1. (AT)
   (a) Let $X$ and $Y$ be compact, oriented manifolds of the same dimension $n$. Define the degree of a continuous map $f : X \to Y$.
   (b) Let $f : \mathbb{C}P^3 \to \mathbb{C}P^3$ be any continuous map. Show that the degree of $f$ is of the form $m^3$ for some integer $m$.
   (c) Show that conversely for any $m \in \mathbb{Z}$ there is a continuous map $f : \mathbb{C}P^3 \to \mathbb{C}P^3$ of degree $m^3$.

2. (A) Let $G$ be a group.
   (a) Prove that, if $V$ and $W$ are irreducible $G$-representations defined over a field $\mathbb{F}$, then a $G$-homomorphism $f : V \to W$ is either zero or an isomorphism.
   (b) Let $G = D_8$ be the dihedral group with 8 elements. What are the dimensions of its irreducible representations over $\mathbb{C}$?

3. (CA) Let $f_n$ be a sequence of analytic functions on the unit disk $\Delta \subset \mathbb{C}$ such that $f_n \to f$ uniformly on compact sets and such that $f$ is not identically zero. Prove that $f(0) = 0$ if and only if there is a sequence $z_n \to 0$ such that $f_n(z_n) = 0$ for $n$ large enough.

4. (AG) Let $K$ be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree $n$ in two variables over $K$. Let $X \subset \mathbb{P}^n$ be the locus of $n^{th}$ powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).
   (a) Show that $X$ and $Y \subset \mathbb{P}^n$ are closed subvarieties.
   (b) What is the degree of $X$?
   (c) What is the degree of $Y$?
5. (DG) Given a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x_1, \ldots, x_n) := f(x_1, \ldots, x_{n-1}) - x_n$$

and consider the preimage $X_f = F^{-1}(0) \subset \mathbb{R}^n$.

(a) Prove that $X_f$ is a smooth manifold which is diffeomorphic to $\mathbb{R}^{n-1}$.

(b) Consider the two examples $X_f$ and $X_g \subset \mathbb{R}^3$ with $f(x_1, x_2) = x_1^2 + x_2^2$ and $g(x_1, x_2) = x_1^2 - x_2^2$. Compute their normal vectors at every point $(x_1, x_2, x_3) \in X_f$ and $(x_1, x_2, x_3) \in X_g$.

6. (RA) Let $K \subset \mathbb{R}^n$ be a compact set. Show that for any measurable function $f : K \to \mathbb{C}$, it holds that

$$\lim_{p \to \infty} \|f\|_{L^p(K)} = \|f\|_{L^\infty(K)}.$$

(Recall that $\|f\|_{L^p(K)} = \left(\int_K |f|^p \, dx\right)^{1/p}$ and that $\|f\|_{L^\infty(K)}$ is the essential supremum of $f$, i.e., the smallest upper bound if the behavior of $f$ on null sets is ignored.)
1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d$.

(a) Let $K_C$ be the canonical bundle of $C$. For what integer $n$ is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then $C$ is not hyperelliptic.

(c) Prove that if $d \geq 5$ then $C$ is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

2. (CA) (The 1/4 theorem). Let $S$ denote the class of functions that are analytic on the disk and one-to-one with $f(0) = 0$ and $f'(0) = 1$.

(a) Prove that if $f \in S$ and $w$ is not in the range of $f$ then

$$g(z) = \frac{wf(z)}{(w - f(z))}$$

is also in $S$.

(b) Show that for any $f \in S$, the image of $f$ contains the ball of radius 1/4 around the origin. You may use the elementary result (Bieberbach) that if $f(z) = z + \sum_{k \geq 2} a_k z^k$ in $S$ then $|a_2| \leq 2$.

3. (A) Find a polynomial $f \in \mathbb{Q}[x]$ whose Galois group (over $\mathbb{Q}$) is $D_8$, the dihedral group of order 8.

