QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 30, 2016 (Day 1)

1. (DG)

(a) Show that if V is a $\mathcal{C}^\infty$-vector bundle over a compact manifold X, then there exists a vector bundle W over X such that $V \oplus W$ is trivializable, i.e. isomorphic to a trivial bundle.

(b) Find a vector bundle W on S^2, the 2-sphere, such that $T^*S^2 \oplus W$ is trivializable.

Solution: Since V is locally trivializable and M is compact, one can find a finite open cover U_i, $i = 1, \ldots, n$, of M and trivializations $T_i : V|_{U_i} \to \mathbb{R}^k$. Thus, each T_i is a smooth map which restricts to a linear isomorphism on each fiber of $V|_{U_i}$. Next, choose a smooth partition of unity $\{f_i\}_{i=1,\ldots,n}$ subordinate to the cover $\{U_i\}_{i=1,\ldots,n}$. If $p : V \to M$ is the projection to the base, then there are maps

$$V|_{U_i} \to \mathbb{R}^k, \qquad v \mapsto f_i(p(v))T_i(v)$$

which extend (by zero) to all of V and which we denote by f_iT_i. Together, the f_iT_i give a map $T : V \to \mathbb{R}^{nk}$ which has maximal rank k everywhere, because at each point of X at least one of the f_i is non-zero. Thus V is isomorphic to a subbundle, $T(V)$, of the trivial bundle, $\mathbb{R}^{nk}$. Using the standard inner product on $\mathbb{R}^{nk}$ we get an orthogonal bundle $W = T(V)^\perp$ which has the desired property.

For the second part, embed S^2 into $\mathbb{R}^3$ in the usual way, then

$$TS^2 \oplus N_{S^2} = T\mathbb{R}^3|_{S^2}$$

where N_{S^2} is the normal bundle to S^2 in $\mathbb{R}^3$. Dualizing we get

$$T^*S^2 \oplus (N_{S^2})^* = T^*\mathbb{R}^3|_{S^2}$$

which solves the problem with $W = (N_{S^2})^*$.

2. (RA) Let (X, d) be a metric space. For any subset $A \subset X$, and any $\epsilon > 0$ we set

$$B_\epsilon(A) = \bigcup_{p \in A} B_\epsilon(p).$$

(This is the "ϵ-fattening" of A.) For Y, Z bounded subsets of X define the *Hausdorff distance* between Y and Z by

$$d_H(Y,Z) := \inf\{\epsilon > 0 \mid Y \subset B_\epsilon(Z), \quad Z \subset B_\epsilon(Y)\}.$$

Show that d_H defines a metric on the set $\tilde{X} := \{A \subset X \big| \, A \text{ is closed and bounded}\}$.

Solution: We need to show that $(\tilde{X}, d_H)$ is a metric space. First, since the sets are bounded, $d_H(Y,Z)$ is well defined for any closed sets Y, Z. Secondly, $d_H(Y,Z) = d_H(Z,Y) \geq 0$ is obvious from the definition. We need to prove that the distance is positive when $Y \neq Z$, and that d_H satisfies the triangle inequality. First, let us show that $d_H(Y,Z) > 0$ if $Y \neq Z$. Without loss of generality, we can assume there is a point $p \in Y \cap Z^c$. Since Z is closed, so there exists $r > 0$ such that $B_r(p) \subset Z^c$. In particular, p is not in $B_r(Z)$. Thus Y is not contained in $B_r(Z)$ and so $d_H(Y,Z) \geq r > 0$.

It remains to prove the triangle inequality. To this end, suppose that Y, Z, W are relevant subsets of X. Fix $\epsilon_1 > d_H(Y,Z), \epsilon_2 > d_H(Z,W)$, then

$$Y \subset B_{\epsilon_1}(Z), \quad Z \subset B_{\epsilon_1}(Y), \quad Z \subset B_{\epsilon_2}(W), \quad W \subset B_{\epsilon_2}(Z)$$

