Qualifying Examination

HARVARD UNIVERSITY
Department of Mathematics
Tuesday, January 19, 2016 (Day 1)

PROBLEM 1 (DG)

Let S denote the surface in $\mathbb{R}^3$ where the coordinates (x, y, z) obey $x^2 + y^2 = 1 + z^2$. This surface can be parametrized by coordinates $t \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ by the map

$$(t, \theta) \to \psi(t,\theta) = (\sqrt{1+t^2}\cos\theta, \sqrt{1+t^2}\sin\theta, t).$$

a) Compute the induced inner product on the tangent space to S using these coordinates.
b) Compute the Gaussian curvature of the metric that you computed in Part a).
c) Compute the parallel transport around the circle in S where $z = 0$ for the Levi-Civita connection of the metric that you computed in Part a).

PROBLEM 2 (T)

Let X be path-connected and locally path-connected, and let Y be a finite Cartesian product of circles. Show that if $\pi_1(X)$ is finite, then every continuous map from X to Y is null-homotopic. (Hint: recall that there is a fiber bundle $\mathbb{Z} \to \mathbb{R} \to S^1$.)

PROBLEM 3 (AN)

Let K be the field $\mathbb{C}(z)$ of rational functions in an indeterminate z, and let $F \subset K$ be the subfield $\mathbb{C}(u)$ where $u = (z^6 + 1)/z^3$.

a) Show that the field extension K/F is normal, and determine its Galois group.
b) Find all fields E, other than F and K themselves, such that $F \subset E \subset K$. For each E, determine whether the extensions E/F and K/E are normal.

PROBLEM 4 (AG)

The nodal cubic is the curve in $\mathbb{CP}^2$ (denoted by X) given in homogeneous coordinates (x, y, z) by the locus $\{zy^2 = x^2(x+z)\}$.

a) Give a definition of a rational map between algebraic varieties.

b) Show that there is a birational map from X to $\mathbb{CP}^1$.

c) Explain how to resolve the singularity of X by blowing up a point in $\mathbb{CP}^2$.

PROBLEM 5 (RA)

Let $\mathbb{B}$ and $\mathbb{L}$ denote Banach spaces, and let $\|\cdot\|_{\mathbb{B}}$ and $\|\cdot\|_{\mathbb{L}}$ denote their norms.

a) Let $L: \mathbb{B} \to \mathbb{L}$ denote a continuous, invertible linear map and let $\mathfrak{m}: \mathbb{B} \otimes \mathbb{B} \to \mathbb{L}$ denote a linear map such that $\|\mathfrak{m}(\phi \otimes \psi)\|_{\mathbb{L}} \le \|\phi\|_{\mathbb{B}} \|\psi\|_{\mathbb{B}}$ for all $\phi, \psi \in \mathbb{B}$. Prove the following assertions:

- *There exists a number* $\kappa > 1$ *depending only on* L *such that if* $a \in \mathbb{B}$ *has norm less than* κ^{-2}*, then there is a unique solution to the equation* $L\phi + \mathfrak{m}(\phi \otimes \phi) = a$ *with* $\|\phi\|_{\mathbb{B}} < \kappa^{-1}$.
- *The norm of the solution from the previous bullet is at most* $\kappa \|a\|_{\mathbb{L}}$.

b) Recall that a Banach space is defined to be a *complete*, normed vector space. Is the assertion of Part a) of the first bullet always true if $\mathbb{B}$ is normed but not complete? If not, explain where the assumption that $\mathbb{B}$ is complete enters your proof of Part a).

PROBLEM 6 (CA)

Fix $a \in \mathbb{C}$ and an integer $n \ge 2$. Show that the equation $az^n + z + 1 = 0$ for a complex number z necessarily has a solution with $|z| \le 2$.

PROBLEM 1 SOLUTION:

Answer to a): The vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \theta}$ along S are

$$\frac{\partial}{\partial t} = \frac{t}{1+t^2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial z} \quad \textit{and} \quad \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} .$$

Since their inner product is $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle = \frac{t^2}{1+t^2} + 1$ $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\rangle = 0$ and $\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\rangle = (1+t^2)$, it follows that the square of the line element for the induced metric is

$$ds^2 = \frac{1+2t^2}{1+t^2} dt\otimes dt + (1+t^2)\, d\theta\otimes d\theta.$$

Answer to b): The 1-forms $e^0 = (\frac{1+2t^2}{1+t^2})^{1/2} dt$ and $e^1 = (1+t^2)^{1/2} d\theta$ are orthonormal. Write The connection matrix of 1-forms is $\mathbb{A} = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}$ with the 1-form Γ obeying

$$de^0 = -\Gamma\wedge e^1 \quad \textit{and} \quad de^1 = \Gamma\wedge e^0.$$

The unique solution is $\Gamma = -\frac{t}{\sqrt{1+2t^2}} d\theta$. The Gauss curvature is denoted by κ and it is defined by writing $d\Gamma$ as $\kappa e^0\wedge e^1$. Thus, $\kappa = -(\frac{1}{1+2t^2})^2$.

Answer to c): Since $\Gamma = 0$ on the $z = 0$ circle, the parallel transport is given by the identity matrix when written using the orthonormal frame $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\}$ for TS at (1,0,0).

