Qualifying Examination
HARVARD UNIVERSITY
Department of Mathematics
Spring 2018 (Day 1)

Problem 1 (RA)
Let X be a subset of [0, 1] with the following two properties:

- For any real number, r, there is $x \in X$ such that $r - x$ is rational.
- For any two distinct $x, y \in X$, the number $x - y$ is irrational.

Prove that X is *not* Lebesgue-measurable.

Problem 2 (DG)
Use (x, y, z) for the Euclidean coordinate functions on $\mathbb{R}^3$ and let a denote the 1-form

$$a = dz + \tfrac{1}{2}(x\,dy - y\,dx).$$

a) Compute da and $a \wedge da$.
b) Prove that the kernel of a defines a smooth, 2-dimensional vector subbundle in $T\mathbb{R}^3$.
c) Suppose that $B \subset \mathbb{R}^3$ is an open ball and that u and w are pointwise linearly independent vector fields in the kernel of a on B. Prove that the commutator of u and v is nowhere in the kernel of a.

Problem 3 (CA)
How many roots of the polynomial $P(z) = z^4 - 6z + 3$ occur where $|z| < 2$?

Problem 4 (T)

Let X_1 and X_2 denote distinct copies of T^2, so each is $S^1 \times S^1$ with S^1 denoting the circle. Define the space X to be the quotient of the disjoint union of X_1 and X_2 by the equivalence relation whereby any point of the form (z, 1) in X_1 is identified with the corresponding (z, 1) in X_2. Compute the cohomology ring of the space X. (Thus, compute $H^*(X; \mathbb{Z})$ and determine its cup-product structure.)

Problem 5 (AG)

Let X denote the affine curve in $\mathbb{A}^2$ where $y^2 - x^3 + x^2 = 0$. Prove that X is singular and that there is a birational morphism from $\mathbb{A}^1$ onto X.

Problem 6 (AN)

Let $G_1, \ldots, G_n$ denote finite groups. For each $m \in \{1, \ldots, n\}$, let $\rho_m: G_m \to GL(V_m)$ denote a finite dimensional, complex representation of G_m. Use χ_m to denote the character of ρ_m. Set $G = G_1 \times \cdots \times G_m$ and $V = V_1 \otimes \cdots \otimes V_m$.

a) Define $\rho: G \to GL(V)$ by the rule $\rho(g_1, \ldots, g_n) = \rho_1(g_1) \otimes \cdots \otimes \rho_n(g_n)$. Write the character of ρ in terms of the characters $\{\chi_m\}_{1 \le m \le n}$.

b) Prove that (V, ρ) is an irreducible representation of G if and only if, for all m, each (V_m, ρ_m) is an irreducible representation of G_m.

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Problem 1 (RA)

a) Let $A_1, A_2, \ldots$ be a countable collection of events in a probability space; and let $A_1^c, A_2^c, \ldots$ denote their respective complements. Prove the following assertion: If $A_1, A_2, \ldots$ are mutually independent, then $A_1^c, A_2^c, \ldots$ are also mutually independent.

(A collection of events $\{A_n\}_{n=1,2,\ldots}$ is said to be mutually independent when any member is independent of the mutual intersection of any finite subcollection of the other events.)

b) Let $A_1, A_2, \ldots$ denote a sequence of mutually independent events in a probability space with the property that the sum of their probabilities is infinite. Prove that with probability one, the event A_n must occur for infinitely many values of the index n.

Problem 2 (DG)

The Euclidean metric on $\mathbb{R}^2$ can be written using the standard rectilinear coordinates (x, y) as $dx \otimes dx + dy \otimes dy$. Let u denote a smooth function on $\mathbb{R}^2$ and let $\mathfrak{g}$ denote the metric $e^{2u}(dx \otimes dx + dy \otimes dy)$. Let ∇ denote the corresponding Levi-Civita covariant derivative for the metric $\mathfrak{g}$, acting on sections of $T^*\mathbb{R}^2$.

a) Write $\nabla(dx)$ and $\nabla(dy)$ in terms of u and its derivatives.

b) Write the scalar curvature of the metric $\mathfrak{g}$ in terms of u and its first and second derivatives.

Problem 3 (CA)

Supposing that a is a positive number, evaluate the integral $\int_0^\infty \frac{\cos^2(x)}{x^2+a^2}\,dx$ using the method of residues.

Problem 4 (T)
Let $X = T^2 \vee S^2$ which is the join of the torus T^2 (which is $S^1 \times S^1$) and the 2-sphere S^2.

a) Describe the universal covering space of X.

b) Compute $\pi_1(X)$.

c) Compute $\pi_2(X)$.

Problem 5 (AG)
Show that for any genus 2 curve, C, there is a divisor on C which has degree greater than zero, but is not linearly equivalent to an effective divisor. (Hint: The Riemann-Roch formula states that $h^0(C,\mathcal{L}) - h^0(C, K_C \otimes \mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C)$ for a line bundle $\mathcal{L}$ on a curve C. Here, K_C denotes the canonical bundle of C and $g(C)$ denotes the genus of C.)

