{Con}(\mathrm{ZFC})$; while we are certainly interested in knowing that our axioms are consistent (for, if they weren't, they would be false), this does not directly affect the sort of problems that mathematicians usually think about. However, we will be considering the independence of *natural* sentences. The idea is to consider sentences of the sort that actually come under mathematical consideration, but turn out to be independent. This distinguishes such sentences from constructions that encode metamathematics, for example. There is no precise, formal criterion for a sentence to be natural, and it may be objected that the distinction is at core meaningless. The Incompleteness Theorems are equivalent to statements about Diophantine equations, after all. However, there seems to be real structure when we discuss natural sentences, enough such that it seems a fruitful pursuit that hopefully will be vindicated by further clarification in the future.

We will see examples of natural independence due to consistency strength in Section 3.1. The other way in which occurrences of natural independence arise will be discussed in Section 2.2. For now, we turn to the picture of the independence phenomenon suggested by the Second Incompleteness Theorem.

By the Second Incompleteness Theorem, it is strictly stronger to assume the consistency of a theory T than to simply assume T. To measure mathematical strength in this way, we introduce the notion of consistency strength:

Definition 2.1. Let S and T be theories. We say that T has *higher consistency strength* than S iff T proves the consistency of S and not vice-versa. If S and T prove each others' consistency, then we say that S and T have the *same consistency strength*.

This notion suggests the following picture.[5] While one cannot have a single theory to found mathematics upon, one can have a hierarchy of theories, in which

[5]This discussion is based on [Koea].

two theories are at the same level iff they have the same consistency strength, and higher levels have higher consistency strengths. Let us call this the *consistency strength hierarchy*. Here we see one advantage of working only with natural theories: The consistency strength hierarchy is well-ordered when restrcting attention to natural theories, while its structure can vary much more wildly without restrictions. Now, if the only independent problems were independent due to consistency strength, then starting with ZFC there is only a straight path upward. In order to resolve independence, one would need to argue that higher levels are consistent, while staying below the levels that are inconsistent. While by the Second Incompleteness Theorem one could never conclusively establish the consistency of a level, while if a level is inconsistent, a proof can be exhibited (namely, the proof of the inconsistency). However, a compelling level of evidence can be gathered to this effect; the situation is not so different in natural science, where in gathering evidence one can never completely place a theory beyond refutation, yet one can provide a compelling case to accept a scientific theory nonetheless. My point here is that independence due to consistency strength alone is not a large problem; this scenario, while not as simple as having a single justified foundational theory, works.

However, this picture does not capture the full extent of the independence problem. The problem is that there may be sentences independent of each level of this hierarchy. Thus, to approach the independence problem, we have two tasks: To accurately climb the consistency strength hierarchy, and to figure out which theories are true among those that are consistent.

2.2 Forcing and Natural Independence

In the previous section, we saw independence arise from problems of consistency strength. However, solving the problem of consistency does not solve the independence problem, for there are sentences that are independent without raising consistency strength. Such sentences can be constructed by means similar to the Incompleteness Theorems.[6] However, such constructions are not our interest in attacking the independence problem; we are, again, concerned with natural independent sentences that do not increase consistency strength. The Continuum Hypothesis is one such sentence, for we have

$$\mathrm{ZFC} \vdash \mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{CH}) \wedge \mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH}).$$

We wish to generally deal with sentences like the Continuum Hypothesis. To do this, especially in the absence of a precise characterization of the natural sentences, we look to the method by which we prove the independence of CH: *forcing*.[7] Forcing is one of the most powerful tools for showing sentences to be independent. In particular, it is the main method for showing independence without increasing consistency strength. Blocking this method is tantamount to solving the independence problem in instances where there is no jump in consistency strength.

As we will be discussing and using it extensively later on, we shall summarize the method in the following sections and develop some facts that we will need. We will also give some examples, including a sketch of the independence of CH.

2.2.1 Basics of Forcing

The key idea of forcing is that it is a method to extend the universe. Starting in V, we find a set G that is "missing" from V, and then add it in, closing under the operations of set theory to form a larger universe $V[G]$. Given a sentence φ that we wish to make true, we carefully choose G so that φ obtains in $V[G]$. Since $G \notin V$, we shouldn't be able to define G in V, and so there should be nothing special about G. For this reason, we call G *generic*. We will not have full access to G while

[6] For example, one can take a Rosser sentence.

[7] For the initiated: Throughout this thesis, we will only consider set forcing, not class forcing.

working in V. Thus, we instead find a way to gain partial access to G, getting just enough information about it in V to figure out what we need about $V[G]$.

However, this intuitive picture fails to make sense in an important way. Since V is the class of all sets, there cannot be a set $G \notin V$. We can formalize this talk, however, by working with a countable transitive model M of ZFC. Since M is countable, there are subsets $G \subseteq M$ such that $G \notin M$. We again pick G so that $M[G]$ will satisfy $\mathrm{ZFC} + \varphi$ for a desired φ. If this works, then, given a model of ZFC, we obtain a model of $\mathrm{ZFC} + \varphi$. Thus, if ZFC is consistent, then so is $\mathrm{ZFC} + \varphi$.[8] If one can, by the same method, show that $\mathrm{ZFC} + \neg\varphi$ is consistent, then φ is independent of ZFC.

While the above gives formally how we work with forcing, the original picture is far more intuitive. Thus, we will often speak of forcing starting with V, and talk about forcing extensions of V, with the understanding that formally, we are talking about a countable transitive model $\overline{V}$ that satisfies ZFC and whatever assumptions we have on V. To further motivate this way of thinking, we observe that the class $\{x \mid x = x\}$ in $\overline{V}$ is the universe of $\overline{V}$; in other words, every model of ZFC internally thinks that it is V, even though with an outside vantage point one may be able to see that it is countable.

We move on to the specifics of the method. Our exposition here is necessarily rushed; we refer the reader to [Kun80], Chapter VII for more details. We need a way to gain partial information on the generic G without it being inside our model. The solution is to use partially ordered sets. Intuitively, the order relation $\leq$ on a partial order $\mathbb{P}$ represents information; the lower in the order an element is, the more information on G it gives. Dense sets in the partial order allow us to control

[8] There is a subtlety here. We would like, on only the assumption of $\mathrm{ZFC} + \mathrm{Con(ZFC)}$, to prove $\mathrm{Con}(\mathrm{ZFC} + \varphi)$. Forcing requires a transitive model of ZFC; but, ZFC + "There exists a transitive model of ZFC" has greater consistency strength than $\mathrm{ZFC} + \mathrm{Con(ZFC)}$. There is a workaround using the Reflection Theorem and the Compactness Theorem, however it is unenlightening to pursue this here; see [Kun80, p. 185] for complete details. We will, as is common practice, simply assume the existence of transitive models from now on, knowing that the bookkeeping can be done.

the generic. We give the definitions.

Definition 2.2. A *forcing notion* is a partial order $(\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$.[9] (When the order relation is clear from context, we omit it). A *condition* is an element of $\mathbb{P}$. A subset $D \subseteq \mathbb{P}$ is *dense* iff for all $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$. D is *dense below* p iff D is dense when we restrict to the set of conditions below p.

With these ideas, we can define generics.

Definition 2.3. Let $\mathbb{P}$ be a forcing notion, and $G \subseteq \mathbb{P}$. G is a *filter* iff it satisfies the following:

1. G is upward closed: For every $p \in G$, if $q \geq p$, then $q \in G$.

2. G is closed under meet: If $p, q \in G$ then there is an $r \in G$ below both p and q

G is a *generic filter* iff G is a filter that intersects every dense subset of $\mathbb{P}$.[10]

To use G to define a new model, we introduce a set of names, and use G to interpret them. The point is that the set of names is the set of all possible objects that could be in $M[G]$. Among these are names $\check{x}$ for each $x \in M$, as well as a name $\dot{G}$ for the generic. We get the extension by having G pick out which objects are actualized, by giving interpretations τ^G to names τ. It is not necessary to go through the construction of the class Name of $\mathbb{P}$-names. We do state the definition of the generic extension.

Definition 2.4. We define the *generic extension* $M[G]$ of M by G by

$$M[G] := \{\tau^G \mid \tau \in \mathrm{Name}\}.$$

[9]Without loss of generality, we may assume that all forcing notions have a maximal element.

[10]We can equivalently define a generic filter to be a filter that intersects every maximal antichain (See Definition 2.5).

If $M \subseteq N$ is an inner model, that is, a transitive model with all the ordinals of N, we say M is a *ground* of N, written $M \subseteq_{\text{gd}} N$, iff there is a partial order $\mathbb{P} \in M$ and a generic filter $G \subseteq \mathbb{P}$ such that $N = M[G]$.

The following lemma captures the basic facts about generic extensions.

Lemma 2.2. *Suppose $M[G]$ is a generic extension of M. Then $M[G]$ is transitive, $M[G] \models \text{ZFC}$, $M \subseteq M[G]$, $G \in M[G]$, and $(\text{On})^M = (\text{On})^{M[G]}$. Furthermore, $M[G]$ is the smallest such model.*

The last major ingredient in forcing is the *forcing relation*, written $p \Vdash \varphi$. This relation captures the idea that a condition gives us partial information about generic extensions by generics that contain the condition. We will not go through the actual definition of the relation here; the reader is referred to [Kun80, p. 195] for details. The main point is contained in the following theorem.

Theorem 2.3 (The Fundamental Theorem of Forcing)**.** *Suppose that M is a transitive model of* ZFC*, and $\mathbb{P} \in M$ is a forcing notion. There is a definable relation $\Vdash$ such that for all conditions p and all $\mathbb{P}$-names $\tau_1, ..., \tau_n$,*

$$p \Vdash \varphi(\tau_1, ..., \tau_n) \iff \textit{For all generic } G \subseteq \mathbb{P} \textit{ such that } p \in G, M[G] \models \varphi(\tau_1^G, ..., \tau_n^G).$$

Finally, we describe some basic facts about the forcing relation.

Lemma 2.4. *1. If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.*

2. If $p \nVdash \varphi$ then there is $q \leq p$ such that $q \Vdash \neg\varphi$.

3. If $p \Vdash \varphi$ then $p \nVdash \neg\varphi$.

2.2.2 Forcing Facts

In this section, we will outline several useful facts about forcing that will be of use to us later. We review the κ chain condition, product forcing, and equivalences between forcing notions. We will not prove any of the lemmas we state here; the reader is referred to [Kun80] Chapter VII.

One upshot of using partial orders to define forcing extensions is that the combinatorics of the partial order can be used to control the properties of the generic extension. A major example is the κ chain condition.

Definition 2.5. Let $\mathbb{P}$ be a partial order. A subset $A \subseteq \mathbb{P}$ is an *antichain* iff for all $p, q \in A$, there is no $r \in \mathbb{P}$ such that $r \leq p, q$. We say that $\mathbb{P}$ has the κ *chain condition* (abbreviated by saying "$\mathbb{P}$ is κ-c.c.") iff for all antichains $A \subseteq \mathbb{P}$, $|A| < \kappa$.

The κ-c.c. is closely related to the following approximation property between an inner model and an extension. We will prove in Section 5.2.1 that this property is equivalent to N being a ground of M by a κ-c.c. forcing notion.

Definition 2.6. Let $N \subseteq M$ be an inner model. We say that N satisfies the κ-*global covering property* for M iff for all functions $f : X \to Y$ with $f \in M$, there is a function $g_f : X \to P(Y)$ such that

1. $g_f \in N$,
2. for all $x \in X$, $f(x) \in g_f(x)$, and
3. for all $x \in X$, $|g_f(x)| < \kappa$.

Remark. It suffices to only consider functions whose domain and range are ordinals, since one can replace the domain and range of an arbitrary function by their cardinalities.

Another property of κ-c.c. forcing is that it must add subsets of κ:

Lemma 2.5. *Suppose $\mathbb{P} \in M$ is a κ-c.c. forcing notion. Let $G \subseteq \mathbb{P}$ be generic. If $M \subsetneq M[G]$, then $P^M(\kappa) \subsetneq P^{M[G]}(\kappa)$.*

We turn to another useful property: weak homogeneity.

Definition 2.7. A partial order $\mathbb{P}$ is *weakly homogeneous* iff for all $p, q \in \mathbb{P}$ there is an automorphism f of $\mathbb{P}$ such that there is $r \in \mathbb{P}$ with $r \leq f(p), q$.

One example of a weakly homogeneous forcing notion is the cardinal collapse:

Definition 2.8. $\mathrm{Coll}(\omega, \kappa)$ is the set of finite partial functions p from ω to κ, ordered by reverse inclusion.

For any n, the set of p such that $n \in \mathrm{dom}(p)$ is dense, so a generic must be a total function, in fact a surjection, from ω to κ in the forcing extension. Thus, this forcing notion collapses a cardinal in the forcing extension to a countable ordinal (recall that with countable transitive models, everything in the model is externally countable). It is weakly homogeneous because we can always take an automorphism that swaps functions whose nth coordinate is a and functions whose nth coordinate is b, and do this to get an arbitrary p to agree with an arbitrary q throughout the shorter one's domain.

We will use the following lemma on weak homogeneity:

Lemma 2.6. *Suppose $\mathbb{P} \in M$ is a weakly homogeneous forcing notion. For any $p \in \mathbb{P}$ and any formula φ, $p \Vdash \varphi$ iff $1_{\mathbb{P}} \Vdash \varphi$.*

Next, we discuss product forcing and iterations. Having passed from M to $M[G]$, it is natural to wonder whether one can find a generic $H \subseteq \mathbb{P}' \in M[G]$ and force again, to add new sets. Doing so corresponds to forcing with a product partial order.

Definition 2.9. Suppose $\mathbb{P}_0$ and $\mathbb{P}_1$ are forcing notions in M. The *product forcing* is given by the following order relation on $\mathbb{P}_0 \times \mathbb{P}_1$:

$$(p_0, q_0) \leq_{\mathbb{P}_0 \times \mathbb{P}_1} (p_1, q_1) \iff (p_0 \leq_{\mathbb{P}_0} p_1) \wedge (q_0 \leq_{\mathbb{P}_1} q_1).$$

Lemma 2.7. *Let $\mathbb{P}_0, \mathbb{P}_1 \in M$ be forcing notions. Let $G \subseteq \mathbb{P}_0 \times \mathbb{P}_1$ be equal to the product $G_0 \times G_1$. The following are equivalent:*

1. *G is generic over M*

2. *G_0 is generic over M and G_1 is generic over $M[G_0]$*

3. *G_1 is generic over M and G_0 is generic over $M[G_1]$*

Furthermore, if any of the above hold, then $M[G] = M[G_0][G_1] = M[G_1][G_0]$.

We turn now to the question of when two forcing notions are equivalent. We do not quite need isomorphisms (in particular, surjections) of partial orders; rather, the type of map we are interested in preserves generics as the important structure on a forcing notion.

Definition 2.10. Let $\mathbb{P}_0$ and $\mathbb{P}_1$ be forcing notions. A function $f : \mathbb{P}_0 \to \mathbb{P}_1$ is a *dense embedding* iff the following hold:

1. f preserves the order: For all $p, q \in \mathbb{P}_0$, if $p \leq q$ then $f(p) \leq f(q)$.

2. f preserves incompatibility: For all $p, q \in \mathbb{P}_0$, if there is no $r \in \mathbb{P}_0$ such that $r \leq p, q$, then there is no such r for $f(p)$ and $f(q)$ in $\mathbb{P}_1$.

3. $f[\mathbb{P}_0]$ is dense in $\mathbb{P}_1$.

The next lemma tells us that dense embeddings do in fact preserve generics, and give a full equivalence between forcing notions.

Lemma 2.8. *Let $f : \mathbb{P}_0 \to \mathbb{P}_1$ be a dense embedding between forcing notions in a transitive model M. For $G \subseteq \mathbb{P}_0$, we set*

$$\hat{f}(G) := \{p \in \mathbb{P}_1 \mid (\exists q \in G)(f(q) \leq p\}.$$

Then f and $\hat{f}$ give a correspondence between $\mathbb{P}_0$ generics and $\mathbb{P}_1$ generics, in that if G is $\mathbb{P}_0$ generic, then $\hat{f}(G)$ is $\mathbb{P}_1$ generic, and if H is $\mathbb{P}_1$ generic, then $f^{-1}[H]$ is $\mathbb{P}_0$ generic. Furthermore, if G is $\mathbb{P}_0$ generic and H is $\mathbb{P}_1$ generic, then $M[G] = M[\hat{f}(G)]$ and $M[H] = M[[f^{-1}[H]]$.

We will use the following equivalence with the collapse forcing:

Lemma 2.9. *If $|\mathbb{P}| \leq \gamma$ and $1_{\mathbb{P}} \Vdash (\exists f)(f : \omega \to \gamma$ is surjective), then there is a dense embedding from (a dense subset of) $\mathbb{P}$ to $\mathrm{Coll}(\omega, \gamma)$.*

2.2.3 The Space of All Forcing Extensions: The Generic Multiverse

It is sometimes useful for us to take a big picture view by thinking of the study of forcing as the study of a space consisting of models of ZFC, specifically, the models produced by considering forcing extensions and grounds. This space is called the Generic Multiverse.

Definition 2.11. The *Generic Multiverse generated by* M_0, denoted $\mathbb{V}_{M_0}$, is the smallest collection of models of ZFC satisfying the following:

1. $M_0 \in \mathbb{V}_{M_0}$.
2. If $M \in \mathbb{V}_{M_0}$ and $M[G]$ is a generic extension of M, then $M[G] \in \mathbb{V}_{M_0}$.
3. If $M \in \mathbb{V}_{M_0}$ and $N \subseteq_{\mathrm{gd}} M$, then $N \in \mathbb{V}_{M_0}$.

The *Generic Multiverse*, denoted $\mathbb{V}$, is the Generic Multiverse generated by V.

The Generic Multiverse gives us all the "possible universes" that we could have due to forcing. As such, it plays a key role in our investigation of the independence problem and its solution: If sentences that are independent due to forcing (such as CH) have no truth value, then for any member W of the Generic Multiverse, there is no reason to believe that the universe of sets is W.[11] That is, there is

[11] See [Woo09].

no privileged point in $\mathbb{V}$. For, if there were, then the truth value of CH or any other such sentence would be the truth value of that sentence in W. The contrapositive of this statement tells us then that the existence of such a privileged point is a reason to believe that the independent questions have answers. This gives a condition on any attempt to fully solve the independence problem: If we claim that some model $\overline{V}$ corresponds to the actual V, then $\overline{V}$ should be such a privileged point in its Generic Multiverse.