PROBLEM 2 SOLUTION:

Here are two solutions:

Solution 1: Let Y denote the space $\times_n S^1$. It is enough to prove that the map from X to Y factors as a map

$$X \xrightarrow{f} \mathbb{R}^n \xrightarrow{(\exp)^{\times n}} Y.$$

To prove this factorization, note that a map f: X $\to$ Y lifts through a cover p: $\tilde{Y} \to Y$ if and only if $f_*(\pi_1(X))$ is a subgroup of $p_*(\pi_1(\tilde{Y}))$ (they are both subgroups of $\pi_1(Y)$). (See, for example Proposition 1.33 in Hatcher's book on algebraic topology.) Since $\pi_1(\mathbb{R}^n) = 0$

and f_* in this case must be the zero homomorphism, this condition is satisfied and so f lifts to some $\tilde{f}$. Because $\mathbb{R}^n$ is contractible, this lift is null-homotopic and any null-homotopy pushes forward to give a null-homotopy of f.

Solution 2: Recall that S^1 (which is $K(\mathbb{Z}, 1)$) classifies integral cohomology classes of degree 1. As a consequence, a map $X \to Y$ is (up to homotopy) determined by an n-tupel of elements in $H^1(X;\mathbb{Z})$. The universal coefficient short exact sequence in this degree is

$$0 \to \mathrm{Ext}(H_0X, \mathbb{Z}) \to H^1(X;\mathbb{Z}) \to \mathrm{Hom}(H_1X, \mathbb{Z}) \to 0.$$

The two end groups are zero: The right most group is zero because $H_1(X;\mathbb{Z})$ is the Abelianization of $\pi_1(X)$ and thus it is a finite group; and finite groups have no non-trivial homomorphisms to $\mathbb{Z}$. The left most group is zero because $H_0(X{:}\,\mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Ext}(\mathbb{Z};\mathbb{Z})$ is trivial since $\mathbb{Z}$ is a free group. Thus $H^1(X;\mathbb{Z}) = 0$ and so all maps from X to Y are homotopic to the constant map.

PROBLEM 3 SOLUTION:

Answer to a) One has $[K{:}\,F] = 6$ because the extension K/F is generated by the solution z of the polynomial equation $z^6 - uz^3 + 1 = 0$ which has degree 6. The Galois group contains the automorphisms $\alpha : z \to 1/z$ and $\beta : z \to \varrho z$, where $\varrho = e^{i2\pi/3} = (-1 + \sqrt{-3})/2$. Since α and β have orders 2 and 3 respectively, the group G generated by α and β has order at least 6. However, $|\mathrm{Gal}(K/F)| \leq [K{:}\,F] = 6$ with equality iff K/F is normal, so K/F must be normal with Galois group G of order 6, which is readily identified with the symmetric group the symmetric group S_3 (for instance, via its permutation action on the set $\{1, \varrho, \varrho^2\}$).

Answer to b) By the fundamental theorem of Galois theory, the intermediate fields E of the Galois extension K/F correspond to subgroups $H \subset G$ by $E = K^H$ (fixed subfield); K/E is always normal with $\mathrm{Gal}(K/E) = H$, while E/F is normal iff $H \trianglelefteq G$. Since F and K are excluded, one need not consider $H = G$ and $H = \{1\}$. The remaining subgroups are $A_3 = \langle\beta\rangle$, which yields the normal extension $\mathbb{C}(z^3)$ of F, and three two-element subgroups which yield non-normal extensions $\mathbb{C}(z + 1/z)$, $\mathbb{C}(z + \varrho z)$, $\mathbb{C}(z + \varrho^2 z)$. (The fact that each of these is indeed the corresponding KE can be confirmed by computing its degree as in Part a).)

PROBLEM 4 SOLUTION:

Answer to a) A rational map from X to Y is an equivalence class of pairs (U, f) where $U \subset X$ is a Zariski dense open subset and $f : U \to Y$ is a regular map. Two pairs (U, f) and (V, g) are equivalent if f = g on the intersection $U \cap V$.

Answer to b) The projection from the point $(0,0,1) \in \mathbb{CP}^2$ to the line where z=0 restricts to a rational map $\mathfrak{p}: X = \{zy^2 = x^2(x + z)\} \to \mathbb{CP}^1$. An inverse is given by the map given in homogeneous coordinates by the rule $(u, v) \to (x = (v^2-u^2)u, y = (v^2-u^2)v, z = u^3)$. This is an inverse since $x^3 = (y^2 - x^2)z$ on X. It follows that $\mathfrak{p}$ is a birational map.

Answer to c) Away from the line z = 0 the blowup of $\mathbb{CP}^2$ at (0,0,1) is given by the locus $\{xt=ys\} \subset \{((x, y),(s, t))\}=\mathbb{C}^2\times\mathbb{CP}^1$. Consider the chart in $\mathbb{CP}^1$ where $s \neq 0$. The blow up of X is defined here by the equations xt = y and $y^2 = x^2(x + 1)$. Substituting for y gives the equation $x^2(t^2 - x - 1) = 0$ which has one irreducible component being the locus x = y = 0 (which is the exceptional curve), and the other being the locus where both $t^2 = x+1$ and xt = y. This is the blow-up of X. In the chart where $t \neq 0$, the blow up of X is defined by the locus where x = ys and $1 = s^2(sy + 1)$. By the Jacobian criterion the curve defined by these equations is nonsingular.

