Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Commutative Algebra

- Let A be a commutative ring, M a finitely generated A-module, and $x_1, \ldots, x_n$ elements of M which generate $M/\mathbf{m}M$ for every $\mathbf{m}$ maximal ideal of A. Show that they generate M. [**Vojta**]
- What is $\mathbb{Z}_{q-1}^*$?
- Give an example of a commutative ring with unity which has prime ideals which are not maximal.
- Give two examples of UFD's which are not PID's.
- Suppose $p(x) \in F[x]$ where F is a field. Let $p(a) = 0$ for $a \in E$, an extension of F. Show that $p(x) = (x - a)q(x)$ for some $q(x) \in F[x]$.
- Show that an element a is irreducible if and only if (a) is maximal, where $a \in R$ and R is a PID.
- Let R be a finite commutative ring free of zero divisors. Show that R has a unit, show that each non-zero element has an inverse. Is the result still true if R is infinite?
- Is there a general class of rings in which maximal ideals and prime ideals are the same?
- Let P be a prime ideal in R such that R/P is a finite ring. Show that P is maximal if R is a commutative ring with unit.
- Show that in a commutative ring with unit every maximal ideal must be prime.
- Describe the ring of endomorphisms of the integers.
- If R is a UFD, prove that $R[X]$ is a UFD.
- Let R be a commutative ring with unity $1 \neq 0$, and let S be a multiplicative subset of R not containing 0. Consider the set of all ideals A which do not intersetct S. Show that a maximal element in this set must be a prime ideal.
- Let R be the ring of real quaternions. Does $R[X]$ satisfy the division algorithm property?
- Distinguish algebraically between $GL(3, \mathbb{R})$ and $GL(2, \mathbb{R})$, not using topology.
- What is a Dedekind ring?
- Prove that $k[X, Y]$ is not Dedekind.

- Let $\Omega \subseteq \mathbb{C}$ be a domain, and $\mathcal{O}(\Omega)$, $\mathcal{O}_F(\Omega)$ the ring of homomorphic functions and the subring of functions with finitely many zeroes. Is $\mathcal{O}(\Omega)$ or $\mathcal{O}_F(\Omega)$ a UFD? What are the primes in these rings?
- Can you give an example of a ring R which is not Cohen-Macauley? [**Ogus**]
- Can you give an example of a ring R which is Cohen-Macauley but not Gorenstein? [**Ogus**]
- For a dimension zero Gorensteins ring, what can you say about the R-module $\operatorname{Hom}_k(R, k)$? $\operatorname{Hom}_R(k, \operatorname{Hom}_k(R, k))$? [**Ogus/Lenstra**]
- Given R a domain, Noetherian, dimension 1, what can we say about $\tilde{R}$, its integral closure? [**Lenstra**]
- Let's prove that in the above case, $\tilde{R}$ is Noetherian. [**Lenstra**]
- Define "Hilbert function". [**Sturmfels**]
- What conditions on a graded ring $S = \oplus_{d=0}^{\infty} S_d$ will assure the agreement of the Hilbert function with a polynomiial? [**Sturmfels**]
- What sorts of invariants appear in the Hilbert polynomial? [**Sturmfels**]
- Given an ideal I, how would one compute its Hilbert function? [**Sturmfels**]
- Define Gröbner basis, term order and initial ideal. [**Sturmfels**]
- How would you calculate $\operatorname{lt}(I)$ where $\operatorname{lt}(I) = \langle \operatorname{lt}(f) \mid f \in I \rangle$ and $\operatorname{lt}(f)$ is the sum of terms of highest degree of f? [**Sturmfels**]
- Give an example of a term order that refines the partial order by degree. [**Casson**]
- Given an example of an ideal $I = \langle f_1, \ldots, f_t \rangle$ such that $\operatorname{lt}(I) \neq \langle \operatorname{lt}(f_1), \ldots, \operatorname{lt}(f_t) \rangle$. [**Sturmfels**]
- Find the ideal $I(X)$ for X the twisted cubic in $\mathbb{A}^3$. [**Sturmfels**]
  - Show that two generators suffice.
  - Find a term order such that these two generators are not a Gröbner basis for $I(X)$ but all three are.
  - Show that $\langle x^2 - yw, xz - y^2, xy - zw \rangle$ generate the ideal of $\overline{X}$, the closure of X in $\mathbb{P}^3$.
  - Compute the Hilbert polynomial of X.
- Tell me about integral extensions. [**Hartshorne**]
- What is a Dedekind domain? [**Hartshorne**]
  - An example of a domain which is noetherian, integrally closed, and not one-dimensional.
  - An example of a domain which is integrally closed, one-dimensional, and not noetherian.
- Suppose that $I \subseteq J$ are ideals of A, B/A is an extension of rings such that $IB = JB$; does it follow that $I = J$? If not, can you give a counterexample? [**Hartshorne**]
  - Is there some hypothesis that makes it work?