There are several conditions we can place on such a point. It should be definable, otherwise we are unable to speak of it. This definition should be generically absolute; that is, the definition should be interpreted the same in W, V, and $V[G]$, where $W \subseteq_{\mathrm{gd}} V \subseteq_{\mathrm{gd}} V[G]$. In order for this to be possible, the privileged point should be minimal; otherwise, if P is the privileged point and $Q \subsetneq_{\mathrm{gd}} P$, then Q cannot interpret the definition of P correctly, for it is missing elements of P. We would like for the privileged point to be a unique minimum, otherwise we would be unable to distinguish between the various minima, and they would all cease to serve their purpose.

We will see in Chapter 5 that if there is such a point, then it must meet these conditions. Thus the central question is whether such a point exists. Under sufficiently strong assumptions, we will see in Chapter 5 that the answer is positive. This becomes part of a larger picture suggesting the solution to the independence problem.

2.3 Recap

We have gone over the independence problem which motivates this thesis. We found that there are two main sources of independence when it comes to natural sentences of mathematics: Consistency strength, and forcing. Thus, a complete solution to the independence problem will address both of these. We will begin discussing approaches which attempt to address these in Chapter 3. However, the first truly compelling solution will only come with the axiom $V = \text{Ultimate } L$ in Chapter 4.

3 Approaches to New Axioms

Having discussed the independence problem, we turn to the ideas, mathematical and philosophical, that have been used to attack the problem.

To solve the independence problem, we require new axioms to be added to ZFC. Such axioms must be capable of solving independent questions. However, we cannot arbitrarily adopt any sentence as an axiom. For, if we could, then we could simply adopt whichever independent sentences as new axioms as we pleased. The question of whether, say, CH is true loses objectivity, as both CH and $\neg\mathrm{CH}$ become equally good axioms. This is no solution to the independence problem; it is merely accepting that one cannot answer independent questions.

Thus, we cannot adopt any independent sentence as a new axiom. We need to justify the new axioms we take on, to argue for their truth. Such arguments are not proofs, for we work with questions that cannot be answered by proof. Thus, such justifications must be partially philosophical. However, to carry weight, they must also be influenced by mathematical facts.

In Section 3.1, we will discuss a first attempt at solving the independence problem: Large cardinal axioms. We will discuss other attempts from inner model theory in Section 3.2

3.1 Large Cardinals

The first approach to the independence problem we shall discuss is via large cardinal axioms. These axioms, while unable to resolve the Continuum Hypothesis (see

Theorem 3.4) have had many successes resolving independent questions.

There is no general, formal characterization for what counts as a large cardinal. Intuitively, a large cardinal axiom asserts the existence of a cardinal number that transcends all the ordinals that ZFC can prove exists, and possibly other large cardinals as well. This is analagous to the Axiom of Infinity, which effectively asserts the existence of ω; without it, the only ordinals that can be proven to exist are the finite ones, whose formation process (iterating successor) is completely transcended by ω. A formal way that large cardinals transcend that which comes before is that they increase consistency strength: If A is any of the standard large cardinal axioms, then $\mathrm{ZFC} + A$ proves the consistency of ZFC (possibly with other large cardinal axioms as well).

Based on the discussion of the previous paragraph, it makes sense to discuss the large cardinals as falling into a hierarchy. In fact, they form a hierarchy in terms of consistency strength. There are several remarkable empirical facts about the large cardinal hierarchy.[12] First, it seems to be well-ordered, with only a few comparisons currently unknown.[13] Second, for any "natural" sentence S, there is a large cardinal axiom A such that $\mathrm{ZFC} + A$ has the same consistency strength as $\mathrm{ZFC} + S$. Furthermore, in many cases there is no known way to prove that two theories have the same consistency strength except by going through the appropriate large cardinal hypothesis.[14] Thus, the structure of the large cardinal hierarchy explains the structure of the consistency strength hierarchy for natural theories.

Let us give some specific examples of large cardinals. These are just a few examples, meant to give the reader a flavor of the principles while developing the specific tools we will need later. For a much more complete picture, the reader is referred to [Kan09].

[12] These cannot be made into theorems without a formal criterion for notions such as "large cardinal" and "natural sentence" however.

[13] See [Kan09, p. 472].

[14] [Koea]

The easiest large cardinal property to state is inaccessibility. Intuitively, inaccessibles have strong closure properties, so strong that the axioms of ZFC cannot escape them. Formally, if κ is inaccessible, then $V_\kappa \models \mathrm{ZFC}$. Thus, $\mathrm{ZFC}+$ "There is an inaccessible cardinal" is of higher consistency strength than ZFC.

Definition 3.1. A cardinal κ is *(strongly) inaccessible* iff κ is regular and strong limit, that is, for all $\gamma < \kappa$, $2^\gamma < \kappa$.

However, inaccessibles are in some sense "small." The other large cardinal notions we will consider all imply the existence of inaccessibles, sometimes to the point of implying the existence of a proper class of them. Furthermore, we will see that by Theorem 3.11, most large cardinals are incompatible with a potential new axiom. Inaccessibles are not incompatible in this way.

To get stronger large cardinal hypotheses, we use a common pattern of formulating hypotheses with *elementary embeddings*. Recall that an elementary embedding is a truth-preserving map: We say $j : M \to N$ is elementary iff for all formulas φ and all $a_0, ..., a_n \in M$, $M \models \varphi(\vec{a})$ iff $N \models \varphi(h(a_0), ..., h(a_n))$. (For simplicity, all elementary embeddings we discuss will be assumed to not be the identity.) We will be taking elementary embeddings from V to inner models (transitive proper class models with all the ordinals) M of V.[15] In order to define large cardinals, we need the notion of the critical point of an elementary embedding.

Definition 3.2. Let $j : V \to M$ be an elementary embedding. If κ is least such that $j(\kappa) > \kappa$,[16] we call κ the *critical point* of j, and we write $\kappa = \mathrm{CRT}(j)$.

With this definition in place, we can define the prototypical large cardinal: the measurable cardinal.

[15] We can restrict the complexity of the formulas for which j is elementary so as not to violate Tarski's theorem by talking about true formulas in V.

[16] There must be such an ordinal for any non-identity elementary embedding: If x is of least rank such that $x \subsetneq j(x)$, then $j(\mathrm{rank}(x)) \neq \mathrm{rank}(x)$.

Definition 3.3. A cardinal κ is *measurable* iff there is an inner model $M \subseteq V$ and an elementary embedding $j : V \to M$ such that $\kappa = \mathrm{CRT}(j)$.

Measurable cardinals can be given a different formulation in terms of *ultrafilters*.

Definition 3.4. Let X be a set. A set $U \subseteq P(X)$ is called an *ultrafilter* on X iff in the partial order $(P(X), \subseteq)$, U is a filter, and for every $Y \in P(X)$, either $Y \in U$ or $X \setminus Y \in U$. The sets in the ultrafilter are said to have *measure 1*, and their complements have *measure 0*. An ultrafilter is called *κ-complete* iff it is closed under $< \kappa$ intersections. An ultrafilter is called *nonprincipal* iff it contains no singletons.

Proposition 3.1. *A cardinal κ is measurable iff there is a κ-complete nonprincipal ultrafilter on κ.*[17]

We shall consider a few stronger large cardinal properties. These are obtained by varying properties of the elementary embeddings and demanding closure conditions on M. Supercompactness will play a central role regarding the new axiom we shall discuss in Section 4. Hyper-huge cardinals shall play a role in our discussion of set-theoretic geology in Section 5.

Definition 3.5. A cardinal κ is *supercompact* iff for each $\lambda > \kappa$ there is an inner model $M \subseteq V$ and an elementary embedding $j : V \to M$ such that $\kappa = \mathrm{CRT}(j)$, $j(\kappa) > \lambda$, and M is closed under λ sequences, that is, ${}^{\lambda}M \subseteq M$.

There is another formulation of supercompactness that uses ultrafilters.

Definition 3.6. An ultrafilter on a subset $P \subseteq P(X)$ is called *fine* iff for all $x \in X$, we have $\{y \in P \mid x \in y\} \in U$. An ultrafilter is called *normal* iff for all functions f, if f presses down on a measure 1 set, that is, $\{\alpha \mid f(\alpha) \in \alpha\} \in U$, then f is constant on a measure 1 set, that is, there is γ such that $\{\alpha \mid f(\alpha) = \gamma\} \in U$.

[17] See [Jec03] Chapter 17.

Proposition 3.2. *A cardinal κ is supercompact iff for all $\lambda > \kappa$, there is a κ-complete normal fine ultrafilter on $P_\kappa(\lambda)$.*[18]

Definition 3.7. A cardinal κ is *hyper-huge* iff for each $\lambda > \kappa$ there is an inner model $M \subseteq V$ and an elementary embedding $j : V \to M$ such that $\kappa = \mathrm{CRT}(j)$, $j(\kappa) > \lambda$, and M is closed under $j(\lambda)$ sequences, that is, ${}^{j(\lambda)}M \subseteq M$.

The following theorem summarizes the relative strength of the large cardinal hypotheses we have defined.

Theorem 3.3. *Every hyper-huge cardinal is supercompact, every supercompact cardinal is measurable, and every measurable cardinal is strongly inaccessible. In particular, if* ZFC + *"There is a hyper-huge cardinal" is consistent, so is* ZFC + *"There is a supercompact cardinal," etc. If κ is hyper-huge, then there is a proper class of inaccessible cardinals.*[19]

One of the most interesting purposes of large cardinal notions is to give new axioms, in the form of asserting, for a large cardinal notion φ, "There is a cardinal κ such that $\varphi(\kappa)$." Kurt Gödel once conjectured that large cardinal axioms alone would suffice to solve the independence problem.[Göd90, p. 151] Large cardinals indeed have been very fruitful. For example, they completely answer the many questions of descriptive set theory that were found independent of ZFC, such as whether the projective sets have regularity properties such as Lebesgue measurability or the perfect set property.[20]

Furthermore, there are reasons for believing these axioms are true extensions of ZFC. We will emphasize one thread: Their fruitful relationship with consistency. That the large cardinal hierarchy explains the structure of the consistency strength hierarchy in itself is evidence. We will argue that the connection runs

[18] See [Jec03] Chapter 20.
[19] [Kan09, p. 487]
[20] For more on the independence problem in descriptive set theory and the solution via large cardinal and determinacy axioms, see [Koeb].

deeper. For most theories T, the consistency of T is not generally strong evidence for its truth because of the existence of other, incompatible theories with the same consistency strength. However, because of the explanatory role of large cardinal axioms regarding the structure of the consistency strength hierarchy, the only reason to accept that a level of the consistency strength hierarchy is consistent is the consistency of the corresponding large cardinal axiom. This is reinforced by the necessity of using large cardinals to show theories have the same consistency strength in many cases. Thus, the only real candidate for truth in a level of consistency strength is ZFC plus the corresponding large cardinal axiom.

So, to establish that a large cardinal axiom A is true, it suffices to establish that $\mathrm{ZFC} + A$ is consistent. Proving more sentences from $\mathrm{ZFC} + A$, strictly speaking, provides inductive evidence for every non-contradiction proved. However, this seems very weak, just as a large amount of numerical evidence for a number-theoretic sentence is weak evidence for that sentence, for counterexamples can (and often do) occur at very large numbers. Instead, one should develop the intuitive picture that results from $\mathrm{ZFC} + A$ by proving deep and central results. If there are tensions, then formally developing those tensions could lead to a proof of contradiction. However, if the picture is coherent, this is reason to believe there is no such proof. One example of a such development here would be Woodin cardinals and the problems of descriptive set theory, which have been alluded to earlier. Another example, often cited, is the inner model theory of some of large cardinals, which studies inner models designed to accommodate the desired large cardinal hypothesis.[21] Such inner models have a striking coherence and order that makes inner model theory particularly powerful for this task.

Henceforth, we will accept large cardinal axioms as true, at least the ones we have stated and plan to work with. Large cardinal axioms thus resolve the in-

[21] See, for example, [Kan09, p. 264].

dependence problem with respect to consistency strength, but what about forcing? The answer, unfortunately, is negative: Forcing still works to show independence results even in the presence of large cardinals, due a theorem of Lévy and Solovay.

Theorem 3.4 (The Lévy-Solovay Theorem)**.** *Suppose $\kappa \in V$ is a cardinal, and $V[G]$ is a generic extension of V with partial order $\mathbb{P}$ such that $|\mathbb{P}| < \kappa$. Then κ is measurable in V iff κ is measurable in $V[G]$.*[22]

Proof. ($\implies$) Let $j : V \to M$ be an elementary embedding with critical point κ. We may assume that $\mathbb{P} \in V_\kappa$ by taking an isomorphism if necessary. Thus, $j(\mathbb{P}) = \mathbb{P}$. We lift j to an elementary embedding. The point is that elementarity preserves being a $\mathbb{P}$-name, so we can use the names to lift the embedding. Let $a \in V[G]$, and let $\dot{a} \in \mathrm{Name}$ be a name for a. We define $j(a) := j(\dot{a})^G$, that is, we send a to its name, and then to a name in M, which we then interpret by $j(G) = G$. Provided that this map is well-defined, it is an elementary embedding: We have

$$\begin{aligned} V[G] \models \varphi(x_1, ..., x_n) &\iff (\exists p \in G)(p \Vdash \varphi(\dot{x}_1, ..., \dot{x}_n)) \\ &\iff j(p) = p \Vdash \varphi(j(\dot{x}_1), ..., j(\dot{x}_n)) \\ &\iff M[G] \models \varphi(j(x_1), ..., j(x_n)). \end{aligned}$$

We check that the map is well-defined. If $\dot{a}_1$ and $\dot{a}_2$ are two names for a, then there is a $p \in G$ such that $p \Vdash \dot{a}_1 = \dot{a}_2$. Then, by elementarity, $j(p) = p \Vdash j(\dot{a}_1) = j(\dot{a}_2)$. So $M[G]$ thinks that they are equal, so $j(\dot{a}_1)^G = j(\dot{a}_2)^G$ as desired. ($\impliedby$) Let U be a κ-complete nonprincipal ultrafilter on κ in $V[G]$. We will show that there is a κ-complete nonprincipal ultrafilter on κ in V. Let p be such that $p \Vdash$"$\dot{U}$ is a $\check{\kappa}$-complete nonprincipal ultrafilter on $\check{\kappa}$". We start with a claim:

Claim. *There is a $q \leq p$ such that for every $X \subseteq \kappa$, either $q \Vdash \dot{X} \in \dot{U}$ or*

[22] This theorem and the following corollary first appear in [LS67]. For the left-to-right direction of the proof, we use the argument from [Jec03, p. 389].

$q \Vdash \check{\kappa} \setminus \dot{X} \in \dot{U}$.

Proof of Claim. Suppose not toward contradiction. For each $q \leq p$, let C_q be a set such that $q \not\Vdash \dot{C}_q \in \dot{U}$ and $q \not\Vdash \check{\kappa} \setminus \dot{C}_q \in \dot{U}$. We use the C_q to determine equivalence classes on κ, by defining

$$\alpha \sim \beta \iff (\forall q \leq p)(\alpha \in C_q \iff \beta \in C_q).$$

There are at most $2^{|\mathbb{P}|}$ equivalence classes. Since κ is inaccessible (in particular, strong limit) in $V[G]$, κ is strong limit in V (being a subset is absolute, and there can only be less subsets of any ordinal in V, so for any $\gamma < \kappa$, $2^\gamma \leq (2^\gamma)^{V[G]} < \kappa$). Thus, $2^{|\mathbb{P}|} < \kappa$. Enumerate the equivalence classes as $\{A_\xi \mid \xi < 2^{|\mathbb{P}|}\}$.

Since $p \Vdash$"$\dot{U}$ is a $\check{\kappa}$-complete nonprincipal ultrafilter on $\check{\kappa}$", p cannot force for all $\xi < 2^{|\mathbb{P}|}$, $A_\xi \notin U$. Otherwise, the intersection of the $\kappa \setminus A_\xi$ would be measure 1; but since the A_ξ are equivalence classes, they are disjoint, and so that intersection is empty. Thus, let ξ and $q \leq p$ be such that $q \Vdash \check{A}_\xi \in \dot{U}$. By construction, either every element of A_ξ is in C_q or is in $\kappa \setminus C_q$; without loss of generality, let us assume the latter, so that $A_\xi \cap C_q = \emptyset$. Then, $q \Vdash \check{C}_q \notin \dot{U}$, which is a contradiction. □

With the claim in hand, we define our ultrafilter on κ in the ground model as follows.

$$U' := \{X \subseteq \kappa \mid q \Vdash \check{X} \in \dot{U}\}.$$

The desired properties follow from the fact that q forces that U is a κ-complete nonprincipal ultrafilter, so it must force that the names of the appropriate sets are in the ultrafilter. It is in fact an ultrafilter by the claim. □

Corollary 3.4.1. ZFC + *"There is a measurable cardinal"* $\nvdash$ CH, $\neg$CH.

Proof. Let κ be measurable. The forcing notion for $\neg$CH has size $\omega_2 < \kappa$ and the forcing notion to collapse 2^ω to ω_1 has size $2^\omega < \kappa$. Thus, by Theorem 3.4, the relative consistency proofs for both work in the presence of measurable cardinals.

□

Remark. This theorem generalizes to other large cardinals, as those can be defined with similar constructions using ultrafilters of some sort or elementary embeddings.

Thus, we turn now to a different source for new axioms: inner model theory.

3.2 Inner Model Theory

We will describe in this section the study of inner models. Recall that if $N \subseteq M$ are models of ZF or ZFC, we say N is an *inner model* of M iff N is transitive and contains all the ordinals of M; we will be concerned in particular with inner models of V. Intuitively, an inner model is a sub-universe of the full universe of sets. While understanding V is difficult due to the independence problem, it turns out that some inner models are much easier to understand. The ultimate hope would be to find a well-understood inner model whose theory is correct, that is, the same as the theory of the full universe. In virtue of understanding that inner model, we would be able to give answers to independent questions, and in virtue of the correctness of the theory of that inner model, these answers would be true. If M is such an inner model, the resulting axiom is $V = M$, the assertion that M is not a proper inner model; M gives the truth, and every set is in M.

In Section 3.2.1, we give some basic results about inner models that will be useful throughout this thesis. We then turn to the Constructible Universe, the prototypical inner model. After that discussion, we will turn to other inner models in Sections 3.2.3 and 3.2.4.