Then $d_H(Y,Z) < \epsilon_1, d_H(Z,W) < \epsilon_2$. Let us prove that $Y \subset B_{\epsilon_1+\epsilon_2}(W)$, the other containment being identical. Fix a point $y \in Y$. By our choice of ϵ_1 there exists a point $z \in Z$ such that $y \in N_{\epsilon_1}(z)$. By our choice of ϵ_2 there exists a point $w \in W$ such that $z \in B_{\epsilon_2}(w)$. Then

$$d(y,w) \leq d(y,z) + d(z,w) \leq \epsilon_1 + \epsilon_2$$

so $y \in B_{\epsilon_1+\epsilon_2}(w)$. This proves the containment. The other containment is identical, by just swapping Y, W. Thus

$$d_H(Y,W) \leq \epsilon_1 + \epsilon_2$$

But this holds for all ϵ_1, ϵ_2 as above. Taking the infimum we obtain the result.

3. (AT) Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the n-torus. Prove that any path-connected covering space $Y \to T^n$ is homeomorphic to $T^m \times \mathbb{R}^{n-m}$, for some m.

Solution: The universal covering space of T^n is $\mathbb{R}^n$, so that any path connected covering space of X is of the form $\mathbb{R}^n/G$, for some subgroup $G \subseteq \pi_1(T^n)$. We have $\pi_1(T^n) = \pi_1(S^1) \times \cdots \times \pi_1(S^1) = \mathbb{Z}^n$, and $\mathbb{Z}^n$ is acting on $\mathbb{R}^n$ by translation. Thus, $G \subseteq \mathbb{Z}^n$ is free. Choose a $\mathbb{Z}$-basis $(v_1, \ldots, v_m)$ of G, and consider the (real!) change of basis taking $(v_1, \ldots, v_m)$ to the first m standard basis vectors $(e_1, \ldots, e_m)$. Hence, G is acting on $\mathbb{R}^n$ by translation in the first m coordinates. Thus,

$$\mathbb{R}^n/G \simeq \mathbb{R}^m/\mathbb{Z}^m \times \mathbb{R}^{n-m} \simeq T^m \times \mathbb{R}^{n-m}.$$

4. (CA)

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Show that the image of f is dense in $\mathbb{C}$.

Solution: Suppose that for some $w_0 \in \mathbb{C}$ and some $\epsilon > 0$, the image of f lies outside the ball $B_\epsilon(w_0) = \{w \in \mathbb{C} \mid |w - w_0| < \epsilon\}$. Then the function

$$g(z) \;=\; \frac{1}{f(z) - w_0}$$

is bounded and homomorphic in the entire plane, hence constant.

5. (A) Let $F \supset \mathbb{Q}$ be a splitting field for the polynomial $f = x^n - 1$.

(a) Let $A \subset F^\times = \{z \in F \mid z \neq 0\}$ be a finite (multiplicative) subgroup. Prove that A is cyclic.

(b) Prove that $G = \mathrm{Gal}(F/\mathbb{Q})$ is abelian.

Solution: For the first part, let $m = |A|$. Suppose that A is not cyclic, so that the order of any element in A is less than m. A is a finite abelian group so it is isomorphic to a product of cyclic groups $A \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_i | n_{i+1}$. In particular, the order of any element in A divides n_k. Hence, for any $z \in A$, $z^{n_k} = 1$. However, the polynomial $x^{n_k} - 1 \in F[x]$ admits at most $n_k < m$ roots in F, which is a contradiction. So, there must be some element in A with order m.

For the second part, since $f' = nx^{n-1}$ and f are relatively prime, f admits n distinct roots $1 = z_0, \ldots, z_{n-1}$. As F is a splitting field of f we can assume that $F = \mathbb{Q}(z_0, \ldots, z_{n-1}) \subseteq \mathbb{C}$. $U = \{z_0, \ldots, z_{n-1}\} \subset F^\times$ is a subgroup of the multiplicative group of units in F and is cyclic; moreover, $Aut(U)$ is isomorphic to the (multiplicative) group of units $(\mathbb{Z}/n\mathbb{Z})^*$. Restriction defines a homomorphism $G \to Aut(U)$, $\alpha \mapsto \alpha_{|U}$; this homomorphism is injective because $F = \mathbb{Q}(z_0, \ldots, z_{n-1})$. In particular, G is isomorphic to a subgroup of the abelian group $(\mathbb{Z}/n\mathbb{Z})^*$.