PROBLEM 5 SOLUTION:

Answer to a) Since L is invertible, its inverse defines a bounded linear map from $\mathbb{L}$ to $\mathbb{B}$ to be denoted by L^{-1}. Using L^{-1}, one can define a map $\mathcal{T}: \mathbb{B} \to \mathbb{B}$ by the rule

$$\mathcal{T}(\phi)=L^{-1}(a - \mathfrak{m}(\phi,\phi)).$$

This is relevant because ϕ is a fixed point of $\mathcal{T}$ (it obeys $\mathcal{T}(\phi) = \phi$) if and only if ϕ obeys the equation $L\phi + \mathfrak{m}(\phi\otimes\phi) = a$. Let c denote the norm of the operator L^{-1}. Then the following are computations:

- $\|\mathcal{T}(\phi)\|_{\mathbb{B}} \leq c(\|a\|_{\mathbb{L}} + \|\phi\|_{\mathbb{B}}^2)$.
- $\|\mathcal{T}(\phi) - \mathcal{T}(\phi')\| \leq 4c(\|\phi\|_{\mathbb{B}}+\|\phi'\|_{\mathbb{B}})\,\|\phi - \phi'\|_{\mathbb{B}}$.

Given $\delta > 0$, let $\mathbb{B}(\delta)$ denote the ball of radius δ about the origin in $\mathbb{B}$. If $E > 0$ and if $\|a\|_{\mathbb{L}} \leq E$ then the top bullet implies that $\mathcal{T}$ maps $\mathbb{B}(\delta)$ to $\mathbb{B}(cE+c\delta^2)$. Thus, if $\delta < (2c)^{-1}$ and if $E < (2c)^{-1}\delta$, then $\mathcal{T}$ maps $\mathbb{B}(\delta)$ to itself. Meanwhile, if $\delta < (8c)^{-1}$ then the lower bullet implies that $\|\mathcal{T}(\phi) - \mathcal{T}(\phi')\| \leq \gamma\|\phi - \phi'\|_{\mathbb{B}}$ for fixed $\gamma < 1$ when $\phi, \phi' \in \mathbb{B}(\delta)$. This implies in turn that $\mathcal{T}$ is a contraction mapping of $\mathbb{B}(\delta)$ to itself. The contraction mapping theorem supplies a unique fixed point of $\mathcal{T}$ in $\mathbb{B}(\delta)$ under these circumstances. Noting again that an element $\phi \in \mathbb{B}$ is a fixed point of $\mathcal{T}$ if and only if ϕ obeys $L\phi + \mathfrak{m}(\phi\otimes\phi) = a$, the top bullet follows if $\|a\|_{\mathbb{L}} \le (16c)^{-1}$. Take κ to be the maximum of $4c^{1/2}$ and $8c$ to obtaine the answer to the first bullet of Part a). The second bullet of Part a) follows directly from the fact that $\phi = \mathcal{T}(\phi)$ and $\|\phi\|_{\mathbb{B}}^2 \le \frac{1}{2}\|\phi\|_{\mathbb{B}}$ because these and the inequality $\|\mathcal{T}(\phi)\|_{\mathbb{B}} \le c(\|a\|_{\mathbb{L}} + \|\phi\|_{\mathbb{B}}^2)$ imply that $\frac{1}{2}\|\phi\|_{\mathbb{B}} \le c\|a\|_{\mathbb{L}}$.

Answer to b) The completeness of B is required. Here is an example: Take $\mathbb{B}$ and $\mathbb{L}$ to be the span of the polynomials functions on [-1, 1] with the norms $\|f\|_{\mathbb{B}} = \|f\|_{\mathbb{L}} = \sup_t |f(t)|$. Take the equation $\phi + \phi^2 = \delta t$ with δ being a small, non-zero number. A solution, must be either $\phi = -\frac{1}{2} + \frac{1}{2}(1 + 4\delta^2 t^2)^{1/2}$ or $\phi = -\frac{1}{2} - \frac{1}{2}(1 + 4\delta^2 t^2)^{1/2}$; but neither is in $\mathbb{B}$. Note that the contraction mapping theorem does not hold if the Banach space in question is not complete because the contraction mapping theorem constructs the desired solution as a limit of a Cauchy sequence in $\mathbb{B}$.

PROBLEM 6 SOLUTION:

There are two cases. First, assume that $|a| < 2^{-n}$. Let D denote the disk where $|z| \le 2$ and let ∂D denote the circle $|z| = 2$. Let $f(z) = az^n + z + 1$ and let $g(z) = z + 1$. On ∂D, the function g - f obeys the inequality $|g(z) - f(z)| = |a|\,|z|^n < 1$. Since this is less than $|g(z)|$ for each $z \in \partial D$, and since g has no zeros on ∂D, none of the members of the 1-parameter family of functions $\{f_\tau = f + \tau(g - f)\}_{\tau\in[0,1]}$ has a zero on ∂D. Therefore, f (which is $f_{\tau=0}$) and g (which is $f_{\tau=1}$) have the same number of zeros (counting multiplicity) in D. This number is 1 (This is Rouche's theorem). Now assume that $|a| \ge 2^{-n}$. By the fundamental theorem of algebra, the function $f(z) = az^n + z + 1$ factors as

$$f(z) = a\prod_{k=1}^{n}(z - \alpha_k)$$

where the $\{\alpha_k\}_{k=1,\dots,n}$ are complex numbers. This implies in particular the identity

$$(-1)^n a \prod_{k=1}^{n}\alpha_k = 1.$$

hence $\prod_{k=1}^{n}|\alpha_k| \le 2^n$. This can happen only if one or more roots α_k are in D.