3.2.1 Basic Facts

We start with some basic results on inner models that will be useful later on. All of these results can be found in [Jec03] Chapter 13. First, we give sufficient conditions for a proper class to be an inner model, using what are known as the Gödel operations.

Definition 3.8. The *fundamental Gödel operations* are the following ten functions:

$$\begin{aligned}
G_1(x,y) &= \{x,y\}\\
G_2(x,y) &= x \times y\\
G_3(x,y) &= \{(u,v) \mid u \in x \wedge v \in y \wedge u \in v\}\\
G_4(x,y) &= x \setminus y\\
G_5(x,y) &= x \cap y\\
G_6(x) &= \bigcup x\\
G_7(x) &= \operatorname{dom}(x) \quad \text{(if } x \text{ is a relation)}\\
G_8(x) &= \{(u,v) \mid (v,u) \in x\}\\
G_9(x) &= \{(u,v,w) \mid (u,w,v) \in x\}\\
G_{10}(x) &= \{(u,v,w) \mid (v,w,u) \in x\}
\end{aligned}$$

We say that a function G is a *Gödel operation* iff G is in the smallest class of functions that contains the fundamental Gödel operations and is closed under composition.

Intuitively, the Gödel operations capture all the most basic ways of building sets. This can be made precise by considering instances of Δ_0-Separation (instances of the Separation Axiom Schema where the formula is Δ_0). Recall that Δ_0 formulas are the simplest formulas in the language of set theory according to the Lévy hierarchy. As it turns out, the Gödel operations completely capture the sets that can be obtained via Δ_0-Separation:

Proposition 3.5. *Let $\varphi(v_1, ..., v_n)$ be a Δ_0 formula. There is a Gödel operation G such that for all $x_1, ..., x_n$,*

$$G(x_1, ..., x_n) = \{y_1, ..., y_n \mid y_1 \in x_1 \wedge ... \wedge y_n \in x_n \wedge \varphi(y_1, ..., y_n)\}.$$

The proof is a straightforward but tedious induction on formula complexity, and we will not go through it here. The reader is referred to [Jec03, p. 177] for details. This proposition has the following useful corollary:

Corollary 3.5.1. *If M is transitive and closed under the Gödel operations then M satisfies all instances of Δ_0-Separation.*

Proof. Let φ be a Δ_0 formula, and $a \in M$. We need to show that $\{x \in a \mid \varphi(x)\} \in M$. By Proposition 3.5, there is a Gödel operation G such that $G(a) = \{x \in a \mid \varphi(x)\}$. Since M is closed under the Gödel operations, $G(a) \in M$, as desired. □

Now we see how to identify inner models using the Gödel operations. Since the Gödel operations capture the ways that we can define sets, one might expect that closing a set under the Gödel operations would be enough to capture all the sets that are needed to model ZFC. We need a little more in order to get everything:

Definition 3.9. Let M be a transitive class. M is *almost universal* iff for all subsets $x \subseteq M$, there is $y \in M$ such that $x \subseteq y$.

Proposition 3.6. *Let M be a transitive class. M is an inner model of* ZF *iff M is closed under the Gödel operations and is almost universal.*

Proof. If M is an inner model, then, since the Gödel operations correspond to Δ_0 formulas, they are absolute, and so M is closed under them. If $x \subseteq M$ is a subset, then it has bounded rank; by the absoluteness of rank, x has bounded rank in M, and so $x \subseteq M_{\mathrm{rank}(x)} \in M$.

Next, we check that these conditions suffice for M to be an inner model. Since M is almost universal, M must be a proper class, otherwise there would have to be a superset of M in M. Since M is a transitive proper class, it must contain all of the ordinals, otherwise there would be a least ordinal $\alpha \notin M$, and as a subclass of V_α we would have that M is a set, contradicting the previous point. Thus, we just need to check the ZF axioms.

Extensionality and Foundation are immediate from transitivity. Because M is closed under the Gödel operations, Pairing and Union follow from applications of $G_1(x, y)$ (taking x and y to their pair) and $G_6(x)$ (taking x to its union). The empty set is in M by taking $G_4(x, x)$ (taking x to $x \setminus x$). To see Infinity, notice that each natural number can be built in M, since we have the empty set, pairs, and unions. Thus, by almost universality, there must be $y \in M$ such that $y \supseteq \omega$. With Separation, one can extract ω, and thus there is an inductive set in M. For Power Set, suppose $x \in M$. Since $P(x) \cap M \subseteq M$, by almost universality there is $y \in M$ containing $P(x) \cap M$. Using Separation, one can extract $P^M(x) = P(x) \cap M$. To see Replacement, let F be a class-sized function on M. Since $F[M] \subseteq M$, by almost universality there is $y \supseteq F[M]$ such that $y \in M$. With Separation, one can then extract $F[M]$.

Thus, we just need to verify Separation. Let $\varphi(v)$ be a formula with n quantifiers. Let $\overline{\varphi}(v, u_1, ..., u_n)$ be the (Δ_0) formula that results from replacing the kth quantifier with a quantifier bounded to range over u_k, for $k < n$. We will show by induction on n that for all $a \in M$ there exist $y_1, ..., y_n \in M$ such that if $x \in a$, then

$$\varphi^M(x) \iff \overline{\varphi}(x, y_1, ..., y_n).$$

Then, since M satisfies Δ_0 Separation by Corollary 3.5.1, $\{x \in a \mid \varphi^M(x)\} \in M$.

If $n = 0$, then $\overline{\varphi}$ is the same as φ. Now suppose the result is true for n, and let $\psi(u, v)$ be a formula with n quantifiers. Without loss of generality, we consider only the existential quantifier. Let φ be $(\exists v)\psi(u, v)$. By definition, $\overline{\varphi}$ is $(\exists v \in u_{n+1})\psi(u, v, u_1, ..., u_n)$. We have the first $y_1, ..., y_n$ by the induction hypothesis. We define y_{n+1} by capturing all the possible witnesses in M by a set in M. We do this by using the Collection schema in V on the formula $\psi^M(u, v)$, thinking of u as the input and v as the output (implicitly we also add as a conjunct $v \in M$ to ensure the witness actually comes from M). We have $\psi^M[a] \subseteq M$ and, for $x \in a$, there is $y \in M$ such that $\psi^M(x, y)$ iff there is a witness in $\psi^M[a]$. By almost universality, let $y_{n+1} \supseteq \psi^M[a]$ be in M. It follows that for $x \in a$ there is $y \in M$ such that $\psi^M(x, y)$ iff there is a witness in y_{n+1}. Thus, using the induction hypothesis,

$$((\exists v)\psi(u, v))^M \iff (\exists v \in y_{n+1})\overline{\psi}(u, v, y_1, ..., y_n)$$

as desired. □

We state another useful proposition for studying inner models. This proposition gives a criterion for two inner models to be equal. Using the Axiom of Choice, we can code all sets by sets of ordinals, and so it suffices to check only sets of ordinals.

Proposition 3.7. *Suppose M and N are transitive models of* ZFC*. If every set of ordinals in M is in N, then $M \subseteq N$; thus, if M and N have the same sets of ordinals, then $M = N$.*

Proof. Since relations on ordinals can be coded by sets of ordinals (using a bijection between $\kappa \times \kappa$ and κ for sufficiently large κ), every relation on ordinals in M is a relation on ordinals in N. Now, let $x \in M_\alpha \subseteq M$. It suffices to show that $M_\alpha \in N$ by the transitivity of N. We code the membership relation restricted to M_α using a relation on $\gamma := |M_\alpha|^M$. Let $\pi : \gamma \to M_\alpha$ be a bijection in M. For $\alpha, \beta \in \gamma$, we set

$\alpha\ E\ \beta$ iff $\pi(\alpha) \in \pi(\beta)$. Since $\in$ is well-founded and extensional, E is as well. We have that $E \in M$, and since N has the same relations on ordinals, $E \in N$. Well-foundedness can be expressed by a Δ_0 formula ("There is an E minimal element" quantifies only over γ), as can extensionality ("For all $x, y, z \in \gamma\ldots$"); thus, E is well-founded and extensional in N. By Mostowski's Collapsing Lemma, there is $T \in N$ such that $(\gamma, E) \cong (T, \in)$. By construction of E, $(M_\alpha, \in) \cong (T, \in)$. By the uniqueness of the Mostowski collapse, $M_\alpha = T \in N$ as desired. □

3.2.2 The Constructible Universe

Let us turn to the prototypical inner model: Gödel's Constructible Universe L. As a first attempt at giving a new axiom using inner model theory, $V = L$ seems to work well; however, it is incompatible with large cardinals.

The definition of L is given in stages, similar to V. However, the power set operation is replaced by a definable version; intuitively, this "thins out" the universe by rejecting arbitrary sets at each level, only allowing the definable ones.

Definition 3.10. Suppose M is a transitive set. The *definable power set* of M, denoted $\mathrm{Def}(M)$, is the set of all subsets of M that are definable in the structure $(M, \in)$, that is, the set of all $x \subseteq M$ such that, for some formula φ in the language of set theory and parameters $a_1, ..., a_n \in M$,

$$x = \{y \in M \mid (M, \in) \models \varphi(y, a_1, ..., a_n)\}.$$

We define the *Constructible Universe*, denoted L, by recursion:

$$L_0 = \emptyset$$
$$L_{\alpha+1} = \mathrm{Def}(L_\alpha)$$
$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \quad \text{(for } \lambda \text{ a limit ordinal.)}$$
$$L = \bigcup_{\alpha\in\mathrm{On}} L_\alpha$$

Remark. Since $L \models \mathrm{ZFC}$ (Lemma 3.8), L must have a rank hierarchy, that is, sets V_α^L such that $L \models$"V_α^L is the set of sets of rank less than α." This is *not* the same as the Constructible hierarchy above; for example, while $L_{\omega+1}$ is countable in L, $V_{\omega+1}^L$ is uncountable in L.

The following lemma lists some basic facts about L. Except for (5), the proofs make use only of basic facts about transitivity, absoluteness, and the definition of L, and we will not prove them here. The reader is referred to [Kun80] Chapter VI for details.

Lemma 3.8.

1. *For all α, L_α is transitive, and so L is transitive.*
2. *For all $\beta \leq \alpha$, $L_\beta \subseteq L_\alpha$.*
3. *If $\alpha \in \mathrm{On}$, then $\alpha \in L_{\alpha+1}$.*
4. $L \models \mathrm{ZF}$.
5. *The Well-Ordering Theorem (and so the Axiom of Choice) holds in L.*

Proof. (5) We construct a well-ordering of each L_α by recursion on the ordinals; since each L_α is transitive and every $x \in L$ appears in some L_α, this is enough. Suppose that we have a well-ordering of L_β for all $\beta < \alpha$; we will construct a well-ordering of L_α. Notice that there are countably many formulas in the language of set theory, and so these are well-ordered by taking a bijection with ω. Now, by definition of L, each element x of L_α is determined by the least β such that $x \in L_{\beta+1}$, the least formula (in the above well-ordering) that defines x in L_β, and the parameters $a_1, ..., a_n \in L_\beta$ that are used in the definition; thus, we identify each $x \in L_\alpha$ with a 3-tuple $(\beta, \varphi, \langle a_1, ..., a_n \rangle)$. Since the lexicographic ordering is a well-ordering when each coordinate is, for all $\beta < \alpha$ the set of finite tuples of elements of L_β is well-ordered by the induction hypothesis. The lexicographic ordering is then a well-ordering of the set of the 3-tuples associated to elements of L_α, which gives us a well-ordering of L_α, as desired. $\square$

Remark. The preceding proof generalizes to any model in which every set is definable using only parameters that can be well-ordered.

To further investigate L, it is useful to know that statements like $V = L$ are expressible in the language of set theory, and to know the Lévy complexity of the statements involved. The next lemma follows directly from formalizing the notions involved in the definition of L (definable power set, union, ordinal, function) in the language of set theory; for full details, once again the reader is referred to [Kun80], Chapter V.

Lemma 3.9. *There is a Σ_1 formula $\mathrm{LVL}(x, \alpha)$ in the language of set theory that expresses "α is an ordinal and $x = L_\alpha$." Let the Axiom of Constructibility, written $V = L$, be the following sentence:*

$$(\forall y)(\exists x)(\exists \alpha)(\mathrm{LVL}(x, \alpha) \wedge y \in x).$$

This is a Π_2 sentence that expresses that every set is in the Constructible Universe (that is, for every y, there is an α such that $y \in L_\alpha$).

The main use of Lemma 3.9 is the following proposition, which gives the invariance of L across transitive models, in particular forcing extensions. Thus, when talking about L, our "vantage point" doesn't matter; we are always talking about the same class.

Proposition 3.10. *L is absolute across transitive models of set theory: If $N \subseteq M$ is an inner model, then $L^N = L^M$. In particular, for every inner model M of V, $L^M = L$.*

Proof. By Lemma 3.9, for all $\alpha \in \mathrm{On}$, the formula that expresses "$x = L_\alpha$" is upwards-absolute. Since $L = \bigcup_\alpha L_\alpha$, if N and M disagree on L, then they disagree on some L_α. But, by the previous, whatever set N thinks is L_α, M must also think is L_α, so disagreement is impossible. □

Corollary 3.10.1.

1. *L is the least inner model.*

2. *If $M[G]$ is a nontrivial generic extension of M, then $M[G] \not\models V = L$.*

3. *If $V = L$ then V is minimal in the Generic Multiverse.*

Proof. (1) Let $N \subseteq V$ be an inner model. We need to show that $L \subseteq N$. Since each L_α can be proven to exist from the ZF axioms alone by the above construction, for all $\alpha \in \mathrm{On}$ there is a set L_α^N such that $N \models \mathrm{LVL}(L_\alpha^N, \alpha)$; thus, there is a class $L^N = \bigcup_\alpha L_\alpha^N \subseteq N$. By Proposition 3.10, $L^N = L$.
(2) By Proposition 3.10, $L^{M[G]} = L^M$. But since $M[G]$ is nontrivial, there is something in $M[G]$ that is not in M, in particular not in $L^{M[G]}$. Thus, $M[G] \not\models V = L$.
(3) This follows directly from (1). □

This last corollary makes L very appealing from the perspective of solving the independence problem. It tells us that the theory $\mathrm{ZFC} + V = L$ is immune to forcing, in the following way. Suppose we want to show φ is independent of

$\text{ZFC} + V = L$; to do so, we need to show that both $\text{ZFC} + V = L$ and $\text{ZFC} + V = L + \neg\varphi$ are consistent. The usual way to show either of these is to start from an arbitrary model M of $\text{ZFC} + V = L$, and then use forcing to construct $M[G]$ such that, say, $M[G] \models \varphi$. However, by Corollary 3.10.1, $M[G] \not\models V = L$, and so we fail to get our consistency result and to show independence. Thus, by the discussion in Section 2.2, adopting $V = L$ is a major step toward solving the independence problem. Combined with large cardinals, this solves the independence problem completely.

However, it is widely believed that $V = L$ is false. While we will return to this topic in Section 4.3, we discuss a major reason here: Incompatibility with large cardinals. $V = L$ implies that all levels of the consistency strength hierarchy after a relatively low level are inconsistent.[23] Of the large cardinal notions we have defined, the weakest that is incompatible with Constructibility is the measurable cardinal, due to Dana Scott.

Theorem 3.11 (Scott's Theorem)**.** *Suppose there is a measurable cardinal. Then* $V \neq L$.[24]

Proof. Suppose not toward contradiction. Let κ be the least measurable cardinal, and let $j : V \to M$ be an elementary embedding with critical point κ. Since M is an inner model and L is the least inner model,

$$L \subseteq M \subseteq V = L.$$

Thus, j is a nontrivial elementary embedding from L to L. Since $L \models$ "κ is the least measurable cardinal," by elementarity $L \models$ "$j(\kappa)$ is the least measurable cardinal." But $\kappa < j(\kappa)$, a contradiction. □

[23]This level is above inaccessibility, hence the earlier remark on the smallness of inaccessibles.
[24][Kan09, p. 49]

Because of Theorem 3.11, the evidence for strong large cardinal axioms is now evidence for $V \neq L$, and so we reject Constructibility. This leaves us with no progress since the beginning of this section; however, there is hope. Throughout the rest of this chapter, we will consider other inner models, including generalizations of L, that potentially could capture the desired features of L while fixing its shortcomings.

3.2.3 Other Inner Models

In this section, we will go over some other inner model theoretic constructions. These constructions will be useful later. However, none of them will give a new axiom capable of solving independence.

First, we consider ordinal definability. The Constructible Universe consisted of sets that were definable in a very strict sense; considering ordinal definability allows us to expand L while retaining this basic idea. However, there is a caveat: we seek a transitive class, and we can't guarantee that every element of an ordinal definable set is ordinal definable. Thus, we restrict our attention to when this does hold.

Definition 3.11. A set is *ordinal definable* iff there is a formula φ and ordinals $\alpha_1, ..., \alpha_n$ such that $y \in x$ iff $\varphi(y, \alpha_1, ..., \alpha_n)$.[25] The class HOD is the class of *hereditarily ordinal definable* sets, that is, the class of sets x such that for all $y \in \text{tc}(\{x\})$, y is ordinal definable.

We check that HOD is an inner model.

Proposition 3.12. HOD *is an inner model of* ZFC.

Proof. By construction, HOD is transitive, and clearly contains all the ordinals. Let G be a Gödel operation and $a, b \in$ HOD. G, as the composition of functions

[25]There is a subtlety with this definition. It requires being able to say whether an arbitrary formula is true, which is impossible by Tarski's Theorem. However, by the Reflection Theorem, for sufficiently large β we can say $y \in x$ iff $V_\beta \models \varphi(y, \alpha_1, ..., \alpha_n)$ to avoid this issue.

expressible by formulas, is expressible by a formula, and so $G(a, b)$ is a set definable with parameters that are in HOD. Combining with the definitions for a and b gives a definition for $G(a, b)$ from ordinal parameters. To see that HOD is almost universal, since every subset of HOD is a subset of $V_\alpha \cap \mathrm{HOD}$ for some α, it suffices to check that for all α, $V_\alpha \cap \mathrm{HOD}$ is hereditarily ordinal definable. We can define it by a formula expressing "x has rank less than α and every element of the transitive closure of $\{x\}$ is ordinal definable."

Notice that every element of HOD is determined by a formula and ordinal paramters (and the level of the rank hierarchy at which we evaluate the formula). By the remark after Lemma 3.8, we have that the Well-Ordering Theorem (and so the Axiom of Choice) holds in HOD. □

As we will see, $V = \mathrm{HOD}$ is not inconsistent with strong larger cardinal axioms.[26] However, it has a limitation that does $V = L$ does not: HOD is not forcing invariant. In fact, any set can be forced into HOD.[27] Thus, the axiom $V = \mathrm{HOD}$ does little work in solving the independence problem. We will see later on, however, that a successful new axiom will make use of being contained in HOD.