6. (AG) Let C and $D \subset \mathbb{P}^2$ be two plane cubics (that is, curves of degree 3), intersecting transversely in 9 points $\{p_1, p_2, \ldots, p_9\}$. Show that $p_1, \ldots, p_6$ lie on a conic (that is, a curve of degree 2) if and only if p_7, p_8 and p_9 are colinear.

Solution: First, observe that we can replace $C = V(F)$ and $D = V(G)$ by any two independent linear combinations $C' = V(a_0F + a_1G)$ and $D' = V(b_0F + b_1G)$. Now suppose that $p_1, \ldots, p_6$ lie on a conic $Q \subset \mathbb{P}^2$. Picking a seventh point $q \in Q$, we see that some linear combination C_0 of C and D contains q and hence contains Q; thus $C_0 = Q \cup L$ for some line $L \subset \mathbb{P}^2$. Replacing C or D with C_0, we see that p_7, p_8 and $p_9 \in L$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday August 31, 2016 (Day 2)

1. (A) Let R be a commutative ring with unit. If $I \subsetneq R$ is a proper ideal, we define the *radical* of I to be

$$\sqrt{I} \ = \ \{a \in R \ | \ a^m \in I \quad \text{for some } m > 0\}.$$

Prove that

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}.$$

Solution: First, we prove for the case $I = 0$. Let $f \in \sqrt{0}$ so that $f^n = 0$, and $f^n \in \mathfrak{p}$, for any prime ideal $\mathfrak{p} \subseteq R$. Let $\mathfrak{p}$ be a prime ideal in R. The quotient ring $R/\mathfrak{p}$ is an integral domain and, in particular, contains no nonzero nilpotent elements. Hence, $f^n + \mathfrak{p} = 0 \in R/\mathfrak{p}$ so that $f \in \mathfrak{p}$.

Now, suppose that $f \notin \sqrt{0}$. The set $S = \{1, f, f^2, \ldots\}$ does not contain 0 so that the localisation R_f is not the zero ring. Let $\mathfrak{m} \subset R_f$ be a maximal ideal. Denote the canonical homomorphism $j : R \to R_f$. As $j(f) \in R_f$ is a unit, $j(f) \notin \mathfrak{m}$. Then $j^{-1}(\mathfrak{m}) \subset R$ is a prime ideal that does not contain f. Hence, $f \notin \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}$.

If $I \subsetneq R$ is a proper ideal, we consider the quotient ring $\pi : R \to S = R/I$. Recall the bijective correspondence

$$\{\text{prime ideals in } S\} \leftrightarrow \{\text{prime ideals in } R \text{ containing } I\} \ , \ \mathfrak{p} \leftrightarrow \pi^{-1}(\mathfrak{p})$$

Then,

$$\sqrt{I} = \pi^{-1}(\sqrt{0_S}) = \pi^{-1}\left(\bigcap_{\mathfrak{p} \subseteq S \text{ prime}} \mathfrak{p}\right) = \bigcap_{\mathfrak{p} \subseteq S \text{ prime}} \pi^{-1}(\mathfrak{p}) = \bigcap_{\substack{\mathfrak{q} \supseteq I \\ \mathfrak{q} \text{ prime}}} \mathfrak{q}.$$

2. (DG) Let $c(s) = (r(s), z(s))$ be a curve in the (x, z)-plane which is parameterized by arc length s. We construct the corresponding rotational surface, S, with parametrization

$$\varphi : (s, \theta) \mapsto (r(s)\cos\theta, r(s)\sin\theta, z(s)).$$

Find an example of a curve c such that S has constant negative curvature -1.

Solution:

$$\frac{\partial \varphi}{\partial s}(s,\theta) = (r'(s)\cos\theta, r'(s)\sin\theta, z'(s))$$
$$\frac{\partial \varphi}{\partial \theta}(s,\theta) = (-r(s)\sin\theta, r(s)\cos\theta, 0)$$

The coefficients of the first fundamental form are:

$$E = r'(s)^2 + z'(s)^2 = 1, \qquad F = 0, \qquad G = r(s)^2$$

Curvature:

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2}{\partial s^2}\sqrt{G} = -\frac{r''(s)}{r(s)}$$

To get $K = -1$ we need to find $r(s)$, $z(s)$ such that

$$r''(s) = r(s),$$
$$r'(s)^2 + z'(s)^2 = 1.$$

A possible solution is $r(s) = e^{-s}$ with

$$z(s) = \int \sqrt{1 - e^{-2t}}dt = \operatorname{Arcosh}(r^{-1}) - \sqrt{1 - r^2}.$$

3. (RA) Let $f \in L^2(0,\infty)$ and consider

$$F(z) = \int_0^\infty f(t)e^{2\pi i z t}dt$$

for z in the upper half-plane.

