Qualifying Exams I, Jan. 2013

1. (Real Analysis) Suppose $f_j, j = 1, 2, \ldots$ and f are real functions on $[0, 1]$. Define $f_j \to f$ in measure if and only if for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu\{x \in [0,1] : |f_j(x) - f(x)| > \varepsilon\} = 0$$

where μ is the Lebesgue measure on $[0, 1]$. In this problems, all functions are assumed to be in $L^1[0, 1]$.

   (a) Suppose that $f_j \to f$ in measure. Does it implies that

$$\lim_{j\to\infty} \int |f_j(x) - f(x)|\mathrm{dx} = 0.$$

   Prove it or give a counterexample.

   (b) Suppose that $f_j \to f$ in measure. Does this imply that $f_j(x) \to f(x)$ almost everywhere in $[0, 1]$? Prove it or give a counter example.

   (c) Suppose that $f_j(x) \to f(x)$ almost everywhere in $[0, 1]$. Does it implies that $f_j \to f$ in measure? Prove it or give a counter example.

2. (Complex analysis) The following questions are independent.

   a) For any $a \in (-1, 1)$, compute

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t}.$$

   b) For any $p > 1$, compute

$$\int_0^{\infty} \frac{\mathrm{d}x}{x^p + 1}.$$

3. (Differential Geometry) The Heisenberg group is the subgroup of $\mathrm{Sl}(3, \mathbb{R})$ composed of the 3×3, upper triangular matrices with 1 on the diagonal, this being the set of matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad (x, y, z) \in \mathbb{R}^3.$$

   This group is observably diffeomorphic to $\mathbb{R}^3$.

   (a) Compute the Lie algebra of the Heisenburg group.

   (b) Exhibit a left-invariant Riemannian metric on the Heisenberg group.

4. (Algebraic Topology) Let $\mathbb{H}$ be the space of quaternions, and denote by $\mathbb{S}^3$ the unit sphere inside $\mathbb{H}$. The quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ acts on $\mathbb{H}$ by left multiplication, and the action preserves the unit sphere $\mathbb{S}^3$. Let X be the quotient space $\mathbb{S}^3/G$. Compute its fundamental group $\pi_1(X)$ and its first homology group $H_1(X, \mathbb{Z})$.

   ($\mathbb{H}$ is spanned by four independent unit vectors $1, i, j, k$ as a real normed vector space. The multiplication is associative and, between two elements of $\mathbb{H}$, it is bilinear and determined by the rules $i^2 = j^2 = k^2 = ijk = -1$, where 1 is the multiplicative identity.)

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5. (Algebra) Let k be a field, and let G be a finite group acting on a vector space V.

   a) If $k = \mathbf{C}$, prove that any subrepresentation $U \subseteq V$ has a G-stable complement, that is, subrepresentation $U' \subseteq V$ such that $V = U \oplus U'$.

   b) Now suppose that $k = \mathbf{Z}/p\mathbf{Z}$ for some prime p, and that G acts doubly transitively on a set X of size n (that is, if $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$ then there exists $g \in G$ such that $g(x) = x'$ and $g(y) = y'$). Let V be the trace-zero subspace of the corresponding permutation representation over k (recall that the permutation representation is the vector space k^S with the natural action of G, so that V is the subspace consisting of vectors whose n coordinates sum to 0). Prove that if $n \equiv 0 \bmod p$ then the trivial subrepresentation generated by $(1, 1, \ldots, 1)$ has no G-stable complement, except for one choice of (p, n, G), and determine that one choice.

6. (Algebraic Geometry) Prove that the following complex algebraic varieties are pairwise nonisomorphic.

   (a) $X_1 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3)$, $X_2 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3 - x)$ and $X_3 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3 - x^2)$.

   (b) $X_1 = \operatorname{Spec} \mathbb{C}[x, y]/(xy^2 + x^2y)$ and $X_2 = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy, yz, zx)$.

   (c) $X_1 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $X_2 = \mathbb{P}^2_{\mathbb{C}}$ and $X_3 =$ the blowing up of X_2 at the point $[0 : 0 : 1]$.

Qualifying Exams II, Jan. 2013

(1) (Real Analysis)

(a) For any bounded positive function f define

$$A(f) := \int_0^1 f(x)\log f(x)\mathrm{d}x, \quad B(f) := \Big(\int_0^1 f(x)\mathrm{d}x\Big)\log\Big(\int_0^1 f(y)\mathrm{d}y\Big).$$

There are three possibilities: (i) $A(f) \geq B(f)$ for all bounded positive functions, (ii) $B(f) \geq A(f)$ for all bounded positive functions, and (iii) $A(f) \geq B(f)$ for some functions while $B(f) \geq A(f)$ for some functions. Decide which possibility is correct and prove your answer. If you use any inequality, state all assumptions of the inequality precisely and clearly.