We will need the following lemma about the HOD of a weakly homogeneous forcing extension (see Definition 2.7).

Lemma 3.13. *Suppose $\mathbb{P}$ is a weakly homogeneous forcing notion, and $G \subseteq \mathbb{P}$ is generic. Then* $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}$.

Proof. It suffices by Proposition 3.7 to show that every set of ordinals in $\mathrm{HOD}^{V[G]}$ is in HOD. Let S be such a set, defined by the formula $\varphi(u, v_1, ..., v_n)$ with ordinal parameters $\beta_1, ..., \beta_n$. Now $\alpha \in S$ iff $\varphi(\alpha, \beta_1, ..., \beta_n)$, and by the Fundamental Theorem of Forcing, this is equivalent to there being a $p \in G$ such that

[26] See Section 4.2.

[27] Doing this goes beyond the forcing machinery we have developed in this thesis. The sketch is to code the set as a set of ordinals $X \subseteq \alpha$, and to use Easton forcing to produce a model where, say, $2^{\aleph_{\beta+1}} > \aleph_{\beta+2}$ iff $\beta \in X$. This gives a definition for X with only an ordinal parameter.

$p \Vdash \varphi(\check{\alpha}, \check{\beta}_1, ..., \check{\beta}_n)$. By Lemma 2.6, this is equivalent to $1_{\mathbb{P}} \Vdash \varphi(\check{\alpha}, \check{\beta}_1, ..., \check{\beta}_n)$. By the definability of the forcing relation, this last statement is expressible by a formula with only ordinal parameters; thus, S is ordinal definable. As the same argument works for every subset of S, $S \in \mathrm{HOD}$, as desired. $\square$

We turn next to generalizations of L. These generalizations allow us to consider L, but expanded to include "missing" sets, such as those that witness large cardinal properties. Consider the following method: Simply add a desired set to L.

Definition 3.12. Let A be a transitive set. We define $L(A)$ as follows.

$$\begin{aligned}
L_0(A) &= A \\
L_{\alpha+1}(A) &= \mathrm{Def}(L_\alpha(A)) \\
L_\lambda(A) &= \bigcup_{\alpha<\lambda} L_\alpha(A) \quad \text{(for } \lambda \text{ a limit ordinal)} \\
L(A) &= \bigcup_{\alpha\in\mathrm{On}} L_\alpha(A)
\end{aligned}$$

Using $L(A)$, we can define extensions of arbitrary transitive models.

Definition 3.13. Suppose M is a transitive model of ZFC, and A a set. We define $M(A)$, the smallest transitive model of ZFC that contains M and has A as a member, by

$$M(A) = \bigcup_{\alpha\in\mathrm{On}} L(M_\alpha \cup \mathrm{tc}(\{A\})).$$

Checking that both of these constructions satisfy ZF is another straightforward use of the Gödel operations.

For our purposes, this method of generalization of L will not be of much use. In some prominent cases, $L(A)$ has a well-understood structure theory. However, it is best understood as a *proper* inner model, in particular, a proper inner model in which the Axiom of Choice fails. In the next section, we will turn to a more fruitful method of generalizing L.

3.2.4 Relative Constructibility

Recall that the problem with $V = L$ is that a measurable cardinal refutes it. This suggests that adjusting L to accommodate a measurable will fix the problem. We do this considering the class of sets *constructible relative to* U, for a κ-complete nonprincipal ultrafilter U witnessing the measurability of a cardinal κ. This material is primarily motivational, and we will skip quickly through it. Our primary point here is that L can be modified to accommodate large cardinals, and that such modifications have been failures in the past.

Definition 3.14. Let A be a set. Suppose M is a transitive set. The *definable power set of* M *with respect to* A, denoted $\mathrm{Def}_A(M)$, is the set of all subsets of M that are definable in the structure $(M, \in, A \cap M)$.[28] That is, we have $x \in \mathrm{Def}_A(M)$ iff there is a formula φ and parameters $a_1, ..., a_n \in M$ such that

$$x = \{y \in M \mid (M, \in, A \cap M) \models \varphi(y, a_1, ..., a_n)\}.$$

We define the class of sets *constructible relative to* A, denoted $L[A]$, by recursion:A

$$\begin{aligned} L_0[A] &= \emptyset \\ L_{\alpha+1}[A] &= \mathrm{Def}_U(L_\alpha[A]) \\ L_\lambda[A] &= \bigcup_{\alpha<\lambda} L_\alpha[A] \quad \text{(for } \lambda \text{ a limit ordinal.)} \\ L[A] &= \bigcup_{\alpha\in\mathrm{On}} L_\alpha[A] \end{aligned}$$

Let κ be a measurable cardinal, and let U be a κ-complete nonprincipal ultrafilter on κ. We are particularly interested in $L[U]$. The following lemma captures the basic features of $L[U]$.

[28]The signature is that of the language of set theory with an additional unary predicate $\dot{A}$ whose intended interpretation is $A \cap M$.

Lemma 3.14.

1. *$L[U] \models \mathrm{ZF}$*

2. *The Well-Ordering Theorem (and so the Axiom of Choice) holds in $L[U]$.*

3. *$U \cap L[U] \in L[U]$ (Amenability).*

4. *$U \cap L[U]$ is a κ-complete nonprincipal ultrafilter on κ in $L[U]$, and so $L[U] \models$ "κ is a measurable cardinal."*

5. *$L[U] \models$ "κ is the only measurable cardinal."*[29]

$L[U]$ functions very similarly to L, and in fact has a rich structure theory, which we will not go into.[30] This structure theory is a major part of what justifies belief in measurable cardinals, as discussed in Section 3.1. However, the key is property (5): There is only one measurable cardinal in $L[U]$. Thus, $L[U]$ implies that the consistency strength hierarchy is inconsistent at levels whose strength is at least that of "There are two measurable cardinals," faring not much better than L. This pattern continues through inner model theory. A break in this pattern appears in the inner model theory for supercompact cardinals, which we discuss next chapter.

3.3 Recap

In this chapter, we have explored some of the approaches to new axioms. We discussed the approach via large cardinal axioms, which is fruitful and well-justified but not strong enough. We explored the potential of inner model theory, but saw conflicts with large cardinal axioms. With compelling reasons to accept large cardinals, we must take the inner model theoretic axioms to be failures. In Chapter 4, we

[29] See [Kan09, p. 261] for more details.
[30] The reader is referred to [Kan09] Chapter 4 for details.

will explore a potential new axiom that corrects the failures of inner model theory up to this point.

4 Ultimate L

In this chapter, we will discuss a recent candidate for a new axiom, due to Hugh Woodin.[Woo01] This new axiom candidate comes as a continuation to the ideas of inner model theory that we discussed in Section 3.2. As we saw, the axiom $V = L$ solves the independence problem as it arises from forcing. Unfortunately, measurable cardinals show that $V = L$ is false. While adjusting L to accommodate a measurable cardinal does not work, a generalization of L could work: The generalization of L to accommodate one supercompact cardinal. This generalization is called Ultimate L. The corresponding axiom candidate is then $V =$ Ultimate L.

The construction of an L-like inner model for a supercompact cardinal is currently unknown, and so it is only conjecture that there is an inner model for one supercompact cardinal. However, many properties of such an inner model have been examined, and many constraints placed. Thus, it is already possible to discuss the consequences of such an axiom, as well as the case for it.

In Section 4.1, we will give the relevant definitions to state the new axiom. We discuss the central ideas of the theory in 4.2, in particular proving that Ultimate L is minimal in the Generic Multiverse and that it is compatible with large cardinals. In Section 4.3, we will discuss some further philosophical considerations, in particular defending our approach to a new axiom by "fixing" L.

It should be noted there is a large theory surrounding Ultimate L, with many separate but important threads. It would be impossible to cover them all here, even devoting ourselves solely to that task. Rather, we will only develop the themes that have run throughout this thesis, large cardinals to solve consistency strength issues and minimality in the Generic Multiverse to solve forcing. Furthermore, the development of these themes here will not be complete. In some places we opt for suboptimal results to avoid the burden of introducing at length new concepts, and we black box some lemmas for the same reason. We hope that because of this the reader is able to see and appreciate the larger picture that can otherwise easily be lost.

4.1 The Axiom $V =$ Ultimate L

In this section, we will state what is conjectured to be the axiom, and a fundamental conjecture relating to it. We start by isolating a class of inner models with properties we expect of an inner model that accommodates a supercompact cardinal. These properties have analogues in $L[U]$.

Definition 4.1. Let $N \subseteq V$ be an inner model and κ a cardinal. We say that N is a *weak extender model* for κ supercompact iff for all $\gamma > \kappa$ there exists a κ-complete normal fine ultrafilter $U \subseteq P_\kappa(\gamma)$ such that

1. $P_\kappa(\gamma) \cap N \in U$ (Concentration)

2. $U \cap N \in N$ (Amenability).

Concentration holds for $L[U]$ since the ambient space on which the ultrafilters live is κ, and amenability holds for $L[U]$ by Lemma 3.14. Intuitively, such models contain everything necessary to witness the supercompactness of κ: They have the ultrafilters (though restricted to only the parts that are actually in N) and the ambient spaces (again properly restricted) are actually measure 1 in these ultrafilters.

To obtain the axiom, we need to define a model that, like $L[U]$, generalizes L, and is a weak extender model for a supercompact cardinal. We need a specific structure theory like that of L and $L[U]$; after all, if there is a supercompact cardinal

κ, then V is trivially a weak extender model for κ supercompact. However, the construction of such an inner model is currently an open question.

While the definition of the model is unknown, there is a schema that is conjectured to be equivalent to the desired axiom. The schema is formulated using methods from the theory surrounding AD^+, a strengthening of the Axiom of Determinacy, and its connections to HOD. To sufficiently develop the theory to properly understand this formulation is far beyond the scope of this thesis, but we state the schema here for interest.

Definition 4.2. The axiom $V = \text{Ultimate } L$ consists of the following two clauses:

1. There is a proper class of Woodin cardinals.

2. For each true Σ_2 sentence φ: There is a universally Baire set $A \subseteq \mathbb{R}$ such that $\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi$.[31]

Conjecture (The Ultimate L Conjecture)**.** *Suppose κ is an extendible cardinal. Then there exists a (Σ_2 definable) weak extender model N for κ supercompact such that $N \subseteq \mathrm{HOD}$ and $N \models V = \text{Ultimate } L$.*

4.2 Central Features of Ultimate L

We will investigate the basics of the theory around Ultimate L. While many there are many threads of importance to the theory, we will focus on two main ideas to give the reader a digestible introduction while matching the themes of the current paper. First, we will establish that Ultimate L is minimal in the Generic Multiverse. Second, we will show that it is compatible with all large cardinal axioms, inheriting the large cardinals that exist in V.

We start by proving that Ultimate L is minimal in the Generic Multiverse,

[31]This is the version of the schema given in [Woo01, p. 111]

just as L. To do this, we will need the following fact that follows from the interactions between Ultimate L and AD^+.

Lemma 4.1. *Suppose $V =$ Ultimate L. If $V \subseteq_{\mathrm{gd}} V[G]$, then $V \subseteq \mathrm{HOD}^{V[G]}$.*[32]

Remark. This immediately implies that $V = \mathrm{HOD}$, by taking a trivial forcing. There is in fact a very close relationship between Ultimate L and HOD suggested by this and the approach to Ultimate L via $\mathrm{HOD}^{L(A,\mathbb{R})}$ that we will not explore further. See, for example, [WDR12] for details.

We now prove the desired result.

Theorem 4.2. *Suppose $V =$ Ultimate L. Then V is minimal in the Generic Multiverse.* [33]

Proof. We use a lemma that we will prove in Section 5.3 after having proven the Downward Directed Grounds Hypothesis.

Lemma. *Let $\{W_i \mid i \in I\}$ be the collection of grounds of V. A model is in the Generic Multiverse iff it is of the form $W_i[G]$ for $G \subseteq \mathbb{P} \in W_i$ generic; that is, the Generic Multiverse consists of the generic extensions of the grounds of V.*

Fix an arbitrary ground W_i, and a generic $G \subseteq \mathbb{P} \in W_i$ such that $W_i[G] = V$. We will show that $V \subseteq W_i$. Fix a cardinal $\gamma > |\mathbb{P}|$. By Lemma 2.9, there is a dense embedding $f \in W_i$ from $\mathbb{P} \times \mathrm{Coll}(\omega, \gamma)$ to $\mathrm{Coll}(\omega, \gamma)$, and so by Lemma 2.8 forcing from V to collapse γ is the same as forcing from W_i with this product. Let $H \subseteq \mathrm{Coll}(\omega, \gamma)$ be generic. By the previous, we have

$$V[H] = W_i[G][H] = W_i[H].$$

It follows that

$$\mathrm{HOD}^{V[H]} = \mathrm{HOD}^{W_i[G][H]} = \mathrm{HOD}^{W_i[H]}.$$

[32][Woo01, p. 117]
[33][Woo01, p. 118]

Since $\mathrm{Coll}(\omega, \gamma)$ is weakly homogeneous, by Lemma 3.13, $\mathrm{HOD}^{W_i[H]} \subseteq W_i$. To finish, by Lemma 4.1, $V \subseteq \mathrm{HOD}^{V[H]}$. Thus, $V \subseteq W_i$, as desired. □

This tells us that Ultimate L is immune to forcing, for no nontrivial forcing extension can satisfy that it is minimal in the Generic Multiverse. Furthermore, we get immunity to forcing strongly by making Ultimate L a privileged point in the Generic Multiverse. Thus, a full solution to the independence problem would be constituted by Ultimate L and large cardinal axioms.

In the previous chapter, at this point a tension between an axiom and large cardinals would cause the picture to break down. However, here we have a new phenomenon: Ultimate L is compatible with large cardinals. The theorem that allows this is the Universality Theorem, which is our next goal. First, we need state without proof the following structural lemma on weak extender models.

Lemma 4.3. *Suppose N is a weak extender model for κ supercompact. Then for each $\lambda > \delta$ and each $a \in V_\lambda$ there exist $\overline{\kappa} < \overline{\lambda} < \kappa$ and $\overline{a} \in V_{\overline{\lambda}}$ and an elementary embedding $j : V_{\overline{\lambda}+1} \to V_{\lambda+1}$ such that*

1. $\mathrm{CRT}(j) = \overline{\kappa}$, $j(\overline{\kappa}) = \kappa$, *and* $j(\overline{a}) = a$
2. $j(N_{\overline{\lambda}}) = N_\lambda$, *and*
3. $j \restriction (N_{\overline{\lambda}}) \in N$.[34]

We can now prove the Universality Theorem. There is a more general form of this theorem that makes use of *extenders*, objects that code sequences of ultrafilters for larger cardinals. However, for our purposes, understanding this version is enough.

[34][Woo01, p. 9]

Theorem 4.4 (The Universality Theorem)**.** *Suppose N is a weak extender model for κ supercompact and $\gamma > \kappa$ is a cardinal in N. Suppose that $j : H^N(\gamma^+) \to H^N(j(\gamma^+))$ is an elementary embedding with $\kappa \leq \mathrm{CRT}(j)$. Then $j \in N$.*[35]

Proof. Let $\lambda > j(\gamma)$ be strong limit. We use Lemma 4.3 with $a = j$. Thus, there exist $\overline{\kappa} < \overline{\lambda} < \kappa$; $\overline{j} \in V_{\overline{\lambda}}$ such that for some $\overline{\gamma}$, $\overline{j}$ is an elementary embedding from $H^N(\overline{\gamma}^+)$ to $H^N(\overline{j}(\overline{\gamma})^+)$; and an elementary embedding $i : V_{\overline{\lambda}+1} \to V_{\lambda+1}$ such that

1. $\mathrm{CRT}(i) = \overline{\kappa}$, $i(\overline{\kappa}) = \kappa$, $i(\overline{j}) = j$, and $i(\overline{\gamma}) = \gamma$;
2. $i(N_{\overline{\lambda}}) = N_\lambda$; and
3. $i \restriction (N_{\overline{\lambda}}) \in N$.

Since $i(\overline{j}) = j$ and $i \restriction (N_{\overline{\lambda}}) \in N$, it suffices to show that $\overline{j} \in N$. We show that the relation $(x, y) \in \overline{j}$ is definable in N. Fix $x \in H^N(\overline{\gamma}^+)$ and $y \in H^N(\overline{j}(\overline{\gamma})^+)$. We have, by elementarity of i,

$$\begin{aligned} (x, y) \in \overline{j} &\iff (i(x), i(y)) \in i(\overline{j}) \\ &\iff (i(x), i(y)) \in j. \end{aligned}$$

Thus, it suffices to compute $j(i(x))$ in N. Since $x \in H^N(\overline{\gamma}^+)$, we only need to consider $i^* := i \restriction H^N(\overline{\gamma}^+) \in N$. By elementarity of j,

$$j(i^*(x)) = j(i^*)(j(x)).$$

Notice that

$$H^N(\overline{\gamma}^+) \subseteq V_{\overline{\gamma}^+} \subseteq N_{\overline{\lambda}}.$$

Since $\overline{\lambda} < \kappa$, j is the identity on $H^N(\overline{\gamma}^+)$, so $j(x) = x$. We are now done, since $j(i^*)$ is a function in N, so N can compute $j(i^*)(x)$, as desired. $\square$

[35][Woo01, p. 11]

Since N absorbs elementary embeddings, and large cardinals are defined in terms of elementary embeddings, N inherits large cardinals:

Corollary 4.4.1. *Let $N \subseteq V$ be a weak extender model for κ supercompact. If $\gamma \geq \kappa$ is measurable, then (γ is measurable)^N.*

Proof. Let γ be measurable, and let $j : V \to M$ be an elementary embedding with critical point γ. We consider j locally: Fix $\lambda > \gamma$, so that j is an elementary embedding from $V_{\lambda+1}$ to $V^M_{j(\lambda)+1}$. By restricting the domain, j is an elementary embedding from $N_{\lambda+1}$ to $N_{j(\lambda)+1} \cap M$. Since $N_{\lambda+1}$ and $H^N(|N_\lambda|^+)$ are mutually interpretable, j can be encoded as an elementary embedding $\hat{j}$ from $H^N(|N_\lambda|^+)$ to $H^N(j(|N_\lambda|)^+)$. By Theorem 4.4, $\hat{j} \in N$, and by undoing the coding, $j \in N$. Thus, γ is the critical point of a nontrivial elementary embedding in N, as desired. □

This theorem tells us that Ultimate L succeeds where other inner models fail: Ultimate L is compatible with large cardinals. In fact, we have something stronger: Ultimate L inherits the large cardinals that exist in V. This agreement suggests that Ultimate L is close to V, giving evidence for $V =$ Ultimate L.