4. (RA)

(a) Let $a_k \geq 0$ be a monotone increasing sequence with $a_k \to \infty$, and consider the ellipse,

$$E(a_k) = \{v \in \ell^2(\mathbb{Z}) : \sum a_kv_k^2 \leq 1\}.$$ 

Show that $E(a_n)$ is a compact subset of $\ell^2(\mathbb{Z})$. 
(b) Let \( \mathbb{T} \) denote the one-dimensional torus; that is, \( \mathbb{R}/2\pi\mathbb{Z} \), or \([0, 2\pi]\) with the ends identified. Recall that the space \( H^1(\mathbb{T}) \) is the closure of \( C^\infty(\mathbb{T}) \) in the norm

\[
\|f\|_{H^1(\mathbb{T})} = \sqrt{\|f\|_{L^2(\mathbb{T})}^2 + \|\frac{d}{dx}f\|_{L^2(\mathbb{T})}^2}.
\]

Use part (a) to conclude that the inclusion \( i : H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \) is a compact operator.

5. (AT) Consider the following topological spaces:

\[ A = S^1 \times S^1 \quad \quad B = S^1 \vee S^1 \vee S^2. \]

(a) Compute the fundamental group of each space.
(b) Compute the integral cohomology ring of each space.
(c) Show that \( B \) is not homotopy equivalent to any compact orientable manifold.

6. (DG) Consider the set

\[
G := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R}^+, \ y \in \mathbb{R} \right\},
\]

and equip it with a smooth structure via the global chart that sends \((x, y) \in \mathbb{R}^+ \times \mathbb{R}\) to the corresponding element of \( G \).

(a) Show that \( G \) is a Lie subgroup of the Lie group \( GL(\mathbb{R}, 3) \).
(b) Prove that the set

\[
\left\{ x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y} \right\}
\]

forms a basis of left-invariant vector fields on \( G \).
(c) Find the structure constants of the Lie algebra \( g \) of \( G \) with respect to the basis in (b).
1. (AT) Let $p : E \to B$ be a $k$-fold covering space, and suppose that the Euler characteristic $\chi(E)$ is defined, nonzero, and relatively prime to $k$. Show that any CW decomposition of $B$ has infinitely many cells.

2. (RA) Let $W$ be Gumbel distributed, that is $P(W \leq x) = e^{-e^{-x}}$. Let $X_i$ be independent and identically distributed Exponential random variables with mean 1; that is, $X_i$ are independent, with $P(X_i \leq x) = \exp(- \max x, 0)$.

Let
\[ M_n = \max_{i \leq n} X_i. \]

Show that there are deterministic sequences $a_n, b_n$ such that
\[ \frac{M_n - b_n}{a_n} \to W \]
in law; that is, such that for any continuous bounded function $F$,
\[ \mathbb{E} F\left( \frac{M_n - b_n}{a_n} \right) \to \mathbb{E} F(W). \]

3. (DG) Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric
\[ g = e^x dx^2 + dy^2. \]

(i) Compute the Christoffel symbols of the Levi-Civita connection for $g$.

(ii) Show that the geodesics are described by the curves $x(t) = 2 \log(At + B)$ and $y(t) = Ct + D$, for appropriate constants $A, B, C, D$.

(iii) Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$, $\gamma(t) = (t, t)$. Compute the parallel transport of the vector $(1, 2)$ along the curve $\gamma$.

(iv) Are there two vector fields $X, Y$ parallel to the curve $\gamma$, such that $g(X(t), Y(t))$ is non-constant?
4. (A) Let $G$ be a group of order 78.
   
   (a) Show that $G$ contains a normal subgroup of index 6.
   
   (b) Show by example that $G$ may contain a subgroup of index 13 that is not normal.

5. (AG) Let $K$ be an algebraically closed field of characteristic 0, and consider the curve $C \subset \mathbb{A}^3$ over $K$ given as the image of the map

$$
\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^3
$$

$$
t \mapsto (t^3, t^4, t^5).
$$

Show that no neighborhood of the point $\phi(0) = (0, 0, 0) \in C$ can be embedded in $\mathbb{A}^2$.

6. (CA) Let $f(z)$ be an entire function such that

   a) $f(z)$ vanishes at all points $z = n, n \in \mathbb{Z}$;

   b) $|f(z)| \leq e^{\pi |\text{Im} z|}$ for all $z \in \mathbb{C}$.

Prove that $f(z) = c \sin \pi z$, with $c \in \mathbb{C}$, $|c| \leq 1$. 