Problem 1 (RA)
Let $X$ be a subset of $[0, 1]$ with the following two properties:
- For any real number, $r$, there is $x \in X$ such that $r - x$ is rational.
- For any two distinct $x, y \in X$, the number $x - y$ is irrational.
Prove that $X$ is not Lebesgue-measurable.

Problem 2 (DG)
Use $(x, y, z)$ for the Euclidean coordinate functions on $\mathbb{R}^3$ and let $a$ denote the 1-form

$$a = dz + \frac{1}{2} (x \, dy - y \, dx).$$

a) Compute $da$ and $a \wedge da$.
b) Prove that the kernel of $a$ defines a smooth, 2-dimensional vector subbundle in $T\mathbb{R}^3$.
c) Suppose that $B \subset \mathbb{R}^3$ is an open ball and that $u$ and $w$ are pointwise linearly independent vector fields in the kernel of $a$ on $B$. Prove that the commutator of $u$ and $v$ is nowhere in the kernel of $a$.

Problem 3 (CA)
How many roots of the polynomial $P(z) = z^4 - 6z + 3$ occur where $|z| < 2$?
Problem 4 (T)
Let $X_1$ and $X_2$ denote distinct copies of $T^2$, so each is $S^1 \times S^1$ with $S^1$ denoting the circle. Define the space $X$ to be the quotient of the disjoint union of $X_1$ and $X_2$ by the equivalence relation whereby any point of the form $(z, 1)$ in $X_1$ is identified with the corresponding $(z, 1)$ in $X_2$. Compute the cohomology ring of the space $X$. (Thus, compute $H^*(X; \mathbb{Z})$ and determine its cup-product structure.)

Problem 5 (AG)
Let $X$ denote the affine curve in $\mathbb{A}^2$ where $y^2 - x^3 + x^2 = 0$. Prove that $X$ is singular and that there is a birational morphism from $\mathbb{A}^1$ onto $X$.

Problem 6 (AN)
Let $G_1, \ldots, G_n$ denote finite groups. For each $m \in \{1, \ldots, n\}$, let $\rho_m: G_m \rightarrow \text{GL}(V_m)$ denote a finite dimensional, complex representation of $G_m$. Use $\chi_m$ to denote the character of $\rho_m$. Set $G = G_1 \times \cdots \times G_n$ and $V = V_1 \otimes \cdots \otimes V_n$.

a) Define $\rho: G \rightarrow \text{GL}(V)$ by the rule $\rho(g_1, \ldots, g_n) = \rho_1(g_1) \otimes \cdots \otimes \rho_n(g_n)$. Write the character of $\rho$ in terms of the characters $\{\chi_m\}_{1 \leq m \leq n}$.

b) Prove that $(V, \rho)$ is an irreducible representation of $G$ if and only if, for all $m$, each $(V_m, \rho_m)$ is an irreducible representation of $G_m$. 
Problem 1 (RA)

a) Let $A_1, A_2, \ldots$ be a countable collection of events in a probability space; and let $A_1^c, A_2^c, \ldots$ denote their respective complements. Prove the following assertion: If $A_1, A_2, \ldots$ are mutually independent, then $A_1^c, A_2^c, \ldots$ are also mutually independent.

(A collection of events $\{A_n\}_{n=1,2,\ldots}$ is said to be mutually independent when any member is independent of the mutual intersection of any finite subcollection of the other events.)

b) Let $A_1, A_2, \ldots$ denote a sequence of mutually independent events in a probability space with the property that the sum of their probabilities is infinite. Prove that with probability one, the event $A_n$ must occur for infinitely many values of the index $n$.

Problem 2 (DG)

The Euclidean metric on $\mathbb{R}^2$ can be written using the standard rectilinear coordinates $(x,y)$ as $dx \otimes dx + dy \otimes dy$. Let $u$ denote a smooth function on $\mathbb{R}^2$ and let $g$ denote the metric $e^{2u} (dx \otimes dx + dy \otimes dy)$. Let $\nabla$ denote the corresponding Levi-Civita covariant derivative for the metric $g$, acting on sections of $T^*\mathbb{R}^2$.

a) Write $\nabla(dx)$ and $\nabla(dy)$ in terms of $u$ and its derivatives.

b) Write the scalar curvature of the metric $g$ in terms of $u$ and its first and second derivatives.

Problem 3 (CA)
Supposing that \( a \) is a positive number, evaluate the integral \( \int_0^\infty \frac{\cos^2(x)}{x^2 + a^2} \, dx \) using the method of residues.

**Problem 4** (T)

Let \( X = T^2 \lor S^2 \) which is the join of the torus \( T^2 \) (which is \( S^1 \times S^1 \)) and the 2-sphere \( S^2 \).

a) Describe the universal covering space of \( X \).

b) Compute \( \pi_1(X) \).

c) Compute \( \pi_2(X) \).

**Problem 5** (AG)

Show that for any genus 2 curve, \( C \), there is a divisor on \( C \) which has degree greater than zero, but is not linearly equivalent to an effective divisor. (Hint: The Riemann-Roch formula states that \( h^0(C, \mathcal{L}) - h^0(C, K_C \otimes \mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C) \) for a line bundle \( \mathcal{L} \) on a curve \( C \). Here, \( K_C \) denotes the canonical bundle of \( C \) and \( g(C) \) denotes the genus of \( C \).)