One might point to another tension between Ultimate L and large cardinals. Recall that part of the reason for believing in measurable cardinals is the analysis of the inner model $L[U]$, the minimal model that contains a measurable cardinal. If Ultimate L does exist, then it is such an inner model for every large cardinal hypothesis stronger than the existence of one supercompact cardinal, *assuming* those large cardinal hypotheses actually obtain. But then, Ultimate L is agnostic toward stronger large cardinal hypotheses: It exists regardless, and is an inner model for them if they obtain, but is not an inner model for them if they do not obtain. Thus, the inner model theory for large cardinal hypotheses stronger than the existence of one supercompact cardinal has no bearing on their truth.

The counter to this is that it is not inner model theory in itself that provides evidence for the consistency and existence of a measurable cardinal. It is structure theory, the building of a coherent picture out of the theory $\mathrm{ZFC} + A$ for a large cardinal axiom A, that gives us reason to accept $\mathrm{ZFC} + A$. That Ultimate L trivializes inner model theory above supercompact cardinals just leads us to seek different consequences of large cardinal axioms to justify these axioms. Thus, we have resolved the apparent tension between Ultimate L and large cardinals, and the combination of the two stand as a solution to the independence problem.

4.3 Further Philosophical Considerations

We have seen that Ultimate L is a very desirable inner model. The potential of having forcing invariance via minimality in the Generic Multiverse alongside all the large cardinals (that actually exist) makes the axiom $V =$ Ultimate L very appealing. However, we still need to consider whether Ultimate L is in fact true. There are many arguments in the literature already for the truth of the potential axiom.[36] We will add to this case in Chapter 5, by discussing a convergent phenomenon in set-theoretic geology. In the present section, we will resolve a potential counterargument against the Ultimate L project.

The motivation for Ultimate L is that L is nice, just incompatible with large cardinals. Ultimate L is tailor-made to resolve this. However, if the main problem with L is not large cardinals, if there are other, more serious issues with the axiom $V = L$, then one may worry that Ultimate L does nothing to address these issues. Moreover, one may worry that its similarity with L might actually lead it to fall to the same problems. We will discuss two views on the problems of Constructibility besides large cardinals that prima facie also count against Ultimate L, and argue that neither actually poses a threat.

The first view we will discuss regards definability. Recall that if $V =$

[36] See [WDR12] and [Woo01].

Ultimate L then $V = \text{HOD}$ by Lemma 4.1. In other words, every set in Ultimate L is definable from ordinals, similar to how every set in L is definable from the lower stages of L. Many see this implication of Constructibility as the main reason to reject it, such as Yiannis Moschovakis:

> The key argument against accepting $V = L$... is that the Axiom of Constructibility appears to restrict unduly the notion of *arbitrary* set of integers; there is no a priori reason why every subset of ω should be definable from ordinal parameters, mush [sic] less by an elementary definition over some countable L_ξ. [Mos09, p. 472]

Notice that Moschovakis dismisses the possibility of ordinal definability for the same reason as Constructibility. Thus, Moschovakis must reject $V =$ Ultimate L on these grounds, and if his argument is correct, the case for the axiom candidate is indeed in trouble.

Let us better understand the ground of this the argument. There is a hint of a restrictiveness argument that we will address later. However, the main line of thought here seems to be that there is "a priori" reason against the definability of all sets of natural numbers. Moschovakis later writes that $V = L$ is unlike determinacy in that there is a "direct intuition" against it.[Mos09, p. 472] Thus, Moschovakis' argument is grounded in some prior belief that "All sets are definable" is false, or at least implausible.

However, such a belief should be overturned in the face of mathematical evidence. There are many mathematical statements that seem implausible, but must be accepted due to mathematical evidence, such as the Banach-Tarski paradox, or the many notorious counterexamples in analysis and topology. One could counter that there is a distinction; in those cases, the unintuitive results are theorems of theories that we accept. However, the distinction then seems to be that the mathematical evidence entails the unintuitive result, whereas in the case of interest one can consistently believe the facts cited as evidence and the intuition (one can accept that the facts cited in favor of $V =$ Ultimate L are theorems of whichever theory, we can in fact accept that theory and so accept that those theorems are true, while denying that $V =$ Ultimate L). However, to adopt such a position would be to deny that these facts are compelling evidence. We rest our case on the prima facie reasonableness of accepting such facts as evidence.

One might counter that Moschovakis' argument does not come from pure intuitions, but rather from intuitions grounded in other mathematical facts, namely the success of large cardinals in the face of Constructibility. However, since ordinal definability is not in conflict with these facts, his full statement is not licensed by them.

Let us turn to another form of argument against Constructibility. This argument is hinted at in the Moschovakis quotation above. It is the idea that $V = L$ is restrictive, and we should be maximizing our theory of sets. We will discuss Penelope Maddy's work as representative of this view.

For Maddy, large cardinals are not the reason to reject Constructibility. Rather, she seeks evidence for large cardinals, and finds it partially in the fact that they recover the true statement $V \neq L$. We can't have it both ways, however, on pain of circularity.[Mad97, p. 110] Thus, she seeks an argument that will not rely on large cardinals in order to argue against Constructibility.

We summarize the argument against Constructibility that Maddy does endorse, as she discusses in [Mad97, p. 210]. One of set theory's roles is foundational. As a foundational discipline, set theory needs to be able to interpret other domains of mathematics. Mathematicians should be allowed to study whatever objects they so choose; therefore, a major goal for a foundation of mathematics should be to maximize the possible objects of study. But, as L is the minimal inner model, there will be sets that cannot be in L.[37] Thus, one should reject Constructibility.

We will present two responses. First, we shall argue that $V \neq L$ is not to be taken as evidence for large cardinal axioms, as Maddy suggests, vindicating our approach of justifying $V \neq L$ via large cardinals and the subsequent move to Ultimate L as a solution. Second, we argue that the argument that Maddy uses against $V = L$ is not a problem for $V =$ Ultimate L, to allay any lingering fears.

We claim that $V \neq L$ is not strong evidence for large cardinals axioms. First, we note that since a measurable suffices to recover $V \neq L$, it is unclear why this would be evidence for the stronger large cardinal axioms. This seems to be an extrapolation beyond what is warranted. Thus, the recovery of $V \neq L$ could at best justify an extension of ZFC of strength close to a measurable. Second, implicit in this argument is the evidence we have already discussed for measurable cardinals.[38] For Maddy points out that her argument carries no weight if one does not believe that the existence of a measurable cardinal is consistent with ZFC, and the other pieces of evidence we have discussed are what support this.[Mad97, p. 216] But, in the discussion of Section 3.1, this evidence was taken as sufficient to establish the existence of measurable cardinals. Thus, the recovery of $V \neq L$ plays no real role in the justification of even measurable cardinals, and so it plays no role in the justification of any large cardinals, solving Maddy's circularity worry.

Next, we claim that a methodological preference for maximizing theories does not refute $V =$ Ultimate L. In order to argue this, we need to clarify what is meant by "maximize." To Maddy, maximization means maximizing the available isomorphism types, which maximizes the number of possible structures that are captured and made available for mathematical study.[Mad97, p. 211] The isomorphism type that L misses is essentially a code for an elementary embedding;

[37] For example, a set coding an elementary embedding from L to L, as in the proof of Theorem 3.11, cannot be in L.

[38] Maddy actually discusses the weaker but closely related assumption that $0^{\#}$ exists.

however, by Theorem 4.4, Ultimate L cannot be missing such an elementary embedding.[39] This suggests that Ultimate L will maximize in this direction. One might attempt to revise the sense of maximization here. John Steel proposes another plausible way of understanding maximization.[Ste14, p. 154] Shifting emphasis away from the objects that mathematicians deal with to the theories from which they prove results, we see that one should attempt to maximize the interpretative power of a foundation, in order to be able to interpret any mathematical theory in the foundation. However, every natural theory can be interpreted by some large cardinal axiom, and so $V =$ Ultimate L does not fail here. Either way, maximization does not pose a problem for $V =$ Ultimate L.

To summarize, we have defended the Ultimate L project against worries that the whole story as we've told it is incorrect, and that the issue is not generalizing L to be compatible with large cardinals. The story holds, and large cardinals must be the reason for $V \neq L$, not the other way around. The argument from anti-definability is at best too weak to refute $V =$ Ultimate L, and the argument from maximization does not pose a threat to $V =$ Ultimate L, for the axiom is maximizing.

4.4 Recap

We have described a potential new axiom that can solve the independence problem, $V =$ Ultimate L. If the Ultimate L conjecture and other related conjectures turn out to be true, then $V =$ Ultimate L will give us immunity to forcing and compatibility with the large cardinals that grant immunity to consistency strength issues. That Ultimate L absorbs large cardinals present in V is already suggestive of the truth of $V =$ Ultimate L. More reasons are given across the literature, such as in

[39] The analogy between L and Ultimate L runs a lot deeper, and involves the closeness of HOD to V. The reader is referred to [WDR12] and [Woo01] for more details.

[Woo01]. We will discuss a new piece of evidence in the next chapter that arises from considerations in set-theoretic geology.

5 Set-theoretic Geology

In this chapter, we will discuss new developments in set-theoretic geology that point toward $V =$ Ultimate L. Recent results in this field indicate that, assuming sufficient large cardinals, there is an important inner model that inherits large cardinals, and furthermore is a privileged point in the Generic Multiverse. These results are consequences of a theorem called the (Strong) Downward Directed Grounds Hypothesis (DDG). In this chapter, we will prove this theorem and discuss these results.

5.1 Preliminaries

We will be concerned in this chapter with *set-theoretic geology*. Recall that forcing is naturally a method for producing outer models. Set-theoretic geology takes the opposite perspective: Starting in V, what can we say about its grounds? Another way to put this is that geology is the study of the lower part of the Generic Multiverse. Recall that a minimal element of the Generic Multiverse could play a special role regarding arguments about set-theoretic truth (Section 2.2.3). A natural question for geology is then whether the Generic Multiverse has a minimal element, and whether it is unique. One may also ask whether V is that minimal element, that is, whether V has proper grounds.

One will note that we are quantifying freely over proper classes ("Does there exist a proper class model that is a proper subclass of V and V is a forcing extension of it?"). One may worry about whether this is formalizable. Even if one takes the countable transitive models approach, we would like to be able to work from within a model. Due to the following theorem, this is possible.

Theorem 5.1 (Ground Model Definability Theorem)**.** *There is a formula $\varphi(x, y)$ in the language of set theory such that if $W \subseteq_{\text{gd}} V$ then there exists $a \in W$ such that $W = \{x \mid \varphi(x, a)\}$.*[40]

Corollary 5.1.1. *There is a formula $\psi(x, y)$ in the language of set theory such that*

1. *Every a (in V) defines a class $W_a := \{x \mid \psi(x, a)\}$, and $a \in W_a$.*
2. *For all a, $W_a \subseteq_{\text{gd}} V$.*
3. *If $U \subseteq V$, then $U \subseteq_{\text{gd}} V$ iff there is an a such that $U = W_a$.*
4. *From ψ one can define a formula $F(x, y, z)$ such that $F(a, G, \mathbb{P})$ iff $\mathbb{P} \in W_a$ is a partial order, $G \subseteq \mathbb{P}$ is generic, and $W_a[G] = V$.*

From these formulas we can define all concepts of interest. Thus, we will continue to liberally use talk of proper class inner models.

While we will not go through the proof or calculation, there is one fact we will need from it. The parameter used in Theorem 5.1 is V_θ^W, for sufficiently large θ (in particular, $\mathbb{P} \in V_\theta^W$). This can be improved to $P^W(\delta)$ for any sufficiently large cardinal δ; it is sufficient for $\delta \geq \gamma^+$ where the ground model satisfies the γ-global covering property for V.[41] Since this definition fully determines W, we get the following:

Corollary 5.1.2. *Suppose $M, N \subseteq_{\text{gd}} V$. If M and N satisfy the γ-global covering property for V and $P^M(\gamma^+) = P^N(\gamma^+)$, then $M = N$.*[42]

When studying the grounds of V and looking for what lies beneath all of them, it is natural to consider their intersection:

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[40] This theorem and the corollary appear in [FHR14, pp. 2,5].

[41] In fact, the parameter can be improved even more with further assumptions. See [FHR14, p. 3].

[42] This formulation combines [FHR14, p. 3] and [Ham16, p. 30].

Definition 5.1. The *mantle*, denoted $\mathbb{M}$, is the intersection of all the grounds of V.

The mantle, as it will turn out, will be the inner model mentioned in the introduction to this chapter. To prove this, we will need the DDG, and so that is what we turn to next.

5.2 The Downward Directed Grounds Hypothesis

In this section, our goal will be to prove a recent theorem of Toshimichi Usuba [Usu16]:

Theorem 5.2 (Strong Downard Directed Grounds Hypothesis)**.** *Let I be a set. Let $\{W_i \mid i \in I\}$ be a set-sized collection of grounds of V. There is a common ground beneath all of the W_i; that is, there is a W such that for all i, $W \subseteq_{\text{gd}} W_i$.*

This theorem, the DDG, is the major tool needed to obtain the main results of this chapter. The key to this theorem is a theorem due to Lev Bukovský that allows one to characterize when a model is a ground of another without reference to a specific partial order. We will prove Bukovský's Theorem in Section 5.2.1. We then turn to the main argument for the DDG in Section 5.2.2.

5.2.1 Bukovský's Theorem

Bukovský's Theorem is remarkable in that it gives a necessary and sufficient condition for being a ground without reference to a specific partial order: The κ-global covering property (See Definition 2.6). This allows us to deal with grounds in great generality, making it possible to prove that $N \subseteq M$ is a ground while prima facie not having enough information to determine a partial order in N that actually gets all and only the sets in M.

As a demonstration of the use of the κ-global covering property, we prove the following lemma on clubs in inner models.

Lemma 5.3. *Suppose $N \subseteq M$ is an inner model, and κ is such that N satisfies the κ-global covering property for M. Let $C \subseteq \kappa$ be club. There is a club $D \in N$ such that $D \subseteq C$.*

Proof. Let $f \in M$ take an ordinal α to the least $\beta > \alpha$ such that $\beta \in C$. By the κ-global covering property, let G_f approximate f in N. Let $G : \kappa \to \kappa$ take α to $\sup G_f(\alpha)$, and for $n < \omega$ let $G^n(\alpha)$ be the result of applying G to α n times. Finally, we define $G^\infty(\alpha)$ to be $\sup_{n<\omega}\{G^n(\alpha) \mid n < \omega\}$. We claim that the closure of $\operatorname{im} G^\infty$ is club and contained in C. First we check that $\operatorname{im} G^\infty$ is unbounded in κ. Let $\alpha < \kappa$. We claim that $G^\infty(\alpha) > \alpha$. Since $f(\alpha) \in G_f(\alpha)$, we have

$$\alpha < f(\alpha) \leq \sup G_f(\alpha) = G(\alpha) \leq G^\infty(\alpha).$$

Next, we check that $\operatorname{im} G^\infty \subseteq C$. This is enough, since C will then contain any limit points of $\operatorname{im} G^\infty$ as C is closed. Let $\alpha < \kappa$; we show that $G^\infty(\alpha) \in C$. We show this by showing that $G^\infty(\alpha)$ is a limit of elements of C, using f. Since $f(\beta) > \beta$ for any $\beta < \kappa$, we have, for any $n < \omega$, $f(G^n(\alpha)) > G^n(\alpha)$. Thus,

$$\sup_{n<\omega}\{G^n(\alpha) \mid n < \omega\} \leq \sup_{n<\omega}\{f(G^n(\alpha)) \mid n < \omega\}.$$

Furthermore, by the above argument, for any $n < \omega$ we have $f(G^n(\alpha)) < G^{n+1}(\alpha)$. Thus,

$$\sup_{n<\omega}\{G^n(\alpha) \mid n < \omega\} \geq \sup_{n<\omega}\{f(G^n(\alpha)) \mid n < \omega\},$$

so

$$G^\infty(\alpha) = \sup_{n<\omega}\{f(G^n(\alpha)) \mid n < \omega\} \in C$$

as desired. □

If κ is regular, we can strengthen the κ-global covering property slightly by allowing the approximation to be a superset of the function output:

Lemma 5.4. *Suppose κ is regular. Let N satisfy the κ-global covering property for M. Then for every ordinal α and every function $f \in M$ with domain α such that for all $\beta < \alpha$, $|f(\beta)| < \kappa$, there is a function G_f such that*

1. *$G_f \in N$,*
2. *for all $\beta < \alpha$, $f(\beta) \subseteq G_f(\beta)$, and*
3. *for all $\beta < \alpha$, $|G_f(\beta)| < \kappa$.*

Proof. By taking bijections, we may assume that $\mathrm{im}(f)$ consists of sets of ordinals. For each β, we cover each element of $f(\beta)$ by an approximation of size less than κ, and use the regularity of κ and the smallness of all sets involved to keep the size of the union less than κ. For each β, enumerate $f(\beta)$ as $\{\gamma_\xi^\beta \mid \xi < \overline{\kappa}_\beta\}$, where $\overline{\kappa}_\beta < \kappa$. Since $f(\beta)$ might have a different size for each β, the enumerations have different lengths $\overline{\kappa}_\beta$. However, since there are $\alpha < \kappa$ many $f(\beta)$ to enumerate, and κ is regular, we can let $\overline{\kappa} = \sup_\beta \overline{\kappa}_\beta < \kappa$, and enumerate (possibly with repetition) everything using $\overline{\kappa}$. Let $\lambda > \sup \bigcup_{\beta<\alpha} f(\beta)$ be sufficiently large so as to contain all the ordinals involved.[43] Now, let $h_\xi : \alpha \to \lambda$ be defined by $h_\xi(\beta) = \gamma_\xi^\beta$, that is, h_ξ takes each β to the ξth element of $f(\beta)$. For each ξ, choose a witness to κ-global covering; that is, let $g_\xi \in N$ be such that $h_\xi(\beta) \in g_\xi(\beta)$ and $|g_\xi(\beta)| < \kappa$. Now define $G_f(\beta) := \bigcup_{\xi<\overline{\kappa}} g_\xi(\beta)$. Since κ is regular, $|G_f(\beta)| < \kappa$ as desired. □

The following lemma connects the κ-global covering property to κ-c.c. forcing:

Lemma 5.5. *Suppose $N \subseteq M$ is an inner model, and suppose N satisfies the κ-global covering property for M. Let $A \in M$ be a set of ordinals. Then there is a κ-c.c. Boolean algebra $\mathbb{B}$ and $G \subseteq \mathbb{B} \setminus \{0_{\mathbb{B}_T}\}$ generic such that $N(A) = N[G]$.*[44]

[43] The only reason to assume that $\mathrm{im}(f)$ consists of sets of ordinals and then to consider λ is to ensure that the codomain of f is a member of N.