(a) Check that the above integral converges absolutely and uniformly in any region $\operatorname{Im}(z) \geq C > 0$.

(b) Show that

$$\sup_{y>0}\int_0^\infty |F(x+iy)|^2 dx = \|f\|^2_{L^2(0,\infty)}.$$

Solution: For $\operatorname{Im}(z) \geq C > 0$ we have

$$|f(t)e^{2\pi i z t}| \leq |f(t)|e^{-2C\pi t}$$

thus with the Cauchy–Schwarz inequality

$$\int_0^\infty |f(t)e^{2\pi i z t}|dt \leq \left(\int_0^\infty |f(t)|^2 dt\right)^{1/2}\left(\int_0^\infty e^{-4C\pi t}dt\right)^{1/2}$$

which proves the claim.

For the second part, Plancherel's theorem gives

$$\int_0^\infty |F(x+iy)|^2 dx = \int_0^\infty |f(t)|^2 e^{-4\pi y t} dt \leq \|f\|^2_{L^2(0,\infty)}$$

and

$$\sup_{y>0} \int_0^\infty |f(t)|^2 e^{-4\pi y t} dt = \int_0^\infty |f(t)|^2 dt$$

by the monotone convergence theorem.

4. (CA) Given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, use contour integration to prove that each of the improper integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$ converges to $\sqrt{\pi/8}$.

Solution: We integrate $e^{-z^2}\,dz$ along a triangular contour with vertices at 0, M, and $(1+i)M$, and let $M \to \infty$. Since e^{-z^2} is holomorphic on $\mathbb{C}$, the integral vanishes. The integral from 0 to M is $\int_0^M e^{-x^2}\,dx$, which approaches $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. The vertical integral approaches zero, because it is bounded in absolute value by

$$\int_0^M |e^{-(M+yi)^2}|\,dy = \int_0^M e^{y^2-M^2}\,dy < \int_0^M e^{M(y-M)}\,dy$$

$$= \int_0^M e^{-Mt}\,dt < \int_0^\infty e^{-Mt}\,dt = \frac{1}{M} \to 0$$

(substituting $t = M - y$ in the middle step). Thus the diagonal integral (with direction reversed, from 0 to $(1+i)\infty$) equals $\frac{1}{2}\sqrt{\pi}$. The change of variable $z = e^{\pi i/4}x$ converts this integral to $e^{\pi i/4}\int_0^\infty e^{-ix^2}\,dx$. Hence

$$\int_0^\infty (\cos x^2 - i\sin x^2)\,dx = \int_0^\infty e^{-ix^2}\,dx = \frac{1}{2}e^{-\pi i/4}\sqrt{\pi} = \frac{1-i}{2\sqrt{2}}\sqrt{\pi}.$$

equating real and imaginary parts yields the required result.

5. (AT)

(a) Let $X = \mathbb{R}P^3 \times S^2$ and $Y = \mathbb{R}P^2 \times S^3$. Show that X and Y have the same homotopy groups but are not homotopy equivalent.

(b) Let $A = S^2 \times S^4$ and $B = \mathbb{C}P^3$. Show that A and B have the same singular homology groups with $\mathbb{Z}$-coefficients but are not homotopy equivalent.

