Qualifying exam, Spring 2006, Day 1

(1) Let $\phi : A \to B$ be a homomorphism of commutative rings, and let $\mathfrak{p}_B \subset B$ be a maximal ideal. Set $A \supset \mathfrak{p}_A := \phi^{-1}(\mathfrak{p}_B)$.

(a) Show that $\mathfrak{p}_A$ is prime but in general non maximal.

(b) Assume that A, B are finitely generated algebras over a field k and ϕ is a morphism of k-algebras. Show that in this case $\mathfrak{p}_A$ is maximal.

(2) Let V be a 4-dimensional vector space over k, and let $Gr^2(V)$ denote the set of 2-dimensional vector subspaces of V. Set $W = \Lambda^2(V)$, and let $\mathbb{P}^5$ be the 5-dimensional projective space, thought of as the set of lines in W.

Define a map of sets $Gr^2(V) \to \mathbb{P}^5$ that sends a 2-dimensional subspace $U \subset V$ to the line $\Lambda^2(U) \subset \Lambda^2(V) = W$.

(a) Show that the above map is injective and identifies $Gr^2(V)$ with the set of points of a projective subvariety of $\mathbb{P}^5$.

(b) Find the dimension of the above projective variety, and its degree.

(3) Are there any non-constant bounded holomorphic functions defined on the complement $\mathbb{C} \setminus I$ of the unit interval

$$I = \{a \in \mathbb{R} \mid 0 \leq a \leq 1\} \subset \mathbb{C}$$

in the complex plane $\mathbb{C}$?

(4) Let X be the topological space obtained by removing one point from a Riemann surface of genus $g \geq 1$. Compute the homotopy groups $\pi_n(X)$.

(5) Let γ be a geodesic curve on a regular surface of revolution $S \subset \mathbb{R}^3$. Let $\theta(p)$ denote the angle the curve forms with the parallel at a point $p \in \gamma$ and $r(p)$ be the distance to the axes of revolution. Prove Clairaut's relation: $r \cos\theta = const.$

(6) Define the function f on the interval $[0, 1]$ as follows. If $x = 0.x_1x_2x_3...$ is the unique non-terminating decimal expansion of $x \in (0, 1]$, define $f(x) = \max_n \{x_n\}$. Prove that f is measurable.

Qualifying exam, Spring 2006, Day 2

(1) Describe irreducible representations of the finite group A_4.

(2) Show that every morphism of projective varieties $\mathbb{P}^2 \to \mathbb{P}^1$ is constant.

(3) Let $g(z)$ be an entire holomorphic function. Define the function $F(z)$ on $\mathbb{C} \setminus [-1,1]$ by

$$F(z) = \int_{-1}^{1} \frac{g(x)}{x-z}\, dx.$$

(a) Show that $F(z)$ is analytic in $\mathbb{C} \setminus [-1,1]$ and can be analytically continued across the open interval $(-1,1)$.
(b) Call $F_-(z)$ and $F_+(z)$ the analytic continuations from below and from above $(-1,1)$ respectively. Calculate $F_+(z) - F_-(z)$ on $(-1,1)$.

(4) Let X be the blow-up of $\mathbb{CP}^2$ at one point. Compute the groups $H^i(X, \mathbb{Z})$.

(5) Let $F(x,y,z)$ be a smooth homogenous function of degree n, i.e. $F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x,y,z)$. Prove that away from the origin the induced metric on the conical surface

$$\Sigma = \{(x,y,z) \mid F(x,y,z) = 0\}$$

has Gaussian curvature equal to 0.

(6) Let $p > 0$. Let l^p denote sequences $\underline{x} = \{x_n\} \in \mathbb{R}^{\mathbb{N}}$ (here $\mathbb{N}$ denotes the set of natural numbers), such that $\sum |x_n|^p$ converges. We define a topology on l^p with the basis

$$B^r(\underline{x}) = \{\underline{y} \mid \sum |y_n - x_n|^p < r\}.$$

For which p does this topology arise from a norm?

Qualifying exam, Spring 2006, Day 3

(1) Let $\zeta = e^{2\pi i/37}$ and let $\alpha = \zeta + \zeta^{10} + \zeta^{26}$. Find (with proof) the degree of $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$.

(2) Let $X \subset \mathbb{A}^n$ be an algebraic subvariety, defined by a non-trivial *homegeneous* ideal $I \subset k[t_1, ..., t_n]$.
(a) Show that the point 0 is contained in X.
(b) Assume that X is non-singular at 0. Show that $X = W$ for some linear subspace $W \subset \mathbb{A}^n$.

(3) Let f be a holomorphic function on $\mathbf{C}$ whose image lands in the upper half plane. Prove that f is constant without using Picard's theorem.

(4) Let f be a continuous map $\mathbf{CP}^n \to \mathbf{CP}^n$.
(a) Prove that if n is even, then f necessarily has a fixed point.
(b) Verify that the map $f : \mathbf{C}^4 \to \mathbf{C}^4$ defined by $f(z_1, z_2, z_3, z_4) = (\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3)$ induces a map $\mathbf{CP}^3 \to \mathbf{CP}^3$ with no fixed points.

(5)
(a) Let f, g be two C^∞-maps between manifolds $X \to Y$. Let ω^k be a closed differential form on Y, and consider $f^*(\omega), g^*(\omega) \in \Omega^k(X)$. Assume that there exists a homotopy between f and g, i.e., a smooth map $h : \mathbb{R} \times X \to Y$ such that $h|_{0\times X} = f$ and $h|_{1\times X} = g$. Show that $f^*(\omega) - g^*(\omega)$ is exact.
(b) Deduce that every closed k-form ($k \geq 1$) on $\mathbb{R}^n$ is exact.

(6) Does there exist a continuous function on the interval $[0, 1]$ such that

$$\int_0^1 x^n f(x)dx = \begin{cases} 1, & n = 1 \\ 0, & n = 2, 3, ... \end{cases}$$