[44] The proof we give here is based on the one given in [FFS16, p. 6].

Proof. We need to produce a Boolean algebra, given nothing except a set of ordinals A. While no Boolean algebra directly related to A sticks out, perhaps a well-known Boolean algebra can be adapted for this task. Consider the Boolean algebra of sentences in a first-order language. One can implement A by adding a unary predicate $\dot{A}$ to the language whose intended interpretation is A. However, the lack of infinite conjunctions and disjunctions can make models more difficult to control. To remedy this, we will consider an infinitary logic, and attempt the same scheme.

Definition 5.2. Let γ be a cardinal. The logic $\mathcal{L}_{\gamma,\omega}$ has the usual formation rules for classical first-order logic, except for the following: If $\overline{\gamma} < \gamma$ and $\{\varphi_\alpha \mid \alpha < \overline{\gamma}\}$ are formulas, then so are the disjunction $\bigvee_{\alpha<\overline{\gamma}} \varphi_\alpha$ and the conjunction $\bigwedge_{\alpha<\overline{\gamma}} \varphi_\alpha$.[45]

Fix a regular $\lambda \in N$ such that $\kappa < \lambda$ and $A \subseteq \lambda$. Working in N, we will define a theory in $\mathcal{L}_{\lambda^+,\omega}$ that will induce an appropriate Boolean algebra; the intended model of this theory will have universe λ, and interpret the unary predicate with A. With this in mind, we will work with the following signature. Let "$\dot{A}$" be a unary predicate (the intended interpretation being A). Let "$\dot{<}$" be a binary predicate (the intended interpretation being $\in$). Finally, let $\{c_\alpha \mid \alpha < \lambda\}$ be the set of constant symbols (the intended interpretation being the ordinals less than λ). The cardinality of the signature is λ, so by embedding it into λ we can "code" the symbols by ordinals less than λ. Let Sent be the set of sentences in this signature in N; that is, Sent is a set of sequences in N of length at most λ of ordinals coding symbols, which satisfy the rules of formation encoded appropriately. A theory in this signature is a set of sentences, that is, subset of Sent in N. We define structures and the satisfaction relation in the usual way, naturally insisting that infinitary conjunctions be true when all conjuncts are true, and infinitary disjunctions be true when at least

[45]For the curious, the second parameter in the logic, the ω, bounds the number of quantifiers allowed in a formula. We will not need infinitely many quantifiers, so this will be kept at ω.

one disjunct is true.

We want to actually talk about λ (up to isomorphism), not some nonstandard model, so we need to restrict the theories we look at to ones that can only be interpreted correctly. Let Good be the set of theories that contain the following sentences:

1. $(\forall x)\left(\bigvee_{\alpha<\lambda} x = c_\alpha\right)$ (The only elements of the model are the ordinals less than λ.)
2. $(\forall x)(\forall y)\left(x \mathrel{\dot{<}} y \equiv \bigvee_{\alpha<\beta<\lambda}(x = c_\alpha \wedge y = c_\beta)\right)$ ("$\dot{<}$" is interpreted by $\in$.)

Given a theory $T \in \mathrm{Good}$, we define an equivalence relation on Sent by setting $\varphi \sim_T \psi$ iff $T \models \varphi \equiv \psi$, that is, we set two sentences to be equivalent if T thinks they're equivalent. Let $\mathbb{B}_T := \mathrm{Sent}/\sim_T$. (For convenience, we will continue to write the elements of $\mathbb{B}_T$ as single sentences instead of equivalence classes.) In moving to $\mathbb{B}_T$ we identify sentences that are provably equivalent. $\mathbb{B}_T$ is a Boolean algebra in the natural way, with $\vee$ for join, $\wedge$ for meet, $\neg$ for complement, $\to$ for the ordering (taking the quotient by $\sim_T$ guarantees antisymmetry), $(\forall x)(x = x)$ for $1_{\mathbb{B}_T}$, and $(\exists x)(x \neq x)$ for $0_{\mathbb{B}_T}$. Furthermore, because we work in $\mathcal{L}_{\lambda^+,\omega}$, $\mathbb{B}_T$ is λ-complete.

If there is a $T \in \mathrm{Good}$ such that the following hold, then we're done:

1. $\langle \lambda, A, \in \rangle \models T$
2. $\mathbb{B}_T$ is κ-c.c.

To see this, let T satisfy (1) and (2). Let $G := \{\varphi \in \mathbb{B}_T \setminus \{0_{\mathbb{B}_T}\} \mid \langle \lambda, A, \in \rangle \models \varphi\}$. We claim that G is a generic filter. Clearly $1_{\mathbb{B}_T} \in G$. If $\varphi, \psi \in G$, then $\varphi \wedge \psi$ is as well, by how satisfaction is defined for conjunction. Suppose $\varphi \in G$ and $\varphi \leq \psi$, that is, $T \models \varphi \to \psi$. Since $\langle \lambda, A, \in \rangle \models T$, we have $\langle \lambda, A, \in \rangle \models \varphi \to \psi$. Thus, $\psi \in G$. Finally, suppose that Q is a maximal antichain; it suffices to show that $G \cap Q \neq \emptyset$. Since $\mathbb{B}_T$ is κ-c.c., $|Q| < \kappa$. Thus, $\bigvee_{\varphi \in Q} \varphi \in \mathbb{B}_T$. We claim that $T \models \bigvee_{\varphi \in Q} \varphi$. Otherwise, this sentence's negation would be incompatible with every element of Q, contrary to the maximality of Q. It follows that $\langle \lambda, A, \in \rangle \models \bigvee_{\varphi \in Q} \varphi$, so $\langle \lambda, A, \in \rangle \models \varphi$ for some $\varphi \in Q$, and so $\varphi \in G \cap Q$ as desired.

Next we claim that $N[G] = N(A)$. A can be defined from G by the sentences in G of the form "$\dot{A}(c_\alpha)$", so $A \in N[G]$ and $N(A) \subseteq N[G]$. Furthermore, G can be defined from A, as the above can be carried out completely in N except for reference to the model $\langle \lambda, A, \in \rangle$, which just needs A. Thus, $G \in N(A)$, so $N[G] \subseteq N(A)$, and we're done.

To finish, we prove the existence of a suitable T. Let $F : \mathrm{Good} \to \mathrm{Sent}$ in M be such that if $\langle \lambda, A, \in \rangle \models \bigvee_{\varphi \in T} \varphi$ then $\langle \lambda, A, \in \rangle \models F(T)$. Let, by the κ-global covering property, $G_F \in N$ approximate F. For a theory T we define

$$\varphi_T \equiv_{\text{df}} \left(\bigvee_{\varphi \in T} \varphi \right) \to \left(\bigvee_{\varphi \in G_F(T)} \varphi \right).$$

Let $T^\infty := \{\varphi_T \mid T \in \mathrm{Good}\}$. We claim that T^∞ satisfies both (1) and (2). By construction of F, if the antecedent of any φ_T is true in $\langle \lambda, A, \in \rangle$, then so is the consequent. Now, suppose Γ is an antichain in $\mathbb{B}_{T^\infty}$; by picking representatives for each element of Γ, we can treat Γ as a theory. We show that $G_F(\Gamma) = \Gamma$, so that the κ-c.c. follows by the cardinality bound on $G_F(\Gamma)$. Suppose for contradiction that $\psi \in \Gamma \setminus G_F(\Gamma)$. Then, $T^\infty \models \psi \to \bigvee_{\varphi \in \Gamma} \varphi$, so $T^\infty \models \psi \to \bigvee_{\varphi \in G_F(\Gamma)} \varphi$. Then, ψ is less than some $\varphi \in G_F(\Gamma) \subseteq \Gamma$ ("$\psi \to \varphi$" has to be true for some φ for the disjunction to be true); since Γ is an antichain, $\psi = \varphi \in G_F(\Gamma)$, a contradiction. Thus $\Gamma = G_F(\Gamma)$, as desired. □

We come now to the theorem. Notice that for any partial order $\mathbb{P}$, there is a κ such that $\mathbb{P}$ is κ-c.c., namely $|\mathbb{P}|^+$; thus, this theorem gives a necessary and sufficient condition to be a ground simpliciter.

Theorem 5.6 (Bukovský)**.** *Let $N \subseteq M$ be an inner model. The following are equivalent:*

1. *N satisfies the κ-global covering property for M.*

2. *There is a κ-c.c partial order $\mathbb{P} \in N$ and a generic $G \subseteq \mathbb{P}$ such that $M = N[G]$; in particular, $N \subseteq_{\mathrm{gd}} M$.*[46]

Proof. (2) $\implies$ (1): Let $f : X \to Y$ be a function in M, and let $\dot{f} \in \mathrm{Name}$ be a $\mathbb{P}$-name such that $\dot{f}^G = f$. By the Fundamental Theorem of Forcing, let $p \in G$ be such that $p \Vdash$ "$\dot{f}$ is a function from $\check{X}$ to $\check{Y}$". For $x \in X$ let $g_f(x)$ be the set of "possible values" of $f(x)$, that is, the set $\{y \in Y \mid (\exists q \leq p)(q \Vdash \dot{f}(\check{x}) = \check{y})\}$. To see that $|g_f(x)| < \kappa$, pick for each $y \in g_f(x)$ a condition q such that $q \Vdash \dot{f}(\check{x}) = \check{y}$. The resulting set of conditions C forms an antichain, so $|C| < \kappa$ by the κ-c.c.

(1) $\implies$ (2): We will show that $N \subseteq_{\mathrm{gd}} N(P^M(\kappa))$, then show that $M = N(P^M(\kappa))$. For the first part, take any bijection from $P^M(\kappa)$ to a set of ordinals A.[47] By Lemma 5.5, $N(A)$ will be a κ-c.c. generic extension of N, and by undoing the coding, $P(\kappa) \in N(A)$ (conversely, $A \in N(P^M(\kappa))$).

Now, suppose for contradiction that $M \neq N(P^M(\kappa))$. Then, since $N(P^M(\kappa)) \subseteq M$, by Proposition 3.7 there is a set of ordinals x such that $x \in M$ and $x \notin N(P^M(\kappa))$. Add x in, that is, consider $N(P^M(\kappa))(x)$. Since any function in $N(P^M(\kappa))(x)$ is in M, and N satisfies the κ-global covering property for M, there is an approximation of every function in $N(P^M(\kappa))(x)$ by a function in $N \subseteq N(P^M(\kappa))$, that is, $N(P^M(\kappa))$ satisfies the κ-global covering property for $N(P^M(\kappa))(x)$. Thus, by Lemma 5.5, $N(P^M(\kappa))(x)$ is a κ-c.c. generic extension of $N(P^M(\kappa))$. But then, by Lemma 2.5, there is $y \subseteq \kappa$ such that $y \in N(P^M(\kappa))(x)$ and $y \notin N(P^M(\kappa))$. But $N(P^M(\kappa))$ contains all subsets of κ, so this is a contradiction. □

[46] The proof of (2) $\implies$ (1) appears in [Kun80, p. 206]. The other direction first appears in [Buk73, p. 43].

[47] The bijection should be definable from the set of ordinals; any bijection in $L[A]$ will work.

We get the following useful corollary on intermediate generic extensions.

Corollary 5.6.1. *Suppose $N \subseteq N' \subseteq M$ and $N \subseteq_{\mathrm{gd}} M$. Then $N \subseteq_{\mathrm{gd}} N'$ and $N' \subseteq_{\mathrm{gd}} M$. Furthermore, let κ be such that the forcing from N to M is κ-c.c., and let $G \subseteq \mathbb{P}$ be the forcing from N' to M. If $\gamma = (2^\kappa)^M$, then we can guarantee that $|\mathbb{P}| \leq (2^\gamma)^{N'}$.*

Proof. Since the forcing from N to M is κ-c.c., by Theorem 5.6 N satisfies the κ-global covering property for M. Since every function in N' is in M, N satisfies the κ-global covering property for N', and thus $N \subseteq_{\mathrm{gd}} N'$. Similarly, since any approximation function in N is in N', N' satisfies the κ-global covering property for M, and hence $N' \subseteq_{\mathrm{gd}} M$.

To get the cardinality bound, we examine the proofs of Lemma 5.5 and Theorem 5.6. By those proofs, we code $P^M(\kappa)$ as a set of ordinals, which we may assume is a subset of $\gamma = (2^\kappa)^M$. The partial order is the Boolean algebra of sentences in a signature of size γ in $\mathcal{L}_{\gamma^+,\omega}$ (possibly with some sentences removed). A sentence is a γ sequence of symbols in the signature, so the set of all sentences in N' has size $(\gamma^\gamma)^{N'} = (2^\gamma)^{N'}$. $\square$

5.2.2 The Main Argument

We turn now to the proof of the DDG (Theorem 5.2).[48]

Proof of Theorem 5.2. Let $\{W_i \mid i \in I\}$ be our set-sized collection of grounds of V; without loss of generality, we may assume that I is well-ordered, and will sometimes write its elements as the ordinals less than $\mathrm{ot}(I)$. We will construct a W contained in all of the W_i such that, for some η, W satisfies η-global covering property for V. Then, by Bukovský's Theorem (Theorem 5.6), $W \subseteq_{\mathrm{gd}} V$. In

[48]The proof we give is in outline the same as the original proof from [Usu16, p. 11], but with simplifications found in [Ham16, p. 32].

defining W, we need to ensure that W has approximations to every function in V. We do this by building W around approximations to a universal function.

Let, for each i, $\mathbb{P}_i \in W_i$ be a partial order such that for some generic $G_i \in \mathbb{P}_i$, $V = W_i[G_i]$. Let κ be regular such that $|I| < \kappa$ and, for all $i \in I$, $|\mathbb{P}_i| < \kappa$ (this is possible because I is a set). Then, every $\mathbb{P}_i$ is κ-c.c., so by Theorem 5.6, every $\mathbb{P}_i$ has the κ-global covering property.

We build our model around a universal approximation in the following claim.

Claim. *For every $\theta > \kappa$ strong limit, there is $A_\theta \subseteq \theta$ such that*

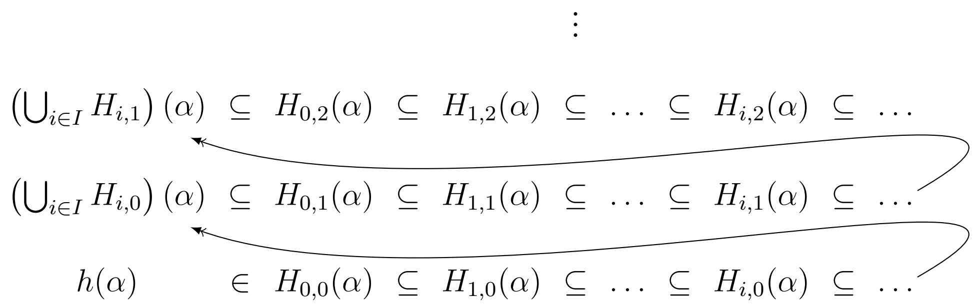
1. *for all $i \in I$, $L[A_\theta] \subseteq W_i$, and*
2. *$(V_\theta)^{L[A_\theta]}$ satisfies the κ^+-global covering property for V_θ.*

Proof of claim. Fix θ. Since $|V_\theta| = \theta$, there are (at most) θ many functions on the ordinals less than θ. Thus, we should be able to code all of these functions into a single one on θ. We can think of such an encoding as laying the functions side by side into a line of length θ; then each function is a "block" in the line. Let $h : \theta \to \theta$ be universal in this way: If $\lambda_0, \lambda_1 < \theta$ and $f : \lambda_0 \to \lambda_1$, then there is $\gamma < \theta$ such that $f(\alpha) = h(\gamma + \alpha)$, so that essentially

$$f = h \restriction \{\alpha \in \theta \mid \gamma_0 \leq \alpha < \gamma + \lambda_0\}.$$

We define a sequence $\{H_{i,\xi} \mid i \in I, \xi < \kappa\}$ by recursion on $i \in I$ and $\xi < \kappa$ such that the union of all the $H_{i,\xi}$ will be an approximation to h. We will maintain that for all ξ, $H_{i,\xi} \in W_i$, and that for all i and ξ, $|H_{i,\xi}| < \kappa$. To ensure that the union is in the intersection of all the W_i, we need to interleave the recursion across the W_i. Thus, for each ξ we will define $H_{i,\xi}$ for all $i \in I$, and use the union of these to obtain $H_{0,\xi+1}$. See Figure 1.

Figure 1: A visualization of the recursion to define the $H_{i,\xi}$, for a fixed $\alpha < \kappa$.



We give the definitions, starting with $H_{0,0}$. By assumption, W_0 satisfies the κ-global covering property for V. Let $H_{0,0} \in W_0$ be such that, for all α, $h(\alpha) \in H_{0,0}(\alpha)$ and $|H_{0,0}(\alpha)| < \kappa$. Suppose that for all $j < i$ we have defined $H_{j,\xi}$. Let $H'_{i,\xi}(\alpha) := \bigcup_{j<i} H_{j,\xi}(\alpha)$. Now, we let, by Lemma 5.4, $H_{i,\xi} \in W_i$ be such that for all α, $H'_{i,\xi}(\alpha) \subseteq H_{i,\xi}(\alpha)$ and $|H_{i,\xi}(\alpha)| < \kappa$. Suppose that for all $\eta < \xi$ and for all $i \in I$ we have defined $H_{i,\eta}$. Let $H'_{0,\xi}(\alpha) := \bigcup_{\substack{i\in I \\ \eta<\xi}} H_{i,\eta}(\alpha)$. Then, by Lemma 5.4, let $H_{0,\xi} \in W_0$ be such that, for all α, $H'_{0,\xi}(\alpha) \subseteq H_{0,\xi}(\alpha)$ and $|H_{0,\xi}(\alpha)| < \kappa$.

