QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 20, 2015 (Day 1)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d.

(a) Let K_C be the canonical bundle of C. For what integer n is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then C is not hyperelliptic.

(c) Prove that if $d \geq 5$ then C is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

2. (A) Let S_4 be the group of automorphisms of a 4-element set. Give the character table for S_4 and explain how you arrived at it.

3. (DG) Let

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^3 - z = 0\}.$$

(a) Prove that M is a smooth surface in $\mathbb{R}^3$.

(b) For what values of $c \in \mathbb{R}$ does the plane $z = c$ intersect M transversely?

4. (RA) Define the Banach space $\mathcal{L}$ to be the completion of the space of continuous functions on the interval $[-1, 1] \subset \mathbb{R}$ using the norm

$$||f|| = \int_{-1}^{1} |f(t)|dt.$$

Suppose that $f \in \mathcal{L}$ and $t \in [-1, 1]$. For $h > 0$, let I_h be the set of points in $[-1, 1]$ with distance h or less from t. Prove that

$$\lim_{h \to 0} \int_{t \in I_h} |f(t)|dt = 0$$

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,

(a) with coefficients in $\mathbb{Z}$?

(b) with coefficients in $\mathbb{Z}/2$?

(c) with coefficients in $\mathbb{Z}/3$?

6. (CA) Let Ω be an open subset of the Euclidean plane $\mathbb{R}^2$. A map $f : \Omega \to \mathbb{R}^2$ is said to be *conformal* at $p \in \Omega$ if its differential df_p preserves the angle between any two tangent vectors at p. Now view $\mathbb{R}^2$ as $\mathbb{C}$ and a map $f : \Omega \to \mathbb{R}^2$ as a $\mathbb{C}$-valued function on Ω.

(a) Supposing that f is a holomorphic function on Ω, prove that f is conformal where its differential is nonzero.

(b) Suppose that f is a nonconstant holomorphic function on Ω, and $p \in \Omega$ is a point where $df_p = 0$. Let L_1 and L_2 denote distinct lines through p. Prove that the angle at $f(p)$ between $f(L_1)$ and $f(L_2)$ is n times that between L_1 and L_2, with n being an integer greater than 1.

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Wednesday January 21, 2015 (Day 2)

1. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2+y^2+z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of X?

(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

2. (A) Let

$$f(t) = t^4 + bt^2 + c \in \mathbb{Z}[t].$$

(a) If E is the splitting field for f over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group D_8.

(b) Given an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Justify.

(c) Give an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Justify.

(d) Give an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to D_8.

3. (CA) Let $a \in (0, 1)$. By using a contour integral, compute

$$\int_0^{2\pi} \frac{dx}{1 - 2a\cos x + a^2}.$$

4. (AG) Let K be an algebraically closed field of characteristic 0 and let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface over K.

(a) Show that Q is rational by exhibiting a birational map $\pi : Q \to \mathbb{P}^{n-1}$.

(b) How does the map π factor into blow-ups and blow-downs?

5. (DG) Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

be the unit sphere centered at the origin in $\mathbb{R}^3$.

(a) Prove that the vector field

$$v = yz\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$ is tangent to S at all points of S, and thus defines a section of the tangent bundle TS.

(b) Let g be the metric on S induced from the euclidean metric on $\mathbb{R}^3$, and let ∇ be the associated, metric compatible, torsion free covariant derivative. The tensor ∇v is a section of $TS \otimes TS^*$. Write ∇v at the point $(0,0,1) \in S$ using the coordinates (x_1, x_2) given by the map $(x_1, x_2) \mapsto (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$ from the unit disc $x_1^2 + x_2^2 < 1$ to S.

6. (RA) Let L be a positive real number.

(a) Compute the Fourier expansion of the function x on the interval $[-L, L] \subset \mathbb{R}$.

(b) Prove that the Fourier transform does not converge to x pointwise on the closed interval $[-L, L]$.

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Thursday January 22, 2015 (Day 3)

1. (DG) The helicoid is the parametrized surface given by

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \to (v \cos u, v \sin u, au)$$

where a is a real constant. Compute its induced metric.

2. (RA) A real valued function defined on an interval $(a, b) \subset \mathbb{R}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

whenever $x, y \in (a, b)$ and $t \in (0, 1)$.

(a) Give an example of a non-constant, non-linear convex function.

(b) Prove that if f is a non-constant convex function on $(a, b) \in \mathbb{R}$, then the set of local minima of f is a connected set where f is constant.

3. (AG) Let K be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree n in two variables over K. Let $X \subset \mathbb{P}^n$ be the locus of n^{th} powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that X and $Y \subset \mathbb{P}^n$ are closed subvarieties.

(b) What is the degree of X?

(c) What is the degree of Y?

4. (AT) Let X be a compact, connected and locally simply connected Hausdorff space, and let $p : \tilde{X} \to X$ be its universal covering space. Prove that $\tilde{X}$ is compact if and only if the fundamental group $\pi_1(X)$ is finite.

5. (CA) Prove that if f and g are entire holomorphic functions and $|f| \leq |g|$ everywhere, then $f = \alpha \cdot g$ for some complex number α.

6. (A) Consider the rings

$$R = \mathbb{Z}[x]/(x^2 + 1) \quad \text{and} \quad S = \mathbb{Z}[x]/(x^2 + 5).$$

(a) Show that R is a principal ideal domain.

(b) Show that S is not a principal ideal domain, by exhibiting a non-principal ideal.