(b) Let $\hat{f}$ denote the Fourier transform of the function f on $\mathbb{R}$. Suppose that $f \in C^\infty(\mathbb{R})$ and

$$\|\hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq \alpha, \qquad \||\xi|^{1+\varepsilon}\hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq \beta$$

for some $\varepsilon > 0$. Find a bound on $\|f\|_{L^\infty}(\mathbb{R})$ in terms of α, β and ε.

(2) (Complex analysis) Is there a conformal map between the following domains? If the answer is yes, give such a conformal map. If it is no, prove it.

a) From $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

b) From the intersection of the open disks $D((0,0),3)$ and $D((0,3),2)$ to $\mathbb{C}$.

c) From $\mathbb{H}\backslash(0,i]$ to $\mathbb{H}$.

d) From $\mathbb{D}$ to $\mathbb{C}\backslash(-\infty,-\frac{1}{4}]$.

(3) (Differential Geometry) View $\mathbb{R}^2 \times \mathbb{C}$ as the product complex line bundle over $\mathbb{R}^2$ and let θ_0 denote the connection on this line bundle whose covariant derivative acts on a given section s as ds with d being the exterior derivative. Let A denote the connection

$$\theta_0 + \frac{i}{1+x^2+y^2}(xdy - ydx).$$

(a) Compute as a function of $r \in (0,\infty)$ the linear map from $\mathbb{C}$ to $\mathbb{C}$ that is obtained by using A to parallel transport a given non-zero vector in $\mathbb{C}$ in the clockwise direction on the circle where $x^2+y^2 = r^2$ from the point $(r,0)$ to itself.

(b) Give a formula for the curvature 2-form of the connection A.

(4) (Algebra) a) Let K/F be a field extension of degree $2n+1$ generated by t. Prove that for every $c \in K$ there exists a unique rational function $f \in F[T]$ such that $\deg(f) \leq n$ and $c = f(t)$. [The *degree* of a rational function f is the smallest d such that $f = P/Q$ for polynomials P, Q each of degree at most d.]

b) Deduce that if $[K : F] = 3$ then $\mathrm{PGL}_2(F)$ acts simply transitively by fractional linear transformations on $K\backslash F$ (the complement of F in K). If $|F| = q < \infty$, compute $|\mathrm{PGL}_2(F)|$ directly, and verify that it equals $|K| - |F|$.

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(5) (Algebraic Topology) Use Z to denote the subset of $\mathbb{R}^2$ that is given using standard polar coordinates (r, θ) by the equation $r = \cos^2(2\theta)$. The set Z is depicted in Figure 1.

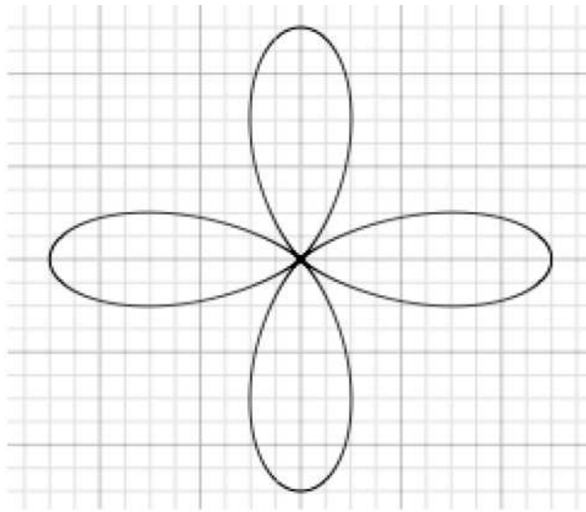

FIGURE 1. The set Z.

(a) Compute the fundamental group of Z.
(b) Let D denote the closed unit disk in $\mathbb{R}^2$ centered at the origin. The boundary of D is the unit circle, this denoted here by ∂D. Parametrize ∂D by the angle $\phi \in [0, 2\pi)$ and let f denote the map from the boundary of ∂D to Z that sends the angle ϕ to the point in Z with polar coordinates $(r = cos^2(2\phi), \theta = \phi)$. Let X denote the space that is obtained from the disjoint union of D and Z by identifying $\phi \in \partial D$ with $f(\phi) \in Z$. Give a set of generators and relations for the fundamental group of X.