Let us define a function H by

$$H(\alpha) := \bigcup_{\substack{i\in I \\ \xi<\kappa}} H_{i,\xi}(\alpha).$$

We claim that for all $i \in I$, $H \in W_i$. The point is that by interleaving the unions, even though the whole recursion is not accessible to a particular W_i, cofinally many stages are. More precisely, we claim that, for any $i \in I$ and for all $\alpha < \theta$, $H(\alpha) = \bigcup_{\xi<\theta} H_{i,\xi}(\alpha)$. By definition of H we have $\bigcup_{\xi<\theta} H_{i,\xi}(\alpha) \subseteq H(\alpha)$. To see the other direction, suppose $x \in H(\alpha)$. Let j, ξ be such that $x \in H_{j,\xi}(\alpha)$. Then $x \in H_{i,\xi+1}(\alpha)$ by construction. It follows that for each α, $H(\alpha) \in W_i$, and thus that $H \in W_i$ as desired.

Since $H \subseteq \theta \times \theta$ and there is a bijection $\theta \times \theta \to \theta$, H is coded by a subset $A_\theta \subseteq \theta$. Since H and A_θ are interdefinable, by the above $L[A_\theta] \subseteq W_i$ for all $i \in I$.

So we just need to see that $(V_\theta)^{L[A_\theta]}$ satisfies the κ^+-global covering property for V_θ. The point here is that we approximated all functions on θ at once using H. To see this, let $\lambda_0, \lambda_1 \in V_\theta$ and $f : \lambda_0 \to \lambda_1$. Then, $\lambda_0, \lambda_1 < \theta$, so there is $\gamma < \theta$ such that $f = h \restriction \{\alpha < \theta \mid \gamma \leq \alpha < \gamma + \lambda_0\}$, where h is the universal function from above. By construction, for any α,

$$h(\alpha) \in H_{0,0}(\alpha) \subseteq H(\alpha).$$

Thus, for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \in H(\alpha)$. Since f deals only with ordinals less than some bound in θ, f has rank less than θ, and so the relevant block of H also has rank less than θ. To finish, we show that for all α, $|H(\alpha)| < \kappa^+$. Each stage of the recursion has cardinality less than κ, and there are $\kappa \cdot |I| = \kappa$ stages to the recursion. This union thus has size at most $\kappa \cdot \kappa = \kappa < \kappa^+$ as desired. $\square$

So, for each $\theta > \kappa$ we have a model, $V_\theta^{L[A_\theta]}$, that satisfies the κ^+-global covering property for V_θ, and so is a κ^+-c.c. ground of V_θ. By Corollary 5.1.2, $V_\theta^{L[A_\theta]}$ is uniquely determined by $P(\kappa^{++})^{L[A_\theta]} \subseteq P(\kappa^{++})$. Since there are set many subsets of $P(\kappa^{++})$ and a proper class of strong limit cardinals greater than κ, by the Pidgeonhole Principle a single subset of $P(\kappa^{++})$ must correspond to a proper class S of strong limits. Furthermore, if $\theta_0 < \theta_1$ are both in S, then $V_\theta^{L[A_{\theta_0}]} \subseteq V_\theta^{L[A_{\theta_1}]}$. For A_{θ_1} encodes all the functions on ordinals less than $\theta_0 < \theta_1$, and so $A_{\theta_0} \in L[A_{\theta_1}]$, which implies that $L[A_{\theta_0}] \subseteq L[A_{\theta_1}]$. By the absoluteness of rank, it follows that $V_\theta^{L[A_{\theta_0}]} \subseteq V_\theta^{L[A_{\theta_1}]}$.

We can now define our common ground. Let

$$W = \bigcup_{\theta \in S} V_\theta^{L[A_\theta]}.$$

By construction, for all $i \in I$, $W \subseteq W_i$. We check that W is in fact a ground of all the W_i.

First, we claim that $W \models \mathrm{ZFC}$. We use Proposition 3.6. We check that W is closed under the Gödel operations (Definition 3.8). Let G be a Gödel operation, and let $x_1, ..., x_n \in W$. Compute $G(x_1, ..., x_n)$. Let $\theta \in S$ be sufficiently large such that the x_i and $G(x_1, ..., x_n)$ are all in $V_\theta^{L[A_\theta]}$, as $L[A_\theta]$ is closed under the Gödel operations, and the output of the functions is absolute. To see that W is almost universal, let $x \subseteq W$. Let $\theta > \mathrm{rank}(x)$ be in S. It follows that $x \subseteq V_\theta \cap L[A_\theta] = V_\theta^{L[A_\theta]} \in W$.

Now, we show that W is a ground. Since there is a η such that every $V_\theta^{L[A_\theta]}$ satisfies η-global covering for V_θ, and every function on ordinals in W is a function on ordinals in some $V_\theta^{L[A_\theta]}$, it follows that there is a η such that W satisfies η-global covering for V, and so by Bukovský's Theorem (Theorem 5.6), $W \subseteq_{\mathrm{gd}} V$. By Corollary 5.6.1, for all i, $W \subseteq_{\mathrm{gd}} W_i$, as desired. □

5.3 Main Results

Having established the DDG, we now prove the results mentioned at the beginning of this section, and discuss their significance. The guiding theme here is that the mantle, $\mathbb{M}$, is special class that exhibits features very close to Ultimate L: Assuming the existence of a hyper-huge cardinal, it is a forcing-invariant inner model, and the minimal element of the Generic Multiverse.

First, we will fulfill a promise from Section 4.2 by proving a lemma on the structure of the Generic Multiverse.

Lemma 5.7. *Let $\{W_i \mid i \in I\}$ be the collection of grounds of V. A model is in the Generic Multiverse iff it is of the form $W_i[G]$ for $G \subseteq \mathbb{P} \in W_i$ generic; that is, the Generic Multiverse consists of the generic extensions of the grounds of V.*[49]

[49][Ham16, p. 44]

Proof. Call the collection of all models of this form C. Clearly every model in C is in the Generic Multiverse. Thus it suffices to show that C is closed under taking forcing extensions and grounds.

If $W_i[G_0][G_1]$ is a forcing extension with $G_0 \subseteq \mathbb{P}_0$ and $G_1 \subseteq \mathbb{P}_1$, then, by Lemma 2.7, $W_i[G_0][G_1] = W_i[H]$ for a generic $H \subseteq \mathbb{P}_0 \times \mathbb{P}_1$, so $W_i[G_0][G_1] \in C$.

If $W \subseteq_{\mathrm{gd}} W_i[G]$, then by the DDG there is a $W' \subseteq_{\mathrm{gd}} V$ such that $W' \subseteq_{\mathrm{gd}} W, W_i$, so $W \in C$. □

Now, we prove a forcing invariance result for the mantle, similar to Theorem 3.10 for L. However, the mantle is particularly special in this regard, for it is the largest possible forcing invariant class.

Theorem 5.8. *The mantle is the largest forcing invariant class; that is, if $G \subseteq \mathbb{P}$ is M-generic, then $\mathbb{M}^M = \mathbb{M}^{M[G]}$, and for any class N such that $N^M = N^{M[G]}$, $N \subseteq \mathbb{M}$.*[50]

Proof. First we show that $\mathbb{M}^M = \mathbb{M}^{M[G]}$. Since every ground of M is a ground of $M[G]$, the intersection of all grounds of $M[G]$ is contained in the intersection of all grounds of M, that is, $\mathbb{M}^{M[G]} \subseteq \mathbb{M}^M$. Now, suppose that $x \notin \mathbb{M}^{M[G]}$. Then, by definition of the mantle, $x \notin M_0$ for some $M_0 \subseteq_{\mathrm{gd}} M[G]$. By the DDG, there is $M_1 \subseteq_{\mathrm{gd}} M_0, M$. But then, $x \notin M_1$, so by again by the definition of the mantle, $x \notin \mathbb{M}^M$. Thus, $\mathbb{M}^M \subseteq \mathbb{M}^{M[G]}$, so $\mathbb{M}^M = \mathbb{M}^{M[G]}$, as desired.

Next, we show that the mantle is the largest such class. To do this, we show that the mantle is the intersection of the Generic Multiverse. This suffices, as any forcing invariant class must have all of its elements in all of the elements of the Generic Multiverse, else it would be changed by forcing. By the above, the mantle is contained in the intersection of the Generic Multiverse. Since every ground of V is in the Generic Multiverse, the intersection of the Generic Multiverse is contained in the intersection of all the grounds of V, which by definition is $\mathbb{M}$. □

[50] [FHR14, p. 9],[Ham16, p. 40]

We can now show that the mantle is in fact a ZFC model.

Proposition 5.9. $\mathbb{M} \models \mathrm{ZFC}$.[51]

Proof. First we show that $\mathbb{M} \models \mathrm{ZF}$ using Proposition 3.6. $\mathbb{M}$ has all the ordinals. It is transitive, for if $x \in \mathbb{M}$, then $x \in W_i$ for all grounds W_i; since for all i, W_i is transitive, $x \subseteq W_i$, and so $x \subseteq \mathbb{M}$. $\mathbb{M}$ is closed under the Gödel operations (Definition 3.8), since for all i, W_i is closed under the Gödel operations, and so the output of any of the operations on any x will be in their intersection. To show that $\mathbb{M}$ is almost universal, let $x \subseteq \mathbb{M}$ be a set. Let $\mathrm{rank}(x) = \alpha$. Then, $x \subseteq V_\alpha \cap \mathbb{M}$. Since $\mathbb{M}$ is first-order definable, and the definition is absolute by Theorem 5.8, by Separation, for all i, $V_\alpha \cap \mathbb{M} \in W_i$, and so $V_\alpha \cap \mathbb{M} \in \mathbb{M}$. Thus, $\mathbb{M} \models \mathrm{ZF}$.

Now, we show that the mantle satisfies the Well-Ordering Theorem. Suppose not toward contradiction. Then, there is an $x \in \mathbb{M}$ such that there for every well-ordering $\preceq$ of x in V, $\preceq \notin \mathbb{M}$. Then, for each $\preceq$ there is $W_i \subseteq_{\mathrm{gd}} V$ such that $\preceq \notin W_i$. If we pick one i for each well-ordering, then there are $|x|^+$ many (in particular, set many) such W_i, so by the DDG there is a common ground W contained in all of them. W cannot have any $\preceq$, so there is no well-ordering of x in W. But W is a ZFC model, so we have a contradiction. $\square$

Finally, we can prove the most interesting consequence of the DDG: The mantle is the unique minimal element of the Generic Multiverse, and it inherits large cardinals from V.

Theorem 5.10. *Assume* ZFC + *"There is a hyper-huge cardinal κ." Then $\mathbb{M}$ is the unique minimal element of the Generic Multiverse.*[52]

Proof. First, we claim that it suffices to show that V has set many grounds. For then, by the DDG, there is a ground that is contained in all other grounds, and so

[51] [FHR14, p. 9]

[52] The proof we give is mostly from [Usu16, p. 15]. The proof that $j[j(\lambda)] \in \overline{M}$ the author learned from conversations with Professor Woodin.

contained in the rest of the Generic Multiverse. It is unique by the DDG: if M_0 and M_1 were distinct minimal elements, by downward directedness there would have to be an N properly contained in both, contradicting minimality. Finally, if this minimum $M \neq \mathbb{M}$, then, since $\mathbb{M}$ is contained in every element of the Generic Multiverse, there must be some $x \in M$ such that $x \notin \mathbb{M}$. But then, there must be some ground M' such that $x \notin M'$. By the DDG, there is a ground properly contained in M and M', contradicting the minimality of M.

The main work in proving that V has set many grounds is the following claim:

Claim. *Suppose* $\overline{V} \subseteq_{\mathrm{gd}} V$. *Then there exists* $\mathbb{P} \in \overline{V}_\kappa$ *and* $G \subseteq \mathbb{P}$ *generic such that* $V = \overline{V}[G]$ *and* $|\mathbb{P}| < \kappa$.

Proof of Claim. We want to use the hyper-hugeness of κ to reflect partial orders back down below κ. The key is that for any λ, there's a j that brings κ above λ, so if there's a partial order above κ (but below $\lambda < j(\kappa)$) that works for sets of rank λ, there's a partial order below κ that works, for arbitrarily large λ. Let us implement this. Let $G_0 \subseteq \mathbb{P}_0$ be such that $\overline{V}[G_0] = V$. Fix $\lambda > \kappa$ an inaccessible cardinal (by Theorem 3.3) such that $G_0, \mathbb{P}_0 \in V_\lambda$. Let $j : V \to M$ be such that $\mathrm{CRT}(j) = \kappa$, $\lambda < j(\kappa)$, and ${}^{j(\lambda)}M \subseteq M$. Let

$$\overline{M} := j(\overline{V}) = \bigcup_{\alpha \in \mathrm{On}} j(\overline{V}_\alpha).$$

We will show that $j(\overline{V}_\lambda) \subseteq_{\mathrm{gd}} M_{j(\lambda)}$ by using Corollary 5.6.1. As a first step toward this, we show that $\overline{V}_{j(\lambda)} \subseteq j(\overline{V}_\lambda)$. Since λ is inaccessible, so is $j(\lambda)$; thus we have

$$\overline{V}_{j(\lambda)} \models \mathrm{ZFC} \text{ and } j(\overline{V}_\lambda) \models \mathrm{ZFC}.$$

So, by Proposition 3.7, it suffices to check that every set of ordinals in $\overline{V}_{j(\lambda)}$ is in

$j(\overline{V}_\lambda)$. To do this, we will use Separation with $j[j(\lambda)]$; thus, we will first show that $j[j(\lambda)] \in \overline{M}$.

It suffices to show that there is $y \in \overline{V}$ with $|y|^{\overline{V}} = j(\lambda)$ such that $j[y] \in \overline{M}$. In particular, showing that there is a club C in $j(\lambda)$ in $\overline{V}$ such that $j[C] \in \overline{M}$ suffices. Let

$$A := \{\alpha < j(\lambda) \mid \operatorname{cof}^{\overline{V}}(\alpha) = \omega\} \text{ and } B := \{\alpha < \sup j[j(\lambda)] \mid \operatorname{cof}^{\overline{M}}(\alpha) = \omega\}$$

be the sets of cofinality ω according to $\overline{V}$ and $\overline{M}$ respectively. Notice that A and B are stationary in $\overline{V}$ and $\overline{M}$ respectively, as one can always take the limit of an ω sequence of elements of a club to produce an element of the club with cofinality ω. We define descending sequences of clubs $\{C_n \mid n < \omega\} \subseteq \overline{V}$ and $\{D_n \mid n < \omega\} \subseteq \overline{M}$, and take the intersections of both sequences to get the desired result. First, we start with a club $D_0 \subseteq \sup j[j(\lambda)] \cap B$. Consider $j^{-1}[D_0]$. This is certainly a member of V, though not necessarily of $\overline{V}$. Furthermore, by elementarity applied to the order relation on the ordinals, it is unbounded in $j(\lambda)$. We take its closure to get a club $C_0'' \in V$. By Lemma 5.3, since $\mathbb{P}_0$ is $|\mathbb{P}_0|^+$-c.c., we can find a club $C_0' \subseteq C_0''$ with $C_0' \in \overline{V}$. Let $C_0 := C_0' \cap A$; since A is stationary in $\overline{V}$, C_0 is club in $\overline{V}$. Now, we check $j[C_0]$. As we took a subset of the preimage of D_0, this will not in general equal D_0, or even be a member of $\overline{M}$. However, it is a member of M, and we can repeat this process: Take its closure to produce D_1'', take a club $D_1' \in \overline{M}$ contained in it, and restrict to the ordinals of cofinality ω to get D_1. We iterate this process through $n < \omega$ to obtain $\{C_n \mid n < \omega\}$ and $\{D_n \mid n < \omega\}$. Now let

$$C := \bigcap_{n<\omega} C_n \text{ and } D := \bigcap_{n<\omega} D_n.$$

Since all the C_n are in $\overline{V}$ and all the D_n are in $\overline{M}$, the intersections are in the respective models.

We claim that $j[C] = D$. First we check $j[C] \subseteq D$. If $j(\alpha) \in j[C]$, then for each $n < \omega$, either $\alpha \in j^{-1}[D_n]$, or α is a limit point of $j^{-1}[D_n]$ in $\overline{V} \cap A$. But, if α is such a limit point, because it has cofinality ω in $\overline{V}$, it is the limit of an ω-sequence $Y := \{\alpha_n \mid n < \omega\} \subseteq j^{-1}[D_n]$. We claim that

$$j(\alpha) = \sup_{n<\omega}\{j(\alpha_n) \mid n < \omega\} = \sup j[Y] \in D_n.$$

The point is that by restricting to cofinality ω, we have guaranteed the continuity of j. Let $Z \subseteq \alpha$ be cofinal such that $\mathrm{ot}(Z) = \omega$. By elementarity, $j(Z)$ is cofinal in $j(\alpha)$. Since $\alpha = \sup Y$, for every $\beta \in Z$ there is n such that $\beta < \alpha_n$. By elementarity, for every $\beta \in j(Z)$ there is n such that $\beta < j(\alpha_n)$; since $j(Z)$ is cofinal, so is $j[Y]$, as desired.

On the other hand, let $\beta \in D$. Then, for each $n+1 < \omega$, either $j^{-1}(\beta) \in C_n$ or β is a limit point of $j[C_n]$ in $\overline{M} \cap B$. But if the latter obtains, then again, because it has cofinality ω in $\overline{M}$, β is the limit of an ω-sequence $Y := \{\beta_n \mid n < \omega\} \subseteq j[C_n]$. We just need that

$$\beta = j(\sup_{n<\omega}\{j^{-1}(\beta_n) \mid n \in \omega\}) = j(\sup j^{-1}[Y]) \in j[C_n].$$

By reasoning similar to the above, we again have continuity because we restricted to cofinality ω, and obtain the desired conclusion.

Thus, $j[C] = D \in \overline{M}$. But since C is club (in particular, unbounded) in $j(\lambda)$, $|C|^{\overline{V}} = j(\lambda)$, and so by undoing bijections we get $j[j(\lambda)] \in D$ as desired.

Now, let $A \in \overline{V}_{j(\lambda)}$ be a set of ordinals. By definition of $\overline{M}$, $j(A) \in \overline{M}$. Since A has rank at most $j(\lambda)$, $A \subseteq j(\lambda)$. By elementarity, $j(x) \in j(A)$ iff $x \in A$; thus, $j[A] = j(A) \cap j[j(\lambda)]$, which is a definable subset of $j[j(\lambda)] \in \overline{M}$ and so by Separation is in $\overline{M}$. Again using Separation and the fact that $j[j(\lambda)] \in \overline{M}$, we have that $j \restriction j(\lambda) \in \overline{M}$. Thus, we can write $A = j^{-1}[j[A]] \in \overline{M}$. Since the rank of A is less than $j(\lambda)$, we have in particular that $A \in \overline{M}_{j(\lambda)} = j(\overline{V}_\lambda)$, and so $\overline{V}_{j(\lambda)} \subseteq j(\overline{V}_\lambda)$ as desired.