Solution: The universal covers of $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are S^2 and S^3, respectively. Moreover, these covers are both 2-sheeted. Hence, we have

$$\pi_1(X) = \pi_1(\mathbb{R}P^3) \times \pi_1(S^2) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$$

$$\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}.$$

Also, $\pi_k(\mathbb{R}P^j) = \pi_k(S^j)$, for $k > 1$, $j = 2, 3$ so that

$$\pi_k(X) = \pi_k(S^2) \times \pi_k(S^3) = \pi_k(Y), \quad k > 1.$$

To show that X and Y are not homotopy equivalent, we show that they have nonisomorphic homology groups. We make use of the following well-known singular homology groups (with integral coefficients)

$$H_0(S^n) = H_n(S^n) = \mathbb{Z}, \quad H_i(S^k) = 0,\ i \neq 0, n,$$

$$H_0(\mathbb{R}P^2) = H_2(\mathbb{R}P^2) = \mathbb{Z},\ H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z},\ H_i(\mathbb{R}P^2) = 0, i \neq 0, 1, 2$$
$$H_0(\mathbb{R}P^3) = \mathbb{Z},\ H_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z},\ H_i(\mathbb{R}P^3) = 0, i \neq 0, 1$$

Now, the Kunneth theorem in singular homology (with $\mathbb{Z}$-coefficients) gives an exact sequence

$$0 \to \bigoplus_{i+j=2} H_i(\mathbb{R}P^3) \otimes_{\mathbb{Z}} H_j(S^2) \to H_2(X) \to \bigoplus_{i+j=1} Tor_1(H_i(\mathbb{R}P^3), H_j(S^2)) \to 0$$

Since $H_k(S^2)$ is free, for every k, we have

$$H_2(X) \simeq \bigoplus_{i+j=2} H_i(\mathbb{R}P^3) \otimes_{\mathbb{Z}} H_j(S^2) = \mathbb{Z}$$

Similarly, we compute

$$H_2(Y) \simeq \bigoplus_{i+j=2} H_i(\mathbb{R}P^2) \otimes_{\mathbb{Z}} H_j(S^3) = \mathbb{Z}/2\mathbb{Z}.$$

In particular, X and Y are not homotopy equivalent.

For the second part, B can be constructed as a cell complex with a single cell in dimensions $0, 2, 4, 6$. Therefore, the homology of B is $H_{2i}(B) = \mathbb{Z}$, for $i = 0, \ldots, 3$, and $H_k(B) = 0$ otherwise.

The Kunneth theorem for singular cohomology (with $\mathbb{Z}$-coefficients), combined with the fact that $H_k(S^n)$ is free, for any k, gives

$$H_k(A) \simeq \bigoplus_{i+j=k} H_i(S^2) \otimes H_j(S^4).$$

Hence, $H_{2i}(A) = \mathbb{Z}$, for $i = 0, \ldots, 3$, and $H_k(A) = 0$ otherwise.

In order to show that A and B are not homotopy equivalent we will show that they have nonisomorphic homotopy groups.

Consider the canonical quotient map $\mathbb{C}^4 - \{0\} \to \mathbb{C}P^3$. This restricts to give a fiber bundle $S^1 \to S^7 \to \mathbb{C}P^3$. The associated long exact sequence in homotopy

$$\cdots \to \pi_{k+1}(\mathbb{C}P^3) \to \pi_k(S^1) \to \pi_k(S^7) \to \pi_k(\mathbb{C}P^3) \to \cdots$$

together with the fact that $\pi_3(S_1) = \pi_4(S^7)$, shows that $\pi_4(\mathbb{C}P^3) = 0$. However, $\pi_4(A) = \pi_4(S^4) = \mathbb{Z}$.

6. (AG)

Let C be the smooth projective curve over $\mathbb{C}$ with affine equation $y^2 = f(x)$, where $f \in \mathbb{C}[x]$ is a square-free monic polynomial of degree $d = 2n$.

(a) Prove that the genus of C is $n - 1$.

(b) Write down an explicit basis for the space of global differentials on C.

Solution: For the first part, use Riemann-Hurwitz: the $2 : 1$ map from C to the x-line is ramified above the roots of f and nowhere else (not even at infinity because $\deg f$ is even), so

$$2 - 2g(C) = \chi(C) = 2\chi(\mathbb{P}^1) - \deg P = 4 - 2n,$$

whence $g(C) = n - 1$.