Qualifying Examination

HARVARD UNIVERSITY

Department of Mathematics

Wednesday, January 20, 2016 (Day 2)

PROBLEM 1 (DG)

Let k denote a positive integer. A non-optimal version of the Whitney embedding theorem states that any k-dimensional manifold can be embedded into $\mathbb{R}^{2k+1}$. Using this, show that any k-dimensional manifold can be immersed in $\mathbb{R}^{2k}$. (Hint: Compose the embedding with a projection onto an appropriate subspace.)

PROBLEM 2 (T)

Let X be a CW-complex with a single cell in each of the dimensions 0, 1, 2, 3, and 5 and no other cells.

a) What are the possible values of $H_*(X; \mathbb{Z})$? (Note: it is not sufficient to consider $H_n(X;\mathbb{Z})$ for each n independently. The value of $H_1(X;\mathbb{Z})$ may constrain the value of $H_2(X;\mathbb{Z})$, for instance.)

b) Now suppose in addition that X is its own universal cover. What extra information does this provide about $H_*(X;\mathbb{Z})$?

PROBLEM 3 (AN)

Let k be a finite field of characteristic p, and n a positive integer. Let G be the group of invertible linear transformations of the k-vector space k^n. Identify G with the group of invertible $n \times n$ matrices with entries in k (acting from the left on column vectors).

a) Prove that the order of G is $\prod_{m=0}^{n-1}(q^n - q^m)$ where q is the number of elements of k.

b) Let U be the subgroup of G consisting of upper-triangular matrices with all diagonal entries equal 1. Prove that U is a p-Sylow subgroup of G.

c) Suppose $H \subset G$ is a subgroup whose order is a power of p. Prove that there is a basis $(v_1, v_2,..., v_n)$ of k^n such that for every $h \in H$ and every $m \in \{1, 2, 3,..., n\}$, the vector $h(v_m) - v_m$ is in the span of $\{v_d: d < m\}$.

PROBLEM 4 (AG)

Let X be a complete intersection of surfaces of degrees a and b in $\mathbb{CP}^3$. Compute the Hilbert polynomial of X.

PROBLEM 5 (RA)

Let $\mathcal{C}^0$ denote the vector space of continuous functions on the interval $[0,1]$. Define a norm on $\mathcal{C}^0$ as follows: If $f \in \mathcal{C}^0$, then its norm (denoted by $\|f\|$) is

$$\|f\| = \sup_{t\in[0,1]}|f(t)| \ .$$

Let $\mathcal{C}^\infty$ denote the space of smooth functions on [0, 1]. View $\mathcal{C}^\infty$ as a normed, linear space with the norm defined as follows: If $f \in \mathcal{C}^\infty$, then its norm (denoted by $\|f\|_*$) is

$$\|f\|_* = \int_{[0,1]} (|\tfrac{d}{dt}f| + |f|)\, dt \ .$$

a) Prove that $\mathcal{C}^0$ is Banach space with respect to the norm $\|\cdot\|$. In particular, prove that it is complete.
b) Let ψ denote the 'forgetful' map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$ that sends f to f. Prove that ψ is a bounded map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$, but not a compact map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$.

PROBLEM 6 (CA)

Let $\mathbb{D}$ denote the closed disk in $\mathbb{C}$ where $|z| \le 1$. Fix $R > 0$ and let $\varphi: \mathbb{D} \to \mathbb{C}$ denote a continuous map with the following properties:

i) φ is holomorphic on the interior of $\mathbb{D}$.
ii) $\varphi(0) = 0$ and its z-derivative, φ', obeys $\varphi'(0) = 1$.
iii) $|\varphi| \le R$ for all $z \in \mathbb{D}$.

Since $\varphi'(0) = 1$, there exists $\delta > 0$ such that φ maps the $|z| < \delta$ disk diffeomorphically onto its image. Prove the following:

a) There is a unique solution in [0, 1] to the equation $2R\delta = (1 - \delta)^3$.
b) Let δ_* denote the unique solution to this equation. If If $0 < \delta < \delta_*$, then φ maps the $|z| < \delta$ disk diffeomorphically onto its image.

PROBLEM 1 SOLUTION:

The desired immersion will come from a projection onto the orthogonal complement of a suitably chosen, nonzero vector in $\mathbb{R}^{2k}$. To find this vector, let M denote the manifold in question and let f denote the embedding of M into $\mathbb{R}^{2k}$. Let g denote the map from TM to $\mathbb{R}^{2k+1}$ that is defined as follows: Supposing that $x \in M$ and $v \in TM|_x$ set $g(x, v) = f_*|_x \cdot v$ where f_* denotes the differential of f. Sard's theorem can be invoked to see that g is not surjective. Let a denote a point that is not in the image of g. (Note that a is necessarily nonzero.) Use π to denote the projection onto the orthogonal complement of a. To see that $\pi \circ f$ is an immersion, let x denote a point in M and let v denote a nonzero vector in $TM|_x$. Suppose for the sake of argument that $(\pi \circ f)_* v$ is zero. If this is so, then the chain rule and the fact that π is linear implies that $f_*|_x \cdot v = ta$ for some nonzero $t \in \mathbb{R}$. This implies in turn that $f_*|_x(t^{-1}v) = a$ which is nonsense because a is in the complement of the image of f_*.

