QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 30, 2016 (Day 1)

1. (DG)

(a) Show that if V is a $\mathcal{C}^\infty$-vector bundle over a compact manifold X, then there exists a vector bundle W over X such that $V \oplus W$ is trivializable, i.e. isomorphic to a trivial bundle.

(b) Find a vector bundle W on S^2, the 2-sphere, such that $T^*S^2 \oplus W$ is trivializable.

2. (RA) Let (X, d) be a metric space. For any subset $A \subset X$, and any $\epsilon > 0$ we set

$$B_\epsilon(A) = \bigcup_{p \in A} B_\epsilon(p).$$

(This is the "ϵ-fattening" of A.) For Y, Z bounded subsets of X define the *Hausdorff distance* between Y and Z by

$$d_H(Y, Z) := \inf \{\epsilon > 0 \mid Y \subset B_\epsilon(Z), \quad Z \subset B_\epsilon(Y)\}.$$

Show that d_H defines a metric on the set $\tilde{X} := \{A \subset X \big| \, A \text{ is closed and bounded}\}$.

3. (AT) Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the n-torus. Prove that any path-connected covering space $Y \to T^n$ is homeomorphic to $T^m \times \mathbb{R}^{n-m}$, for some m.

4. (CA)

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Show that the image of f is dense in $\mathbb{C}$.

5. (A) Let $F \supset \mathbb{Q}$ be a splitting field for the polynomial $f = x^n - 1$.

(a) Let $A \subset F^\times = \{z \in F \mid z \neq 0\}$ be a finite (multiplicative) subgroup. Prove that A is cyclic.

(b) Prove that $G = \mathrm{Gal}(F/\mathbb{Q})$ is abelian.

6. (AG) Let C and $D \subset \mathbb{P}^2$ be two plane cubics (that is, curves of degree 3), intersecting transversely in 9 points $\{p_1, p_2, \ldots, p_9\}$. Show that $p_1, \ldots, p_6$ lie on a conic (that is, a curve of degree 2) if and only if p_7, p_8 and p_9 are colinear.

QUALIFYING EXAMINATION

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Wednesday August 31, 2016 (Day 2)

1. (A) Let R be a commutative ring with unit. If $I \subseteq R$ is a proper ideal, we define the *radical* of I to be

$$\sqrt{I} \;=\; \{a \in R \mid a^m \in I \quad \text{for some } m > 0\}.$$

Prove that

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \mathfrak{p}\text{ prime}}} \mathfrak{p}.$$

2. (DG) Let $c(s) = (r(s), z(s))$ be a curve in the (x, z)-plane which is parameterized by arc length s. We construct the corresponding rotational surface, S, with parametrization

$$\varphi : (s, \theta) \mapsto (r(s)\cos\theta, r(s)\sin\theta, z(s)).$$

Find an example of a curve c such that S has constant negative curvature -1.

3. (RA) Let $f \in L^2(0, \infty)$ and consider

$$F(z) = \int_0^\infty f(t)e^{2\pi i z t}dt$$

for z in the upper half-plane.

(a) Check that the above integral converges absolutely and uniformly in any region $\mathrm{Im}(z) \geq C > 0$.

(b) Show that

$$\sup_{y>0} \int_0^\infty |F(x + iy)|^2 dx = \|f\|^2_{L^2(0,\infty)}.$$

4. (CA) Given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, use contour integration to prove that each of the improper integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$ converges to $\sqrt{\pi/8}$.

5. (AT)

(a) Let $X = \mathbb{R}P^3 \times S^2$ and $Y = \mathbb{R}P^2 \times S^3$. Show that X and Y have the same homotopy groups but are not homotopy equivalent.

(b) Let $A = S^2 \times S^4$ and $B = \mathbb{C}P^3$. Show that A and B have the same singular homology groups with $\mathbb{Z}$-coefficients but are not homotopy equivalent.

6. (AG)

Let C be the smooth projective curve over $\mathbb{C}$ with affine equation $y^2 = f(x)$, where $f \in \mathbb{C}[x]$ is a square-free monic polynomial of degree $d = 2n$.

(a) Prove that the genus of C is $n - 1$.

(b) Write down an explicit basis for the space of global differentials on C.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 1, 2016 (Day 3)

1. (AT) Model S^{2n-1} as the unit sphere in $\mathbb{C}^n$, and consider the inclusions

$$\begin{array}{ccccccc} \cdots & \to & S^{2n-1} & \to & S^{2n+1} & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathbb{C}^n & \to & \mathbb{C}^{n+1} & \to & \cdots. \end{array}$$

Let S^∞ and $\mathbb{C}^\infty$ denote the union of these spaces, using these inclusions.

(a) Show that S^∞ is a contractible space.

(b) The group S^1 appears as the unit norm elements of $\mathbb{C}^\times$, which acts compatibly on the spaces $\mathbb{C}^n$ and S^{2n-1} in the systems above. Calculate *all* the homotopy groups of the homogeneous space S^∞/S^1.

2. (AG) Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. Show that if

$$\binom{k+d}{k} > (k+1)(n-k)$$

then X does not contain any k-plane $\Lambda \subset \mathbb{P}^n$.

3. (DG) Let $\mathcal{H}^2 := \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Equip $\mathcal{H}^2$ with a metric

$$g_\alpha := \frac{dx^2 + dy^2}{y^\alpha}$$

where $\alpha \in \mathbb{R}$.

(a) Show that $(\mathcal{H}^2, g_\alpha)$ is incomplete if $\alpha \neq 2$.

(b) Identify $z = x + iy$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, consider the map $z \mapsto \frac{az+b}{cz+d}$. Show that this defines an isometry of $(\mathcal{H}^2, g_2)$.

(c) Show that $(\mathcal{H}^2, g_2)$ is complete. (Hint: Show that the isometry group acts transitively on the tangent space at each point.)

4. (RA)

(a) Let H be a Hilbert space, $K \subset H$ a closed subspace, and x a point in H. Show that there exists a unique y in K that minimizes the distance $\|x - y\|$ to x.

(b) Give an example to show that the conclusion can fail if H is an inner product space which is not complete.

5. (A)

(a) Prove that there exists a unique (up to isomorphism) nonabelian group of order 21.

(b) Let G be this group. How many conjugacy classes does G have?

(c) What are the dimensions of the irreducible representations of G?

6. (CA) Find (with proof) all entire holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the conditions:

1. $f(z+1) = f(z)$ for all $z \in \mathbb{C}$; and
2. There exists M such that $|f(z)| \leq M \exp(10|z|)$ for all $z \in \mathbb{C}$.