(6) (Algebraic Geometry) Let f and g be irreducible homogeneous polynomials in $S = \mathbb{C}[X_0, X_1, X_2, X_3]$ of degrees 2 and 3, respectively. For parts (a) and (b), combinatorial polynomials (such as $\binom{T}{2} = T(T-1)/2$) are acceptable in the final answer.
(a) Compute the Hilbert polynomial of $X = \mathrm{Proj}(S/(g))$ embedded in $P = \mathbb{P}^3_{\mathbb{C}} = \mathrm{Proj}(S)$.
(b) Compute the Hilbert polynomial of $Y = \mathrm{Proj}(S/(f, g))$ embedded in P.
(c) Assuming in addition that Y is nonsingular, use your answer for part (b) to compute its geometric genus

$$\dim_{\mathbb{C}} \Gamma(Y, \Omega^1_{Y/\mathbb{C}}).$$

Qualifying Exams III, Jan. 2013

(1) (Real Analysis) Assume that $X_1, X_2, \ldots$ are independent random variables uniformly distributed on $[0,1]$. Let $Y^{(n)} = n \inf\{X_i, 1 \le i \le n\}$. Prove that $Y^{(n)}$ converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{E}\left(f(Y^{(n)})\right) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^+} f(u)e^{-u}\mathrm{d}u.$$

(2) (Complex Analysis) The following questions are independent.

a) Describe all harmonic functions on the plane $\mathbb{R}^2$ that are bounded from above.

b) Let $h : \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \to \mathbb{C}$ be holomorphic. Assume that $|h(z)| \le 1$ for any $z \in \mathbb{H}$ and i is a zero of h of order $m \ge 1$. Prove that, for any $z \in \mathbb{H}$,

$$|h(z)| \le \left|\frac{z-i}{z+i}\right|^m.$$

(3) (Differential Geometry) Use (t, x, y, z) to denote the Euclidean coordinates for $\mathbb{R}^4$. Let $t \mapsto a(t)$ denote a strictly positive function on $\mathbb{R}$. A Riemannian metric on $\mathbb{R}^4$ is given by the quadratic form:

$$g = dt \otimes dt + a(t)^2\Big(dx \otimes dx + dy \otimes dy + dz \otimes dz\Big).$$

Compute the components of the Riemann curvature tensor of g using the orthonormal basis $\{dt, a\,dx, a\,dy, a\,dz\}$ for $\mathrm{T}^\star\mathbb{R}^4$.

(4) (Algebraic Topology) Let $K \subset \mathbb{R}^3$ denote a knot, this being a compact, connected, dimension 1 submanifold.

(a) Compute the homology of the complement in $\mathbb{R}^3$ of any given knot K.

(b) Figure 1 shows a picture of the trefoil knot.

Figure 1. The trefoil knot.

Sketch on this picture a curve or curves in the complement of K that generate(s) the first homology of $\mathbb{R}^3 - K$.

(c) A Seifert surface for a knot in $\mathbb{R}^3$ is a connected, embedded surface with boundary, with the knot being the boundary (we do not impose orientability here). By way of an example, view the unit circle in the xy plane as a knot in $\mathbb{R}^3$. This is called the unknot. The unit disk in the xy plane is a Seifert surface for the unknot.

(i) Compute the second homology of the complement in $\mathbb{R}^3$ of any given Seifert surface for the unknot.

(ii) Sketch a Seifert surface for the unknot whose complement is not simply connected.

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(5) (Algebra) Let k be a finite field of cardinality q, and let $L = k(t)$, the field of rational functions over k in an indeterminate t. Set $x = t^q - t$, $K = k(x)$, and $F = k(x^{q-1})$.

a) Prove that L/K is a Galois extension with $\mathrm{Gal}(L/K) = (k, +)$ (the additive group of k).

b) Prove that L/F is Galois. What is $\mathrm{Gal}(L/F)$, and how does $\mathrm{Gal}(L/F)$ act on t?

(6) (Algebraic Geometry) Let X_0 be the affine plane curve defined by the equation

$$y^3 - 3y = x^5$$

over the complex numbers, and let X be the projective smooth model of X_0.

(a) Show that X_0 is nonsingular.

(b) Find all $a \in \mathbb{C}$ for which the polynomial $P_a(y) = y^3 - 3y - a$ has repeated roots. For each such a, factor the polynomial $P_a(y)$.

(c) Let $\pi : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ be the unique extension of the coordinate map $x : X_0 \longrightarrow \mathbb{A}^1_{\mathbb{C}}$. Describe the ramification divisor of π and compute its degree.

(d) Compute the genus of X by applying Hurwitz's theorem to $\pi : X \longrightarrow \mathbb{P}^1$.