PROBLEM 2 SOLUTION:

<u>Answer</u> <u>to</u> <u>a)</u> The cellular chain complex for X must be of the form

$$\begin{array}{ccccccccccccccc}
0 & \to & C_5X & \to & C_4X & \to & C_3X & \to & C_2X & \to & C_1X & \to & C_0X & \to & 0 \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
0 & \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \xrightarrow{a} & \mathbb{Z} & \xrightarrow{b} & \mathbb{Z} & \xrightarrow{c} & \mathbb{Z} & \to & 0.
\end{array}$$

Since X is connected, it must have $H_0(X; \mathbb{Z}) = \mathbb{Z}$, so the map c must be zero. The only other restriction is that the sequence form a complex, so $b \circ a = 0$; but since $b \circ a$ is multiplication by some integer, either $a = 0$ or $b = 0$. In the case $a = 0$ and $b \neq 0$, the homology groups take the form

$$\begin{array}{cccccc}
H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\
\| & \| & \| & \| & \| & \| \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}/b & \mathbb{Z}.
\end{array}$$

$$\begin{array}{cccccc}
H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\
\| & \| & \| & \| & \| & \| \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}/b & \mathbb{Z}.
\end{array}$$

In the case $a \neq 0$ and $b = 0$, the homology groups take the form

$$\begin{array}{cccccc} H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\ \| & \| & \| & \| & \| & \| \\ \mathbb{Z} & 0 & 0 & \mathbb{Z}/a & \mathbb{Z} & \mathbb{Z}. \end{array}$$

In the remaing $a = 0 = b$ case, they take the form

$$\begin{array}{cccccc} H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\ \| & \| & \| & \| & \| & \| \\ \mathbb{Z} & 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}. \end{array}$$

<u>Answer</u> <u>to</u> <u>b)</u> The assertion that X is its own universal cover is the same as the assertion $\pi_1(X) = 0$. But, since $H_1(X) = \pi_1(X)^{ab}$, this means $H_1(X) = 0$. The only case where this is possible is when $a = 0$ and $b \neq 0$. Moreover, since $\mathbb{Z}/b = 0$ in this case, b must be a multiplicative unit: $b = \pm 1$.

PROBLEM 3 SOLUTION:

<u>Answer</u> <u>to</u> <u>a)</u> The elements of G are in bijection with ordered bases $(v_1,...,v_n)$ of k^n (the map takes each matrix to its columns). For each $j \in \{0, 1, 2,..., n-1\}$, once v_i for all $i \leq j$ has been chosen, then there are $q^n - q^j$ choices for the index $(j+1)$ basis element because any of the q^n elements of k^n except the q^m linear combinations of $v_1, \ldots, v_j$ will do. Hence the number of possible bases is $\prod_{m=0}^{n-1} (q^n - q^m)$.

<u>Answer</u> <u>to</u> <u>b)</u> Each factor $q^n - q^j$ is q^j times an integer not divisible by p because it is congruent to -1 modulo q, and q is a multiple of p. Hence the number of elements in G is qd times some integer not divisible by p, where $d = \sum_{j=0}^{n-1} j$. But q^d is the order of U because there are d entries above the diagonal, and a power of p. Hence U is a p-Sylow subgroup of G.

<u>Answer</u> <u>to</u> <u>c)</u> U consists of the matrices h that satisfy the desired property with respect to the standard basis of unit vectors. Hence the matrices h that satisfy this property for the basis $(v_1,...,v_n)$ constitute the subgroup of G obtained by conjugating U by the matrix with columns $v_1, \ldots, v_n$. But by Sylow's second theorem H is contained in a conjugate of U.

PROBLEM 4 SOLUTION:

Let $S = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the homogeneous coordinate ring of $\mathbb{CP}^3$. The coordinate ring of X is of the form S/(f, g) for some irreducible polynomials f and g of degrees a, b respectively. There is a four-term exact sequence of graded modules

$$0 \to S(-a-b) \to S(-a) \otimes S(-b) \to S \to S/(f,g) \to 0$$

with maps given by multiplication with f and g. Hence the Hilbert polynomial of X is

$$\begin{aligned} P_X(z) &= \binom{z+3}{3} - \binom{z+3-a}{3} - \binom{z+3-b}{3} + \binom{z+3-a-b}{3} \\ &= ab\left(z + \frac{4-a-b}{2}\right) \end{aligned}$$

PROBLEM 5 SOLUTION:

Answer to a) One has to show that a Cauchy sequence $\{f_n\}_{n=1,2,...}$ in $\mathcal{C}^0$ converges to a continuous function. To do this, note that for each $t \in [0, 1]$, the sequence $\{f_n(t)\}_{n\in\{1,2,...\}}$ is a Cauchy sequence in $\mathbb{R}$ so it converges. Let $f(t)$ denote the limit. The assignment $t \to f(t)$ defines a function on [0, 1]. The task is to prove that this function is continuous. This means the following: Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(t')| < \varepsilon$ when $|t - t'| < \delta$. To find δ, first fix N so that $|f_n(t) - f_m(t)| < \frac{1}{3}\varepsilon$ for all $t \in [0, 1]$ and all pairs n, $m > N$. This implies that $|f_n(t) - f(t)| \le \frac{1}{3}\varepsilon$ for all t. Such N exists because $\{f_n\}_{n\in\{1,2,...\}}$ is a Cauchy sequence in $\mathcal{C}^0$. To continue, take $n > N$ and fix δ so that $|f_n(t) - f_n(t')| < \frac{1}{3}\varepsilon$ when $|t' - t| < \delta$. It then follows by the triangle inequality that

$$|f(t) - f(t')| \le |f(t) - f_n(t)| + |f(t') - f_n(t')| + |f_n(t') - f_n(t)| < \varepsilon.$$