- Can you give an example of a surjective morphism of rings which is not finite? [**Hartshorne**]
- State Nakayama's Lemma [**Bergman**]
    - Give an example of a ring and a nonzero ideal that satisfy the hypothesis. [**Bergman**]
- Prove, possibly using Nakayama's Lemma, that if M is an $n \times n$ matrix over a local ring, with coefficients in the maximal ideal I, then $I + M$ is invertible. [**Bergman**]
- Show that a PID has dimension 0 or 1.
- Let A be a noetherian valuation ring which is not a field. Show that A is a dvr.
- Let $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be ideals of a commutative ring A such that $\mathbf{a}_i + \mathbf{a}_j = (1)$ for all $i \neq j$. Show that

$$\prod_{i=1}^{r} \mathbf{a}_i = \bigcap_{i=1}^{r} \mathbf{a}_i.$$

- Let M be an A-module, $S \subset A$ a multiplicative subset of A. Do we have

$$S^{-1}\mathrm{Ann}(M) = \mathrm{Ann}S^{-1}(M)?$$

  If not, give a counterexample, and if yes prove it. What if M is finitely generated?
- Let M be an A-module, $\mathbf{m}$ a maximal ideal of A. Prove that $M/\mathbf{m}M \cong M_{\mathbf{m}}/\mathbf{m}M_{\mathbf{m}}$.
- Let M be a finitely generated A-module. If $M = \mathbf{m}M$ for every maximal ideal $\mathbf{m}$ of A show that $M = 0$.
- Let M be a finitely generated A-module. Suppose that $x_1, \ldots, x_n$ generate $M_{\mathbf{m}}$ for every maximal ideal $\mathbf{m}$ of A. Show they generate M.
- Let $B \supset A$ be commutative rings, with B integral over A. Let $x \in A$, $x \in B^*$. Show that $x \in A^*$.
- Prove that any ideal in a Dedekind ring is generated by at most two elements.
- Show that $\mathbb{Z}_p$ is a dvr.
- Show that an Artin ring has only finitely many maximal ideals.
- Let $A \supset B$ be rings with B integral over A. Let $\mathbf{q}$ be a prime ideal of B, and let $\mathbf{p} = \mathbf{q} \cap A$. Show that $\mathbf{q}$ is maximal in B if an only if $\mathbf{p}$ is maximal in A.
- Show that if $f \colon A \to A$ is a surjective endomorphism of a noetherian ring A, then f is an automorphism.
- Give an example of an ideal in $k[x, y, z]$ whose associated primes are (x, y) and (x, y, z). [**Eisenbud**]
- Give an example of a surface that has only one singular point but is not normal. [**Eisenbud**]
- In the example $k[x, y]/(y^2 - x^3) = R$, prove that it's not integrally closed. State Serre's Criterion. Write down all the primes of R. Can you see them in the picture

of R? What parts of Serre's criterion does R satisfy? What parts does it not satisfy? Prove that $R_{(x,y)}$ is not a dvr.

(Hint: to show that the maximal ideal cannot be generated by 1 element, think of Nakayama's lemma) [**Eisenbud**]

- How do you decide if a given polynomial belongs to an ideal I? [**Sturmfels**]
- State the main theorem of Elimination Theory. Why is it called this? [**Sturmfels**]
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