Since $\overline{V}_\lambda \subseteq V_\lambda$, by elementarity $j(\overline{V}_\lambda) = \overline{M}_{j(\lambda)} \subseteq M_{j(\lambda)}$. By closure under $j(\lambda)$ sequences, M is correct about $V_{j(\lambda)}$ (which is defined as the union of a $j(\lambda)$-length sequence of power sets, which can be associated to binary sequences of length less than $j(\lambda)$), and so $M_{j(\lambda)} = V_{j(\lambda)}$. In sum, we have

$$\overline{V}_{j(\lambda)} \subseteq j(\overline{V}_\lambda) \subseteq M_{j(\lambda)} = V_{j(\lambda)}.$$

Since $\overline{V} \subseteq_{\mathrm{gd}} V$ with $\mathbb{P}_0$ and G_0 both in $V_{j(\lambda)}$, we get $\overline{V}_{j(\lambda)} \subseteq_{\mathrm{gd}} V_{j(\lambda)}$. By Corollary 5.6.1 and the above, we must have that $j(\overline{V}_\lambda) \subseteq_{\mathrm{gd}} M_{j(\lambda)}$, as desired.

Furthermore, let $\mathbb{P}_1$ be the partial order, and let

$$\gamma := (2^{|\mathbb{P}_0|^+})^{M_{j(\lambda)}} = (2^{|\mathbb{P}_0|^+})^M.$$

By Corollary 5.6.1, we have

$$|\mathbb{P}_1| \leq (2^\gamma)^{j(\overline{V}_\lambda)} = (2^\gamma)^{j(\overline{V})} = (2^\gamma)^{\overline{M}}.$$

Since $\overline{M}$ and M are inner models of V, we have

$$\gamma = (2^{|\mathbb{P}_0|^+})^M \leq \gamma' := 2^{|\mathbb{P}_0|^+}$$

and

$$(2^\gamma)^{\overline{M}} \leq 2^\gamma \leq 2^{\gamma'}.$$

Since λ is inaccessible (in particular, strong limit) and $|\mathbb{P}_0| < \lambda$, we have that $\gamma' < \lambda$. Thus

$$|\mathbb{P}_1| \leq 2^{\gamma'} < \lambda.$$

We may assume, taking an isomorphism if necessary, that $\mathbb{P}_1 \in \overline{M}_{j(\kappa)} = j(\overline{V}_\kappa)$. Thus, M satisfies the following:

> There exists a partial order $\mathbb{P}_1 \in j(\overline{V}_\kappa)$ and $G_1 \subseteq \mathbb{P}_1$ generic such that $j(\overline{V}_\lambda)[G_1] = M_{j(\lambda)}$ and $|\mathbb{P}_1| < j(\kappa)$.

By elementarity, we have the following:

> There exists a partial order $\mathbb{P} \in \overline{V}_\kappa$ and $G \subseteq \mathbb{P}$ generic such that $\overline{V}_\lambda[G] = V_\lambda$ and $|\mathbb{P}| < \kappa$.

Thus, for every sufficiently large inaccessible λ, there is a partial order $\mathbb{P}_\lambda \in \overline{V}_\kappa$ and a generic $G_\lambda \subseteq \mathbb{P}_\lambda$ such that $\overline{V}_\lambda[G_\lambda] = V_\lambda$. Notice that the set containing the partial orders, $\overline{V}_\kappa$, does not change with λ. There are at most $|\overline{V}_\kappa|$ partial orders, and for each partial order $\mathbb{P}$ there are at most $2^{|\mathbb{P}|}$ generics; thus, there are only set many possible forcings in $\overline{V}_\kappa$. By Theorem 3.3, there is a proper class of inaccessible cardinals. Thus, by the Pidgeonhole Principle, one partial order $\mathbb{P} \in \overline{V}_\kappa$ and one generic $G \subseteq \mathbb{P}$ must work for most of the inaccessibles: There is a proper class I of inaccessible λ such that $\overline{V}_\lambda[G] = V_\lambda$. Thus, as $\overline{V} = \bigcup_{\lambda \in I} \overline{V}_\lambda$ and $V = \bigcup_{\lambda \in I} V_\lambda$, we must have $\overline{V}[G] = V$, as desired. □

We now finish by showing that V has set many grounds. The key is that for any $W \subseteq_{\text{gd}} V$, there is a partial order $\mathbb{P} \in V_\kappa$ that witnesses this (as, for any ground W, $W_\kappa \subseteq V_\kappa$); in particular, the ambient space in which all the partial orders for all the grounds live is a fixed set. There are thus at most $|V_\kappa| = \kappa$ many partial orders. Furthermore, for each partial order $\mathbb{P}$, there are at most $2^{|\mathbb{P}|}$ many generic filters. Identifying the partial orders and generics is not enough, as multiple models could use the same partial order and generic. By Corollary 5.1.2, a ground W using partial order $\mathbb{P}$ is uniquely identified by $P^W(|\mathbb{P}|^{++})$. For any W, $P^W(|\mathbb{P}|^{++}) \subseteq P(|\mathbb{P}|^{++})$ by transitivity; thus, there are only $2^{2^{|\mathbb{P}|^{++}}}$ possibilities for the power set of $|\mathbb{P}|^{++}$.

Multiplying these three parameters gives an upper bound on the total number of possible ground models, and so there are set many grounds, as desired. □

Corollary 5.10.1. *Assume* ZFC + *"There is a hyper-huge cardinal κ." If $\lambda > \kappa$ is a measurable cardinal, then* (λ *is a measurable cardinal*)$^{\mathbb{M}}$.[53]

Proof. By the proof of Theorem 5.10, there is a partial order $\mathbb{P}$ and $G \subseteq \mathbb{P}$ such that $V = \mathbb{M}[G]$, with $|\mathbb{P}| < \kappa$. By Theorem 3.4, the measurability of λ transfers from V to $\mathbb{M}$. □

Remark. Corollary 5.10.1 generalizes with the Lévy-Solovay theorem to other large cardinals.

These theorems suggest that the mantle is very special among the models in the Generic Multiverse. It is definable, forcing invariant, and minimal, just as we required of a privileged point. One may counter these conditions are only necessary for a model to be a privileged point; if there is to be a privileged point, it must have these properties, and so must be the mantle. But, it is still possible that truth in the mantle has no more claim to set-theoretic truth than truth in any other model in the Generic Multiverse. However, Corollary 5.10.1 gives us more: It suggests that $\mathbb{M}$ is similar to V in an important way, by sharing large cardinals. This is a point that makes the mantle preferable to many other universes in the Generic Multiverse, such as those that collapse large cardinals to ω and so destroy their large cardinal properties.

Even more important is the similarity of the properties of the mantle to those of Ultimate L. By Theorem 4.2, Ultimate L is minimal in the Generic Multiverse, and so must be the mantle of its Generic Multiverse. Furthermore, $\mathbb{M}$ and Ultimate L both absorb large cardinals from V above a distinguished large cardinal κ (hyperhuge in the former, extendible in the latter). Thus, the case for Ultimate L

[53] [Ham16, p. 54]

is intertwined with the case for a privileged point in the Generic Multiverse; the evidence for Ultimate L becomes evidence for the mantle being a privileged point, and the observations we have made here become part of the case for Ultimate L. That the pictures suggested by Ultimate L and set-theoretic geology converge is even more compelling as evidence for those pictures because they come from different domains. The study of Ultimate L is motivated by inner model theory, while set-theoretic geology is motivated by forcing. We can see this in the flavor of the proofs: In Chapter 4, they were much more about the properties of weak extender models and connections to AD^+ theory, while in this chapter, forcing methods have dominated. That these distinct domains point to the same conclusion suggests that there is something to that conclusion.[54]

5.4 Recap

From set-theoretic geology, we can prove that there is an inner model of V that is a privileged point in the Generic Multiverse, namely the mantle. The mantle is furthermore close to V in that it inherits the large cardinals that exist in V, similar to Ultimate L. That both inner model theory and set-theoretic geology point to such a inner model is compelling evidence for $V =$ Ultimate L.

6 Conclusion

The independence problem complicates the picture of mathematical inquiry. However, in spite of the seeming intractability of problems that cannot be solved by our standard means, there is hope. The axiom $V =$ Ultimate L, assuming the promised inner model actually exists, could wipe away the independence phenomenon. Fur-

[54]A similar point has been made about the convergence of large cardinal theory and determinacy theory in descriptive set theory, which has been considered strong evidence for the correctness of both approaches. See [Koeb] for details.

thermore, there are reasons to believe that this axiom candidate is true, among them the conclusions of our inquiry into set-theoretic geology.

7 Appendix

7.1 Notation

Most notation that we use is standard. We use $\subseteq$ for subset with possible equality and $\subsetneq$ for proper subset. We do introduce a new symbol, $\subseteq_{\mathrm{gd}}$, in the text for grounds of a model. We use $P(x)$ for the power set of x. We use $\bigcup x$ to denote the union $\bigcup_{y \in x} y$. We use $x \setminus y$ for set difference. We use $|x|$ to denote the cardinality of x. We denote the transitive closure of x by $\mathrm{tc}(x)$. We denote the order type of a well-ordered set x by $\mathrm{ot}(x)$. If f is a function, then we use $f(x)$ to denote the application of the function to x, and $f[x]$ to denote $\{y \mid (\exists z \in x)(f(z) = y)\}$. We use $\mathrm{rank}(x)$ to denote the rank of x. We use $\mathrm{cof}(\alpha)$ to denote the cofinality of α. We use $\wedge, \vee, \rightarrow, \equiv$ as formal logical symbols. If M is a model of the language of set theory, we relativize formulas, constants, and functions to M with a subscript M. If $M = V$, then generally we will omit the superscript. If M is an inner model of V, then we write M_α for $V_\alpha^M = M \cap V_\alpha$.

We assume the von Neumann construction of the ordinals, so that the order relation on the ordinals is the membership relation. We use On to denote the class of ordinals. We identify the cardinals and the initial ordinals, so $\aleph_\alpha = \omega_\alpha$. We identify the natural numbers and the finite ordinals. We use ω to denote the least infinite ordinal, which is the set of natural numbers.

7.2 The ZFC Axioms

Here is a list of the ZFC axioms, written informally, along with some useful related principles.

- **Extensionality**: Two sets are equal iff they have the same elements.

- **Foundation**: The membership relation is well-founded.

  This is equivalent to the assertion that for all x, there is α such that $x \in V_\alpha$.

- **Empty Set**: The empty set $\emptyset$ exists.

- **Pairing**: For any two sets x and y, the set $\{x, y\}$ exists.

- **Union**: For any set x, $\bigcup x$ exists.

- **Power set**: For any set x, $P(x)$ exists.

- **Infinity**: There is an inductive set, that is, a set I such that $0 \in I$ and if $x \in I$ then $x \cup \{x\} \in I$.

  This implies that ω exists.

- **Separation**: If φ is a predicate, and a a set, then the set $\{x \in a \mid \varphi(x)\}$ exists.

- **Replacement**: If F is a class-sized function, that is, a two-place predicate such that for all x there is a unique y such that $F(x, y)$, then for any set a, $F[a]$ exists.

  This implies the weaker principle **Collection**: If F is a two-place predicate, then for any a there exists a superset of $F[a] = \{y \mid (\exists x \in a)F(x, y)\}$.

- **Choice**: For any set I of sets, there is a choice function on I: a function $f : I \to \bigcup I$ such that for all $i \in I$, $f(i) \in i$.

  This is equivalent to the **Well-Ordering Theorem**: Every set can be well-ordered.

7.3 The Ordinals

We will not give an exposition of the ordinals from scratch; the interested reader is referred to [Jec03] or [Kun80] for an introduction. Suffice it to say, they are the generalization of the natural numbers to the transfinite, and the isomorphism types of the all the well-orders. Thus, we can do induction and recursion with ordinal indices.

The *cofinality* of an ordinal α is the least ordinal β such that there is a subset of α of order type β that is *cofinal* (unbounded). If $\mathrm{cof}(\alpha) = \alpha$, then α is *regular*, otherwise it is *singular*.

The ordinals are well-ordered, so we may consider the order topology on them. Being closed in this topology is equivalent to having the supremum of every bounded subset, as these are the limit points. A subset of an ordinal is *club* iff it is closed and unbounded in that ordinal. A subset of an ordinal is *stationary* iff it intersects every club. If the ambient space is κ, then the intersection of less than κ many clubs is club, and the intersection of a stationary set and a club is club.

7.4 The Universe of Sets

The universe of sets is typically denoted V. Since our variables range over sets, this is the definable class $\{x \mid x = x\}$. Using the Axiom of Foundation, V can also be built up as a hierarchy, known as the rank hierarchy:

$$
\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= P(V_\alpha) \\
V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \quad (\lambda \text{ a limit ordinal}) \\
V &= \bigcup_{\alpha\in\mathrm{On}} V_\alpha
\end{aligned}
$$

We can think of this as starting with the empty set and building up all the sets in stages. The least α such that $x \in V_{\alpha+1}$ is called the *rank* of x.

The initial segments V_α of the universe start to look more and more like V, in that for any finite collection of true sentences, one can find a β such that V_β satisfies those sentences. This is the Reflection Theorem. Thus, one can often assume that one has everything one needs for a purpose by taking sufficiently large β.

By $H(\kappa)$ we mean the sets of hereditary cardinality at most κ:

$$\{x \mid |\mathrm{tc}(\{x\})| < \kappa\}.$$

The $H(\kappa)$ are interdefinable with levels of the rank hierarchy, and are sometimes nicer to work with than a corresponding V_β.

7.5 Transitive Models and Absoluteness

Let us turn to the formulas of set theory. We can classify formulas by complexity as follows. A formula is Δ_0 iff all of its quantifiers are bounded (that is, of the form "$\forall x \in a$" or "$\exists x \in a''$). It is Σ_1 iff it is of the form $(\exists x)\varphi$ where φ is Δ_0. The negation of a Σ_1 formula is a Π_1 formula. Inductively, a formula is Σ_{n+1} iff it is of the form $(\exists x)\varphi$ where φ is Π_n, and it is Π_{n+1} iff it is the negation of a Σ_{n+1}

formula.

Transitive models (models M whose universe is transitive, that is, if $y \in x \in M$ then $y \in M$) are particularly important. This is because Δ_0 formulas are *absolute* between transitive models: Their truth values do not change between transitive models. As a corollary, if $M \subseteq N$ are transitive, then if M satisfies a Σ_1 formula φ, so does N, and if N satisfies a Π_1 formula ψ, so does M. We say that Σ_1 formulas are *upward absolute* and Π_1 formulas *downward absolute*. Examples of absolute statements include statements about membership, rank, and the subset relation.

Mostowski's collapsing lemma allows us to produce transitive models. It says that if E is an *extensional* (for all x and y, $x = y$ iff for all z, zEx iff zEy) and well-founded relation on M, then there is a unique transitive model isomorphic to (M, E).

References

[Buk73] Lev Bukovský. “Characterization of generic extensions of models of set theory”. In: *Fundamenta Mathematica* 83 (1973), pp. 35–46.

[FFS16] Sy David Friedman, Sakaé Fuchino, and Hiroshi Sakai. “On the set-generic multiverse”. In: *Proceedings of the Singapore Programme on Sets and Computations*. 2016. URL: http://www.logic.univie.ac.at/~sdf/papers/joint.sakae.hiroshi.pdf.

[FHR14] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. “Set-theoretic Geology”. 2014. URL: https://arxiv.org/abs/1107.4776v2.

[Göd90] Kurt Gödel. “Remarks before the Princeton bicentennial conference on problems in mathematics”. In: *Kurt Gödel Collected Works*. Ed. by Solomon Feferman et al. Vol. 2. Oxford: Oxford University Press, 1990, pp. 150–153.

[Ham16] Joel David Hamkins. *Recent advances in set-theoretic geology*. Presented at the Harvard Logic Colloquium. 2016. URL: http://jdh.hamkins.org/wp-content/uploads/2016/09/Harvard-Logic-Colloquium-2016.pdf.

[Jec03] Thomas Jech. *Set Theory*. Third Millennium Edition. Berlin: Springer-Verlag, 2003.

[Kan09] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. 2nd ed. Berlin: Springer, 2009.

[Koea] Peter Koellner. “Independence and Large Cardinals”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2011. URL: https://plato.stanford.edu/archives/sum2011/entries/independence-large-cardinals/.

[Koeb] Peter Koellner. "Large Cardinals and Determinacy". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Spring 2014. URL: http://plato.stanford.edu/archives/spr2014/entries/large-cardinals-determinacy/.

[Kun80] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. Amsterdam: Elsevier Science Publishers, 1980.

[LS67] Azriel Lévy and Robert Solovay. "Measurable cardinals and the continuum hypothesis". In: *Israel Journal of Mathematics* 5.4 (1967), pp. 234–248.

[Mad97] Penelope Maddy. *Naturalism in Mathematics*. Oxford: Oxford University Press, 1997.

[Mos09] Yiannis Moschovakis. *Descriptive Set Theory*. 2nd ed. 2009. URL: http://www.math.ucla.edu/~ynm/lectures/dst2009/dst2009.pdf.

[Ste14] John Steel. "Gödel's Program". In: *Interpreting Gödel: Critical Essays*. Ed. by Juliette Kennedy. Cambridge: Cambridge University Press, 2014, pp. 153–179.

[Usu16] Toshimichi Usuba. "The Downward Directed Grounds Hypothesis and Very Large Cardinals". 2016.

[Woo01] W Hugh Woodin. "The 19th Midrasha Mathematicae Lectures". 201?

[Woo09] W Hugh Woodin. "The Continuum Hypothesis, the generic-multiverse of sets, and the Ω Conjecture". 2009. URL: http://logic.harvard.edu/EFI_Woodin_TheContinuumHypothesis.pdf.

[WDR12] W Hugh Woodin, Jacob Davis, and Daniel Rodríguez. "The HOD Dichotomy". In: *Appalachian Set Theory: 2006–2012*. Ed. by James Cummings and Ernest Schimmerling. Cambridge: Cambridge University Press, 2012, pp. 397–419.