Answer to b) The map ψ is bounded because for all t, one has the identity

$$f(t) = \int_0^1 \left(\int_r^t \frac{d}{ds} f(s)ds + f(r)\right) dr \;,$$

and thus $|f(t)| \le \| f \|_*$ for all t. It is not a compact map. To prove this, fix a smooth function on $[0, \infty)$ that is equal to 1 near t = 0 and equal to 0 for $t > \frac{1}{2}$. Call this function

f. Define $f_n(t) = f(nt)$. This function is smooth on [0, 1]. The sequence $\{f_n(t)\}$ has bounded $\|\cdot\|_*$ norm but it has no convergent sequence in $\mathcal{C}^0$.

PROBLEM 6 SOLUTION:

Answer to a) The function $f(\delta) = 2R\delta/(1-\delta)^3$ has strictly positive derivative and therefore defines a diffeomorphism from [0, 1) to [0, ∞). It follows from this that there is a single point where f is equal to 1.

Answer to b) To obtain the asserted lower bound for δ, note that φ maps the disk where $|z| < \delta$ diffeomorphically to its image if it is 1-1 on this disk and if $|\varphi'| > 0$ on this disk. The Cauchy integral formula is used to see when this happens. Here is Cauchy's formula:

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{1}{z - w} \varphi(w)dw \,.$$

Differentiating this, one sees that $|\varphi''|$ on the $|z| < \delta$ disk is bounded by $2R(1-\delta)^{-3}$. This implies that

$$|\varphi' - 1| < 2R\delta(1-\delta)^{-3} \quad \textit{where } |z| < \delta.$$

If $\varphi' > 0$, then φ is a local diffeomorphism. This is the case when $\delta < \delta_*$ with δ_* being the solution in (0, 1) to the equation $2R\delta_*(1-\delta_*)^{-3} = 1$. Meanwhile, if z, z´ have norm less than δ, then $|\varphi(z) - \varphi(z')| \ge (1 - 2R\delta(1-\delta)^{-3})|z - z'|$ which is a positive multiple of $|z - z'|$ precisely when $\delta < \delta_*$.

Qualifying Examination

HARVARD UNIVERSITY

Department of Mathematics

Thursday, January 21, 2016 (Day 3)

PROBLEM 1 (DG)

Recall that a symplectic manifold is a pair (M, ω), where M is a smooth manifold and ω is a closed nondegenerate differential 2-form on M. (The 2-form ω is called the symplectic form.)

a) Show that if H: M → R is a smooth function, then there exists a unique vector field, to be denoted by X_H, satisfying $\iota_{X_H}\omega = dH$. (Here, ι denotes the contraction operation.)

b) Supposing that t > 0 is given, suppose in what follows that the flow of X_H is defined for time t, and let ϕ_t denote the resulting diffeomorphism of M. Show that $\phi_t{}^*\omega = \omega$.

c) Denote the Euclidean coordinates on $\mathbb{R}^4$ by (x_1, y_1, x_2, y_2) and use these to define the symplectic form $\omega_0 = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. Find a function H: $\mathbb{R}^4 \to \mathbb{R}$ such that the diffeomorphism $\phi_{t=1}$ that is defined by the time t = 1 flow of X_H fixes the half space where $x_1 \leq 0$ and moves each point in the half space where $x_1 \geq 1$ by 1 in the y_2 direction.

PROBLEM 2 (T)

Let X denote a finite CW complex and let f: X → X be a self-map of X. Recall that the Lefschetz trace of f, denoted by $\tau(f)$, is defined by the rule

$$\tau(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_n : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q}))$$

with f_n denoting the induced homomorphism. Use $\tau(\cdot)$ to answer the following:

a) Does there exist a continuous map from $\mathbb{RP}^2$ to itself with no fixed points? If so, give an example; and if not, give a proof.

b) Does there exist a continuous map from $\mathbb{RP}^3$ to itself with no fixed points? If so, give an example; and if not, give a proof.

PROBLEM 3 (AN)

Let A be the ring $\mathbb{Z}[\sqrt[5]{2016}\,] = \mathbb{Z}[X]/(X^5 - 2016)$. Given that 2017 is prime in $\mathbb{Z}$, determine the factorization of $2017 \cdot A$ into prime ideals of A.

PROBLEM 4 (AG)

a) State a version of the Riemann–Roch theorem.

b) Apply this theorem to show that if X is a complete nonsingular curve and $P \in X$ is any point, there is a rational function on X which has a pole at P and is regular on $X-\{P\}$.