Problem 6 (AN)
Let k denote a finite field of 2^f elements for some positive integer f.

a) Prove that the map from k to itself given by $x \to x^2 + x$ is a homomorphism of additive groups. Assuming this, then prove that exactly 2^{f-1} elements of k can be written as $x^2 + x$ for some $x \in k$.

b) Prove that any given $a \in k$ can be written as $x^2 + x$ for some $x \in k$ if and only if $\sum_{i=0}^{f-1} a^{2^i} = 0$.

Qualifying Examination
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Problem 1 (RA)
For f: $\mathbb{R} \to \mathbb{R}$ a Lebesgue measurable function, let $\|f\|$ denote the norm,

$$\|f\| = \int_{\mathbb{R}} |f|\, d\mu$$

where $d\mu$ is the Lebesgue measure. Let $L^1(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of Lebesgue measurable functions f: $\mathbb{R} \to \mathbb{R}$ with $\|f\| < \infty$ (we identify functions that are equal almost everywhere). If f, g are functions in $L^1(\mathbb{R})$, define their convolution (an $\mathbb{R}$-valued function on $\mathbb{R}$ denoted by $f*g$) by the following rule:

$$(f*g)(x) = \int_{\mathbb{R}} f(x-t)\, g(t)\, dt \quad \text{if} \quad \int_{\mathbb{R}} |f(x-t)||g(t)|\, dt \textit{ is finite; and } (f*g)(x) = 0 \textit{ otherwise.}$$

Prove the following: There is no function $\mathfrak{e} \in L^1(\mathbb{R})$ such that $\mathfrak{e} * f = f$ for all $f \in L^1(\mathbb{R})$.

Here are two hints: First, keep in mind that a function is Lebesgue measurable when, for any real number E, the set of points in $\mathbb{R}$ where the function is less than E is Lebesgue measurable. Second, consider the sequence $\{f_n\}_{n=1,2,..}$ of functions on $\mathbb{R}$ which is defined as follows: For any given positive integer n, set $f_n(x) = 1$ if $-\frac{1}{n} \le x \le \frac{1}{n}$, and set $f_n(x) = 0$ otherwise.

Problem 2 (DG)
View the 4-dimensional sphere (denoted by S^4) as the 1-point compactification of $\mathbb{R}^4$, thus $\mathbb{R}^4 \cup \infty$. A complex, rank 2 vector bundle over S^4 (to be denoted by $\mathbb{E}$) can be defined as follows: Cover the sphere S^4 by the two open sets $\mathbb{R}^4$ and $(\mathbb{R}^4 - 0) \cup \infty$. A map (to be denoted by g) from their intersection (which is $\mathbb{R}^4 - 0$) to the group SU(2) (the group of 2×2 unitary matrices with determinant 1) is defined by first writing the Euclidean coordinates of any given $x \in \mathbb{R}^4$ as (x_1, x_2, x_3, x_4) and using these coordinate functions to define $g(x)$ for $x \in \mathbb{R}^4 - 0$ by

$$g(x) = \frac{1}{|x|}\begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} .$$

The vector bundle $\mathbb{E}$ is the quotient of the product $\mathbb{C}^2$ bundle over the $\mathbb{R}^4$ part of S^4, and the product $\mathbb{C}^2$ bundle over the complement in S^4 of $0 \in \mathbb{R}^4$ by the equivalence relation that identifies pairs $(x, s_0) \in \mathbb{R}^4 \times \mathbb{C}^2$ and $(y, s_1) \in ((\mathbb{R}^4 - 0) \cup \infty)$ when x and y are in $\mathbb{R}^4 - 0$ and $x = y$ and $s_1 = g(x)\, s_0$. (The projection map from $\mathbb{E}$ to S^4 sends the equivalence class of any (x, s) for $x \in \mathbb{R}^4$ to x; and it sends the equivalence class of (∞, s) to ∞.)

a) Write a connection on this vector bundle.

b) Compute the curvature 2-form of your connection.

Problem 3 (CA)

a) Prove that $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ defines a meromorphic function on $\mathbb{C}$ with poles only at the points in $\mathbb{Z}$.

b) Prove that $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}$.

Problem 4 (T)

a) Construct a connected, topological space X such that $\pi_1(X)$ is generated by two elements, denoted by a and b, subject to the relations $a^3 = 1$ and $b^3 = 1$.

b) For which $q \geq 1$, is $H_q(X; \mathbb{Z})$ independent of your choice of X?

Problem 5 (AG)
The *twisted cubic* (to be denoted by X) is the image of the map from $\mathbb{P}^1$ to $\mathbb{P}^3$ defined using homogeneous coordinates by the rule $[s:t] \to [s^3 : s^2t : st^2 : t^3]$. It is also the locus in $\mathbb{P}^3$ where the three polynomials $\{z_0z_3 - z_1z_2, z_0z_2 - z_1^2, z_1z_3 - z_2^2\}$ are simultaneously zero. Prove that the Hilbert polynomial of the twisted cubic, $\wp_X$, obeys $\wp_X(n) = 3n+1$.

Problem 6 (AN)
Prove that there is a unique positive integer $n \leq 10^{2017}$ such that the last 2017 digits of n^3 are $0000 \cdots 00002017$ (with all 2005 digits represented by $\cdots$ being zeros as well).