For the second, let $\omega_0 = dx/y$. This differential is holomorphic, with zeros of order $g - 1$ at the two points at infinity. (Proof by local computation around those points and the roots of P, which are the only places where holomorphy is not immediate; dx has a pole of order -2 at infinity but $1/y$ has zeros of order n at the points above $x = \infty$, while $2y\,dy = P'(x)\,dx$ takes care of the Weierstrass points.) Hence the space of holomorphic differentials contains

$$\Omega := \{P(x)\,\omega_0 \mid \deg P < g\},$$

which has dimension g. Thus Ω is the full space of differentials, with basis $\{\omega_k = x^k\omega_0,\ k = 0, \ldots, g - 1\}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 1, 2016 (Day 3)

1. (AT) Model S^{2n-1} as the unit sphere in $\mathbb{C}^n$, and consider the inclusions

$$\begin{array}{ccccccc} \cdots & \to & S^{2n-1} & \to & S^{2n+1} & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathbb{C}^n & \to & \mathbb{C}^{n+1} & \to & \cdots . \end{array}$$

Let S^∞ and $\mathbb{C}^\infty$ denote the union of these spaces, using these inclusions.

(a) Show that S^∞ is a contractible space.

(b) The group S^1 appears as the unit norm elements of $\mathbb{C}^\times$, which acts compatibly on the spaces $\mathbb{C}^n$ and S^{2n-1} in the systems above. Calculate *all* the homotopy groups of the homogeneous space S^∞/S^1.

Solution: The shift operator gives a norm-preserving injective map $T : \mathbb{C}^\infty \to \mathbb{C}^\infty$ that sends S^∞ into the hemisphere where the first coordinate is zero. The line joining $x \in S^\infty$ to $T(x)$ cannot pass through zero, since x and $T(x)$ cannot be scalar multiples, and hence the linear homotopy joining x to $T(x)$ shows that T is homotopic to the identity. However, since $T(S^\infty)$ forms an equatorial hemisphere, there is a also a linear homotopy from T to the constant map at either of the poles.

For the second part, because S^1 acts properly discontinuously on S^∞, the quotient sequence

$$S^1 \to S^\infty \to S^\infty/S^1$$

forms a fiber bundle. The homotopy groups of S^1 are known: $\pi_1 S^1 \cong \mathbb{Z}$ and $\pi_{\neq 1} S^1 = 0$ otherwise. Since S^∞ is contractible, the long exact sequence of higher homotopy groups shows that $\pi_2(S^\infty/S^1) = \mathbb{Z}$ and $\pi_{\neq 2}(S^\infty/S^1) = 0$ otherwise.

2. (AG) Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. Show that if

$$\binom{k+d}{k} \; > \; (k+1)(n-k)$$

then X does not contain any k-plane $\Lambda \subset \mathbb{P}^n$.

Solution: For the first, let $\mathbb{P}^N$ be the space of all hypersurfaces of degree d in $\mathbb{P}^n$, and let

$$\Gamma \; = \; \{(X, \Lambda) \in \mathbb{P}^N \times \mathbb{G}(k,n) \;\mid\; \Lambda \subset X\}.$$

The fiber of Γ over the point $[\Lambda] \in \mathbb{G}(k,n)$ is just the subspace of $\mathbb{P}^N$ corresponding to the vector space of polynomials vanishing on Λ; since the space of polynomials on $\mathbb{P}^n$ surjects onto the space of polynomials on $\Lambda \cong \mathbb{P}^k$, this is a subspace of codimension $\binom{k+d}{k}$ in $\mathbb{P}^N$. We deduce that

$$\dim \Gamma \;=\; (k+1)(n-k) + N - \binom{k+d}{k};$$

in particular, if the inequality of the problem holds, then $\dim \Gamma < N$, so that Γ cannot dominate $\mathbb{P}^N$.

3. (DG) Let $\mathcal{H}^2 := \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Equip $\mathcal{H}^2$ with a metric

$$g_\alpha := \frac{dx^2 + dy^2}{y^\alpha}$$

where $\alpha \in \mathbb{R}$.

(a) Show that $(\mathcal{H}^2, g_\alpha)$ is incomplete if $\alpha \neq 2$.

(b) Identify $z = x + iy$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, consider the map $z \mapsto \frac{az+b}{cz+d}$. Show that this defines an isometry of $(\mathcal{H}^2, g_2)$.

(c) Show that $(\mathcal{H}^2, g_2)$ is complete. (Hint: Show that the isometry group acts transitively on the tangent space at each point.)