PROBLEM 5 (RA)

Let $\wp$ denote a probability measure for a real valued random variable with mean 0. Denote this random variable by x. Suppose that the random variable $|x|$ has mean equal to 2.

a) Given $R > 2$, state a non-trivial upper bound for event that $x \geq R$. (The trivial upper bound is 1.)

b) Give a non-zero lower bound for the standard deviation of x.

c) A function f on $\mathbb{R}$ is Lipshitz when there exists a number $c \geq 0$ such that

$$|f(p) - f(p')| \leq c|p - p'| \quad \text{for any pair } p, p' \in \mathbb{R}.$$

Let $\hat{\wp}$ denote the function on $\mathbb{R}$ whose value at a given $p \in \mathbb{R}$ is the expectation of the random variable e^{ipx}. (This is the *characteristic function of* $\wp$.) Give a rigorous proof that $\hat{\wp}$ is Lipshitz and give an upper bound for c in this case.

d) Suppose that the standard deviation of x is equal to 4. Let N denote an integer greater than 1, and let $\{x_1, \ldots, x_N\}$ denote a set of independent random variables each with probabilities given by $\wp$. Use S_N to denote the random variable $\frac{1}{N}(x_1 + \cdots + x_N)$. The central limit theorem gives an integral that approximates the probability of the event where $S_N \in [-1,1]$ when N is large. Write this integral.

PROBLEM 6 (CA)

Let $H \subset \mathbb{C}$ denote the open right half plane, thus $H = \{z = x + iy: x > 0\}$. Suppose that $f: H \to \mathbb{C}$ is a bounded, analytic function such that $f(1/n) = 0$ for each positive integer n. Prove that $f(z) = 0$ for all z.

(Hint: Consider the behavior of the sequence of functions $\{h_N(z) = \prod_{n=1}^{N} \frac{z - 1/n}{z + 1/n}\}_{N=1,2,\ldots}$ on H and, in particular, on the positive real axis.}

PROBLEM 1 SOLUTION:

Answer to a) To say that ω is non-degenerate is to say that the contraction operation defines a vector bundle isomorphism between TM and T*M.

Answer to b) The definition of the Lie derivative is such that $\frac{\partial}{\partial t}(\phi_t{}^*\omega) = \phi_t{}^*(\mathfrak{L}_{X_H}\omega)$ with $\mathfrak{L}_{X_H}\omega$ denoting the Lie derivative of ω along the vector field X_H. Cartan's formula for $\mathfrak{L}_{X_H}\omega$ is $\mathfrak{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}d\omega$ and both of these terms are zero. Thus, $\phi_t{}^*\omega$ is independent of t and thus equal to its value at t = 0 which is ω.

Answer to c) Choose a smooth function $f: \mathbb{R} \to [0, 1]$ so that $f(s) = 0$ for $s \leq 0$ and $f(s) = 1$ for $s \geq 1$. The function sending $(x_1, y_1, x_2, y_2) \to H(x_1, y_1, x_2, y_2) = -f(x_1)\, x_2$ has the desired properties because $X_H = 0$ for $x_1 \leq 0$ and $X_H = \frac{\partial}{\partial y_2}$ for $x_1 \geq 1$.

PROBLEM 2 SOLUTION:

Answer to a) The Lefschetz trace theorem states that if $\tau(f) \neq 0$, then f must have a fixed point. To see that $\tau(f)$ is never zero, note first that the rational homology of $\mathbb{RP}^2$ is zero except for $H_0(\mathbb{RP}^2;\mathbb{Q})$, which is $\mathbb{Q}$. Since f_0 is multiplication by 1, it $\tau(f)$ is never zero.

Answer to b) In this case, the non-zero rational homology is in dimensions 0 and 3, each being isomorphic to $\mathbb{Q}$. As a consequence, the argument used for $\mathbb{RP}^2$ can not be used here. In fact, there is a self-map with no fixed points and it is constructed momentarily. It is instructive to consider first the case of $\mathbb{RP}^1$ which is S^1, where a rotation by angle π has no fixed points. Now viewing $\mathbb{RP}^1$ as $(\mathbb{R}^2\text{–}0)/\mathbb{R}^*$, then this rotation through angle π is depicted using homogeneous coordinates $[x_1, x_2]$ as the map $[x_1, x_2] \to [x_2, -x_1]$ which can't have a fixed point because there is no non-zero real number λ and $(x_1, x_2) \in \mathbb{R}^2\text{–}0$ with $x_2 = \lambda x_1$ and $x_1 = -\lambda x_2$. To mimick this for $\mathbb{RP}^3$, write $\mathbb{RP}^3$ as $(\mathbb{R}^4\text{–}0)/\mathbb{R}^*$ and then define the desired self map using homogeneous coordinates $[x_1, x_2, x_3, x_4]$ by the rule whereby $[x_1, x_2, x_3, x_4] \to [x_2, -x_1, x_4, -x_3]$. This has no fixed points because there is no non-zero real number λ and $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4\text{–}0$ such that $x_2 = \lambda x_1$, $x_1 = -\lambda x_2$, $x_4 = \lambda x_3$ and $x_3 = -\lambda x_4$.

PROBLEM 3 SOLUTION:

2017A is the product of the prime ideals (2017, X + 1) and (2017, $X^4 - X^3 + X^2 - X + 1$). In general, if the polynomial P(X) factors modulo a prime p into distinct irreducibles $\{P_i\}$ then the ideal pZ[X]/(P(X)) is the product of ideals (p, P_i). In our case, p = 2017 and $P = X^5 - 2016 \equiv X^5 + 1 \bmod p$. The roots of $X^5 + 1$ in an algebraic closure of Z/pZ are the set $\{-1, -w, -w^2, -w^3, -w^4\}$ where w is a nontrivial 5th root of unity. The irreducible factors correspond to orbits of the permutation $x \to x^p$ of those roots. Clearly -1 is a fixed point, and since $p \equiv 2 \bmod 5$ the remaining roots fall in to a single orbit

$$-w \to -w^2 \to -w^4 \to -w^3 \to -w.$$

Hence the irreducible factors of $X^5 +1$ mod p are X + 1 and $(X^5 +1)/(X +1)$ which is the polynomial $X^4 - X^3 + X^2 - X + 1$.

