Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Also available from the MGSA is an extended syllabus and sample Qual questions with answers that Danny Calegari wrote. His questions on 3-manifolds are included here, but his write-up also includes solutions to them.

Low-Dimensional Topology and Manifolds.

- Given a 4-manifold made up of a 0-handle and some 2-handles, what does the determinant of the intersection form tell you about the boundary?
- What can you say about the components of the complements of a PL-embedded 2-sphere in S^3?
- How about of a torus in S^3?
- Prove that if one component U of the complement of the torus in S^3 has fundamental group $\mathbb{Z}$, the closure of U is homeomorphic to a solid torus.
- How can you weaken the condition on U?
- Sketch a proof of the Reidemeister-Singer Theorem.
- Sketch a proof of Milnor's theorem on unique decomposition of 3-manifolds into prime factors.
- What is a spin structure? [**Kirby**]
- If a manifold is spin, how many different spin structures does it have? [**Casson**]
- Give an example of a 4-manifold which is not spin. [**Kirby**]
- Draw your favorite knot (not the unknot) K and compute $\pi_1(S^3 - K)$. [**Kirby**]
- Why does the Wirtinger presentation method of computing $\pi_1(S^3-K)$ work? [**Kirby**]
- Why is $\pi_1(S^3 - K)$ not $\mathbb{Z}$ if K is the trefoil? [**Stallings**]
- Why does $\pi_1(S^3 - K)$, with K the trefoil, have a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$? [**Casson**]
- Define Heegaard splittings.
- Define handlebody.
- Do Heegaard splittings always exist? Why?
- Consider $+k$ surgery on a trefoil. Can you relate this to a Heegaard splitting? [**Kirby**]
- How do you define Dehn surgery?

- Why is it enough to specify a homotopy class of essential curve (in Dehn surgery)? [**Casson**]
- Define Haken manifold.
- Define incompressible surface.
- Define $\mathbb{P}^2$-irreducible.
- Can you give an example of a Haken manifold with a separating incompressible surface? [**Casson**]
- One without boundary?
- Suppose M is non-Haken, specifically, that it is closed, orientable, irreducible, but does not contain an incompressible surface. Does every embedded torus in M necessarily bound a solid torus? [**Casson**]
- Suppose that M is an orientable compact irreducible manifold with $\pi_1(M)$ free. What is M?
- Consider a $+k$ surgery on a trefoil. How can this be related to a Heegard splitting?
- Why is it sufficient to specify a homotopy class of essential curves in a torus to define a Dehn surgery?
- Give a surgery presentation for a homology 3-sphere.
- Give a surgery on a knot which gives a reducible manifold.
- Give an example of a Haken manifold with a separating incompressible surface.
- Show that a manifold M is Haken iff it is incompressible and its fundamental group can be written non-trivially as $\pi_1(M) = A *_B C$ or as $\pi_1(M) = A*_B$.
- Give an example of a non-Haken manifold.
- Show that if M^3 is prime and orientable then $\pi_2(M) = 0$ or M^3 is an S^2 bundle over S^1.
- Show that any Solv manifold is virtually Haken.
- Show that any knot has a Seifert surface.
- If two 3-manifolds have the same π_1 and π_2 is zero, are they homotopy equivalent?
- Show that any closed compact hyperbolic 3-manifold contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroups.
- What is the Euler number of the S^1-bundle determined by the natural Seifert fibered structure on teh Poincaré homology sphere?
- Let $K \subset S^3$ be a fibered knot. What surgeries on K give a 3-manifold which fibers over S^1?
- Give an example of a non-prime 3-manifold admitting a geometric structure.
- Give an example of a homeomorphism Ψ from the punctured torus to itself whose mapping torus does not admit a hyperbolic structure.
- What is a Dehn surgery presentation for the quotient of S^3 by the group of order 48 giving the truncated cube space?
- Why does the Mostow Rigidity theorem fail for infinite volume manifolds?

- Give an explicit construction of a closed hyperbolic 3-manifold.
- Suppose that α and β are simple closed non-intersecting curves on a hyperbolic surface S. Show that their geodesic representatives α' and β' do not intersect.
- Show that no closed compact surface in $\mathbb{R}^3$ can have strictly negative curvature everywhere.
- Show that there are $\mathbb{R}^3$ hyperbolic structures on a pair of pants with geodesic boundary.
- Which one of Moise's theorem did you mean? [**Kirby**]
- What is Waldhausen's theorem? [**Kirby**]
- What about one with an infinite fundamental group? [**Kirby**]
- Explain why any 3-manifold with an infinite H_1 is sufficiently large. (i.e. Haken) [**Kirby**]
- Why does an element of the first cohomology of M correspond to a map from M to S^1? [**Kirby**]
- Is there a similar result for higher cohomology groups? [**Kirby**]
- List all double covers of $\mathbb{RP}^2 \vee S^1$ [**Casson**]
- Compute (geometrically) the Alexander polynomial for the figure eight knot and relate it to the homology of (finite) cyclic covers of the knot complement. [**McMullen?**]
- Let M be a closed, oriented 3-manifold. Show irreducible implies prime. What if M is not oriented?
- Let $T^2 \subset S^3$ be an embedded torus. Show that this torus compresses. [**Stallings**]