Solution: For the first part, consider the geodesic $\gamma(t)$ with $\gamma(0) = (0,1)$, and $\gamma'(0) = \frac{\partial}{\partial y}$. In order for $(\mathcal{H}^2, g_\alpha)$ to be complete, this geodesics must exist for all $t \in (-\infty, \infty)$. By symmetry, this geodesic must be given by

$$\mathbf{x}(t) = (0, y(t)).$$

Furthermore, $\mathbf{x}(t)$ must have constant speed, which we may as well take to be 1. Thus $\frac{(\dot{y})^2}{y^\alpha} = 1$, or in other words,

$$\dot{y} = y^{\alpha/2}.$$

If $\alpha \neq 2$, then the solution to this ODE is

$$y(t) = \left((1 - \frac{\alpha}{2})t + 1\right)^{1/(1-\frac{\alpha}{2})}$$

thus, this geodesics persists only as long as $(1-\frac{\alpha}{2})t+1 \geq 0$. This set is always bounded from one side. Note that when $\alpha = 2$, we get $\mathbf{x}(t) = (0, e^t)$, which exists for all time.

(b) To begin, note that $dz \otimes d\bar{z} = dx \otimes dx + dy \otimes dy$, so we can write the metric as

$$g_2 = \frac{4dz \otimes d\bar{z}}{|z - \bar{z}|^2}$$

Let $A \in SL(2, \mathbb{R})$, we compute

$$A^* dz = \frac{adz}{cz + d} - c\frac{(az + b)dz}{(cz + d)^2} = (ad - bc)\frac{dz}{(cz + d)^2} = \frac{dz}{(cz + d)^2}$$

and so $A^* d\bar{z} = \frac{d\bar{z}}{(c\bar{z}+d)^2}$. It remains to compute

$$A^* z - A^* \bar{z} = \frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d} = \frac{z - \bar{z}}{|cz + d|^2},$$

where we have used that $A \in SL(2, \mathbb{R})$. Putting everything together we get

$$A^* g_2 = \frac{4dz \otimes d\bar{z}}{|cz + d|^4} \cdot \frac{|cz + d|^4}{|z - \bar{z}|^2} = g_2,$$

and so $SL(2, \mathbb{R})$ acts by isometry.

(c) By the computation from part (a), we know that the geodesic– let's call it $\gamma_0(t)$– through the point $(0, 1)$ in the direction $(0, 1)$ exists for all time. Let $z = x + iy$ be any point in $\mathcal{H}^2$. By an isometry, we can map this point to $z = iy$. Without loss of generality, let us assume $y = 1$. It suffices to show that we can find $A \in SL(2, \mathbb{R})$ so that $A(i) = i$, and $A_* V = (0, 1)$, where V is any unit vector in the tangent space $T_i \mathcal{H}^2$, for then the geodesic through i with tangent vector V will be nothing but $A^{-1}(\gamma_0(t))$, and hence will exist for all time. First, observe that $A(i) = i$, if and only if $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Consider the rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A straightforward computation shows that, in complex coordinates,

$$A_* V = \frac{1}{(\cos\theta + i \sin\theta)^2} V = e^{-2\sqrt{-1}\theta} V,$$

that is, $A_* : T_i \mathcal{H}^2 \to T_i \mathcal{H}^2$ acts as a rotation. Since θ is arbitrary, and the rotations act transitively on S^2, we're done.

4. (RA)

(a) Let H be a Hilbert space, $K \subset H$ a closed subspace, and x a point in H. Show that there exists a unique y in K that minimizes the distance $\|x - y\|$ to x.

(b) Give an example to show that the conclusion can fail if H is an inner product space which is not complete.

Solution: (a): If $y, y' \in K$ both minimize distance to x, then by the parallelogram law:

$$\|x - \frac{y+y'}{2}\|^2 + \|\frac{y-y'}{2}\|^2 = \frac{1}{2}(\|x-y\|^2 + \|x-y'\|^2) = \|x-y\|^2$$

But $\frac{y+y'}{2}$ cannot be closer to x than y, by assumption, so $y = y'$.

