QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, February 25, 1997 (Day 1)

1. Factor the polynomial $x^3 - x + 1$ and find the Galois group of its splitting field if the ground field is:

$$\text{a) } \mathbf{R}, \qquad \text{b) } \mathbf{Q}, \qquad \text{c) } \mathbf{Z}/2\mathbf{Z}\,.$$

2. Let A be the $n \times n$ (real or complex) matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1/n & 1/n & 1/n & \dots & 1/n \end{pmatrix}.$$

Prove that as $k \to \infty$, A^k tends to a projection operator P onto a one-dimensional subspace. Find $\ker P$ and $\operatorname{Image} P$.

3. a. Show that there are infinitely many primes p congruent to 3 mod 4.
b. Show that there are infinitely many primes p congruent to 1 mod 4.

4. a. Let L_1, L_2 and $L_3 \subset \mathbf{P}^3_{\mathbf{C}}$ be three pairwise skew lines. Describe the locus of lines $L \subset \mathbf{P}^3_{\mathbf{C}}$ meeting all three.
b. Now let L_1, L_2, L_3 and $L_4 \subset \mathbf{P}^3_{\mathbf{C}}$ be four pairwise skew lines. Show that if there are three or more lines $L \subset \mathbf{P}^3_{\mathbf{C}}$ meeting all four, then there are infinitely many.

5. a. State the Poincaré duality and Kunneth theorems for homology with coefficients in $\mathbf{Z}$ (partial credit for coefficients in $\mathbf{Q}$).
b. Find an example of a compact 4-manifold M whose first and third Betti numbers are not equal, that is, such that $H^1(M, \mathbf{Q})$ and $H^3(M, \mathbf{Q})$ do not have the same dimension.

6. Compute

$$\int_0^\infty \frac{\log x}{x^2 + b^2}\, dx$$

for b a positive real number.

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Wednesday, February 26, 1997 (Day 2)

1. Define a metric on the unit disc $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$ by the line element

$$ds^2 \;=\; \frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^p}\,.$$

Here (r, θ) are polar coordinates and p is any real number.
   a. For which p is the circle $r = 1/2$ a geodesic?
   b. Compute the Gaussian curvature of this metric.

2. Let $\mathcal{C}$ be the space $\mathcal{C}[0, 1]$ of continuous real-valued functions on the closed interval $[0, 1]$, with the sup norm

$$||f||_\infty \;=\; \max_{t\in[0,1]} f(t)\,.$$

Let $\mathcal{C}^1$ be the space $\mathcal{C}^1[0, 1]$ of $\mathcal{C}^1$ functions on $[0, 1]$ with the norm

$$||f|| \;=\; ||f||_\infty + ||f'||_\infty\,.$$

Prove that the natural inclusion $\mathcal{C}^1 \subset \mathcal{C}$ is a compact operator.

3. Let M be a compact Riemann surface, and let f and g be two meromorphic functions on M. Show that there exists a polynomial $P \in \mathbf{C}[X, Y]$ such that $P(f(z), g(z)) \equiv 0$.

4. Let $S^3 = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 = 1\}$. Let p be a prime and m an integer relatively prime to p. Let ζ be a primitive p^{th} root of unity, and let the group G of p^{th} roots of unity act on S^3 by letting $\zeta \in G$ send (z, w) to $(\zeta z, \zeta^m w)$. Let $M = S^3/G$.
   a. compute $\pi_i(M)$ for $i = 1$, 2 and 3.
   b. compute $H_i(M, \mathbf{Z})$ for $i = 1$, 2 and 3.
   c. compute $H^i(M, \mathbf{Z})$ for $i = 1$, 2 and 3.

5. Let d be a square-free integer. Compute the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. Give an example where this ring is not a principal ideal domain, and give an example of a non-principal ideal.

6. Prove that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \, .$$

QUALIFYING EXAMINATION
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Thursday, October 17, 1996 (Day 3)

1. Let $\alpha : (0,1) \to \mathbf{R}^3$ be any regular arc (that is, α is differentiable and α' is nowhere zero). Let $\mathbf{t}(u)$, $\mathbf{n}(u)$ and $\mathbf{b}(u)$ be the unit tangent, normal and binormal vectors to α at $\alpha(u)$. Consider the *normal tube of radius* ϵ around α, that is, the surface given parametrically by

$$\phi(u,v) \;=\; \alpha(u) + \epsilon\cos(v)\mathbf{n}(u) + \epsilon\sin(v)\mathbf{b}(u)\,.$$

a. For what values of ϵ is this an immersion?

b. Assuming that α itself has finite length, find the surface area of the normal tube of radius ϵ around α.

The answers to both questions should be expressed in terms of the curvature $\kappa(u)$ and torsion $\tau(u)$ of α.

2. Recall that a *fundamental solution* of a linear partial differential operator P on $\mathbf{R}^n$ is a distribution E on $\mathbf{R}^n$ such that $PE = \delta$ in the distribution sense, where δ is the unit Dirac measure at the origin. Find a fundamental solution E of the Laplacian on $\mathbf{R}^3$

$$\Delta \;=\; \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

that is a function of $r = |x|$ alone. Prove that your fundamental solution indeed satisfies $\Delta E = \delta$.

Hint: Use the appropriate form of Green's theorem.

3. The group of rotations of the cube in $\mathbf{R}^3$ is the symmetric group S_4 on four letters. Consider the action of this group on the set of 8 vertices of the cube, and the corresponding permutation representation of S_4 on $\mathbf{C}^8$. Describe the decomposition of this representation into irreducible representations.

4. Suppose a_i, $i = i, \ldots, n$ are positive real numbers with $a_1 + \ldots + a_n = 1$. Prove that for any nonnegative real numbers $\lambda_1, \ldots, \lambda_n$,

$$\sum_{i=1}^{n} a_i \lambda_i^2 \geq \left(\sum_{i=1}^{n} a_i \lambda_i \right)^2$$

with equality holding only if $\lambda_1 = \ldots = \lambda_n$.

5. a. For which natural numbers n is it the case that every continuous map from $\mathbf{P}^n_{\mathbf{C}}$ to itself has a fixed point?

b. For which n is it the case that every continuous map from $\mathbf{P}^n_{\mathbf{R}}$ to itself has a fixed point?

6. Fermat proved that the number $2^{37} - 1 = 137438953471$ was composite by finding a small prime factor p. Suppose you know that $200 < p < 300$. What is p?