PROBLEM 4 SOLUTION:

Answer to a) Let X be a complete non-singular curve of genus g. Let K denote the canonical divisor. If D is any divisor on X, let $\ell(D) = \dim(H_0(X, \mathcal{O}_X(D)))$. The Riemann-Roch theorem asserts that $\ell(D) - \ell(K-D) = \deg(D) + 1 - g$.

Answer to b) Fix a point $Q \neq P$ and let D denote the divisor $2P - Q$. Choose a positive integer n such that $n > \max\{2g - 2, 0\}$. Noting that $n = \deg(nD)$ and that $\deg(K) = 2g-2$, it follows that $\deg(K - nD) < 0$. This implies that $\ell(K-D) = 0$. Therefore, the Riemann–Roch theorem applied to nD implies that $\ell(nD) = n + 1 - g$ which is greater than 1. This means that there is an effective divisor (to be denoted by D´) and a rational function on X (to be denoted by f) such that $nD + (f) = D'$. Rewriting this gives $(f) = D' - 2nP + nQ$ so f has poles only at P.

PROBLEM 5 SOLUTION:

Answer to a) The event in question is $\int_{x \geq R} \wp$. This is no smaller than $\frac{1}{R} \int_{x \geq R} |x| \wp$ which in turn is no greater than $\frac{2}{R}$.

Answer to b) The square of the standard deviation is the square root of the expectation of the random variable x^2. Since

$$\int_{\mathbb{R}} |x| \wp \le \left(\int_{\mathbb{R}} \wp\right)^{1/2} \left(\int_{\mathbb{R}} x^2 \wp\right)^{1/2} \qquad (*)$$

(which is proved momentarily), and since $\int_{\mathbb{R}} \wp = 1$, it follows that $(\int_{\mathbb{R}} x^2 \wp)^{1/2} \ge 2$. To prove (∗), note that for any $t \in (0,\infty)$, the expectation of $(t - t^{-1}x)^2$ is the sum

$$t^2 \int_{\mathbb{R}} \wp - 2\int_{\mathbb{R}} |x| \wp + t^{-2} \int_{\mathbb{R}} x^2 \wp .$$

This is non-negative for any $t \in (0, 1)$ since it is the expectation of a positive random variable. The assertion that it is non-negative for the case $t = (\int_{\mathbb{R}} x^2 \wp)^{1/4} (\int_{\mathbb{R}} \wp)^{-1/4}$ is (∗).

Answer to c) Supposing that p, p´ ∈ $\mathbb{R}$, then

$$\hat{\wp}(p) - \hat{\wp}(p') = \int_{\mathbb{R}} (e^{ixp} - e^{ixp'}) \wp . \qquad (**)$$

Noting that $e^{ixp} - e^{ixp'} = i\,x \int_p^{p'} e^{ixq}\,dq$ by the fundamental theorem of calculus, it follows that $|e^{ixp} - e^{ixp'}| \le |x|\,|p - p'|$. This understood, then (∗∗) leads to the bound

$$|\hat{\wp}(p) - \hat{\wp}(p')| \le \left(\int_{\mathbb{R}} |x| \wp\right) |p - p'| = 2\,|p - p'| .$$

Answer to d) The random variable S_N has mean 0 and standard deviation equal to $N^{-1/2}$ times the standard deviation of x, thus $4N^{-1/2}$. (The expecation of $S_N^{\,2}$ is the that of $N^{-2}\sum_{i,k=1,\ldots,N} x_i x_k$. Only the i = k terms are non-zero (because x has mean zero), there are N of them and each is the expectation of x^2 which is 16.) Denote this standard deviation of S_N by σ_N for the moment. The central limit theorem approximates the probability in question by $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-x^2/2\sigma_N^2}\,dx$ where σ_N again denotes $4N^{-1/2}$.

PROBLEM 6 SOLUTION

This is a form of Jensen's inequality. To elaborate, fix B so that $|f(z)| \leq B$ for all $z \in H$. For each integer N, define

$$F_N(z) = f(z)/h_N(z) = f(z)\prod_{n=1}^{N}\frac{z + 1/n}{z - 1/n} .$$

This function is analytic on H because the poles at z = 1, 2, 3,..., N are matched by zeros of f. Moreover, the absolute value of each of the factors $(z + 1/n)/(z - 1/n)$ approaches 1 as $\mathrm{Re}(z) \to 0$ (uniformly in $\mathrm{Im}(z)$), and also approaches 1 as $|z| \to \infty$. Hence $|F_n(z)| \leq B$ for all $z \in H$ by virtue of the maximum modulus principle (the norm of an analytic function can not take on a local maximum). With the preceding understood, note that for any fixed, positive real z, the factor $\prod_{n=1}^{N}\frac{z + 1/n}{z - 1/n}$ becomes unbounded as $N \to \infty$. Hence its product with $f(z)$ cannot remain bounded unless $f(z) = 0$ on the real axis. But a holomorphic function on any domain has discrete zeros, so $f(z)$ must be everywhere 0.