Let $C = \inf_{y \in K} \|x - y\|$, then $0 \leq C < \infty$ because K is non-empty. We can find a sequence $y_n \in K$ such that $\|x - y_n\| \to C$, which we want to show is Cauchy. The midpoints $\frac{y_n+y_m}{2}$ are in K by convexity, so $\|x - \frac{y_n+y_m}{2}\| \geq C$ and using the parallelogram law as above one sees that $\|y_n - y_m\| \to 0$ as $n, m \to \infty$. By completeness of H the sequence y_n converges to a limit y, which is in K, since K is closed. Finally, continuity of the norm implies that $\|x - y\| = C$.

(b): For example choose $H = C([0,1]) \subset L^2([0,1])$, K the subspace of functions with support contained in $[0, \frac{1}{2}]$, and and $x = 1$ the constant function.

If f_n is a sequence in K converging to $f \in H$ in L^2-norm, then

$$\int_{1/2}^{1} |f|^2 = 0$$

thus f vanishes on $[1/2, 1]$, showing that K is closed. The distance $\|x - y\|$ can be made arbitrarily close to $1/\sqrt{2}$ for $y \in K$ by approximating $\chi_{[0,1/2]}$ by continuous functions, but the infimum is not attained.

5. (A)

(a) Prove that there exists a unique (up to isomorphism) nonabelian group of order 21.

(b) Let G be this group. How many conjugacy classes does G have?

(c) What are the dimensions of the irreducible representations of G?

Solution: Let G be a group of order 21, and select elements g_3 and g_7 of orders 3 and 7 respectively. The subgroup generated by g_7 is normal — if it weren't, then g_7 and $x g_7 x^{-1}$ witnessing nonnormality would generate a group of order

49. In particular, we have $g_3 g_7 g_3^{-1} = g_7^j$ for some nonzero $j \in \mathbb{Z}/7$. Now we use the order of g_3:

$$\begin{aligned} g_7 &= g_3 g_3 g_3 \cdot g_7 \cdot g_3^{-1} g_3^{-1} g_3^{-1} \\ &= g_3 g_3 (g_7^j) g_3^{-1} g_3^{-1} \\ &= g_3 (g_7^{j^2}) g_3^{-1} \\ &= g_7^{j^3}, \end{aligned}$$

and hence $j^3 \equiv 1 \pmod 7$. This is nontrivially solved by $j = 2$ and $j = 4$, and these two cases coincide: if for instance $g_3 g_7 g_3^{-1} = g_7^2$, then by replacing the generator g_3 with g_3^2 we instead see

$$g_3^2 g_7 (g_3^2)^{-1} = g_3 g_7^2 g_3^{-1} = g_7^4.$$

We have the following conjugacy classes of elements:

- $\{e\}$ forms a class of its own.
- $\{g_7, g_7^4, g_7^2\}$ and $\{g_7^3, g_7^5, g_7^6\}$ form classes by our choice of j.
- Any element of order 3 generates a Sylow 3–subgroup, all of which are conjugate as subgroups. However, there cannot be an x with $x g_3 x^{-1} = g_3^2$, since G has only elements of odd order. Hence, there are two final conjugacy classes, each of size 7: those elements conjugate to g_3 and those conjugate to g_3^2.

These five conjugacy sets give rise to five irreducible representations, which must be of dimensions 1, 1, 1, 3, and 3 (since these square-sum to $|G| = 21$).

6. (CA) Find (with proof) all entire holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the conditions:

1. $f(z+1) = f(z)$ for all $z \in \mathbb{C}$; and
2. There exists M such that $|f(z)| \leq M \exp(10|z|)$ for all $z \in \mathbb{C}$.

Solution: The functions satisfying these conditions are precisely the $\mathbb{C}$-linear combinations of $e^{-2\pi i z}$, 1, and $e^{2\pi i z}$. Indeed such f is readily seen to satisfy the two conditions. Conversely (1) means that f descends to a function of $q := e^{2\pi i z} \in \mathbb{C}^*$, say $f(z) = F(q)$, and then by (2) there is some M' such that $|F(q)| \leq M' \max(|q|^{-5/\pi}, |q|^{5/\pi})$ for all q, whence qF and $q^{-1}F$ have removable singularities at $q = 0$ and $q = \infty$ respectively, etc.