**Problem 6** (AN)

Let \( k \) denote a finite field of \( 2^f \) elements for some positive integer \( f \).

a) Prove that the map from \( k \) to itself given by \( x \mapsto x^2 + x \) is a homomorphism of additive groups. Assuming this, then prove that exactly \( 2^{f-1} \) elements of \( k \) can be written as \( x^2 + x \) for some \( x \in k \).

b) Prove that any given \( a \in k \) can be written as \( x^2 + x \) for some \( x \in k \) if and only if \( \sum_{i=0}^{f-1} a^i = 0 \).
Problem 1 (RA)
For $f : \mathbb{R} \to \mathbb{R}$ a Lebesgue measurable function, let $\|f\|$ denote the norm,

$$\|f\| = \int |f| \, d\mu$$

where $d\mu$ is the Lebesgue measure. Let $L^1(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ with $\|f\| < \infty$ (we identify functions that are equal almost everywhere). If $f$, $g$ are functions in $L^1(\mathbb{R})$, define their convolution (an $\mathbb{R}$-valued function on $\mathbb{R}$ denoted by $f \ast g$) by the following rule:

$$(f \ast g)(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt \quad \text{if} \quad \int_{\mathbb{R}} |f(x - t)| |g(t)| \, dt \text{ is finite; and } (f \ast g)(x) = 0 \text{ otherwise.}$$

Prove the following: There is no function $\epsilon \in L^1(\mathbb{R})$ such that $\epsilon \ast f = f$ for all $f \in L^1(\mathbb{R})$.

Here are two hints: First, keep in mind that a function is Lebesgue measurable when, for any real number $E$, the set of points in $\mathbb{R}$ where the function is less than $E$ is Lebesgue measurable. Second, consider the sequence $\{f_n\}_{n=1,2,\ldots}$ of functions on $\mathbb{R}$ which is defined as follows: For any given positive integer $n$, set $f_n(x) = 1$ if $-\frac{1}{n} \leq x \leq \frac{1}{n}$, and set $f_n(x) = 0$ otherwise.

Problem 2 (DG)
View the 4-dimensional sphere (denoted by $S^4$) as the 1-point compactification of $\mathbb{R}^4$, thus $\mathbb{R}^4 \cup \infty$. A complex, rank 2 vector bundle over $S^4$ (to be denoted by $E$) can be defined as follows: Cover the sphere $S^4$ by the two open sets $\mathbb{R}^4$ and $(\mathbb{R}^4 - 0) \cup \infty$. A map (to be denoted by $\varrho$) from their intersection (which is $\mathbb{R}^4 - 0$) to the group $SU(2)$ (the group of $2 \times 2$ unitary matrices with determinant 1) is defined by first writing the Euclidean coordinates of any given $x \in \mathbb{R}^4$ as $(x_1, x_2, x_3, x_4)$ and using these coordinate functions to define $\varrho(x)$ for $x \in \mathbb{R}^4 - 0$ by
The vector bundle $\mathcal{E}$ is the quotient of the product $\mathbb{C}^2$ bundle over the $\mathbb{R}^4$ part of $S^4$, and the product $\mathbb{C}^2$ bundle over the complement in $S^4$ of $0 \in \mathbb{R}^4$ by the equivalence relation that identifies pairs $(x, s_0) \in \mathbb{R}^4 \times \mathbb{C}^2$ and $(y, s_1) \in ((\mathbb{R}^4 - 0) \cup \infty)$ when $x$ and $y$ are in $\mathbb{R}^4 - 0$ and $x = y$ and $s_1 = g(x)s_0$. (The projection map from $\mathcal{E}$ to $S^4$ sends the equivalence class of any $(x, s)$ for $x \in \mathbb{R}^4$ to $x$; and it sends the equivalence class of $(\infty, s)$ to $\infty$.)

a) Write a connection on this vector bundle.

b) Compute the curvature 2-form of your connection.

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**Problem 3 (CA)**

a) Prove that $\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$ defines a meromorphic function on $\mathbb{C}$ with poles only at the points in $\mathbb{Z}$.

b) Prove that $\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}$.

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**Problem 4 (T)**

a) Construct a connected, topological space $X$ such that $\pi_1(X)$ is generated by two elements, denoted by $a$ and $b$, subject to the relations $a^3 = 1$ and $b^3 = 1$.

b) For which $q \geq 1$, is $H_q(X; \mathbb{Z})$ independent of your choice of $X$?
**Problem 5 (AG)**
The twisted cubic (to be denoted by $X$) is the image of the map from $\mathbb{P}^1$ to $\mathbb{P}^3$ defined using homogeneous coordinates by the rule $[s:\!t] \to [s^3 : s^2t : st^2 : t^3]$. It is also the locus in $\mathbb{P}^3$ where the three polynomials $\{z_0z_3 - z_1z_2, z_0z_2 - z_1^2, z_1z_3 - z_2^2\}$ are simultaneously zero. Prove that the Hilbert polynomial of the twisted cubic, $\mathcal{P}_X$, obeys $\mathcal{P}_X(n) = 3n+1$.

**Problem 6 (AN)**
Prove that there is a unique positive integer $n \leq 10^{2017}$ such that the last 2017 digits of $n^3$ are 0000 \ldots 00002017 (with all 2005 digits represented by \ldots being zeros as well).