QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2015 (Day 1)

1. (A) The integer 8871870642308873326043363 is the 13^{th} power of an integer n. Find n.

2. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree 4.

(a) Describe the canonical bundle of C in terms of line bundles on $\mathbb{P}^2$. What are the effective canonical divisors on C?

(b) What is the genus of C? Explain how you obtain this formula.

(c) Prove that C is not hyperelliptic.

3. (DG) Let M be a C^∞ manifold, TM its tangent bundle, and $T^{\mathbb{C}}M = \mathbb{C} \otimes_{\mathbb{R}} TM$ the complexified tangent bundle. An almost complex structure on M is a C^∞ bundle map $J : TM \to TM$ such that $J^2 = -1$.

(a) Show that an almost complex structure naturally determines, and is determined by, each of the following two structures:
i) the structure of a complex C^∞ vector bundle – i.e., with fibres that are complex vector spaces – on TM compatible with its real structure.
ii) a C^∞ direct sum decomposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ with $T^{0,1}M$ = complex conjugate of $T^{1,0}M$.

(b) Show that every almost complex manifold is orientable.

(c) If S is a C^∞, orientable, 2-dimensional, Riemannian manifold, construct a natural almost complex structure on S in terms of its Riemannian structure, but one that depends only on the underlying conformal structure of S.

(d) Does the almost complex structure constructed in (c) determine the conformal structure of S? You need NOT give a detailed answer to this question; a heuristic one- or two-sentence answer suffices.

4. (RA) In this problem V denotes a Banach space over $\mathbb{R}$ or $\mathbb{C}$.

(a) Show that any finite dimensional subspace $U_0 \subset V$ is closed in V.

(b) Now let $U_1 \subset V$ a closed subspace, and $U_2 \subset V$ a finite dimensional subspace. Show that $U_1 + U_2$ is closed in V.

5. (AT) Consider the following three topological spaces:

$$A = \mathbb{H}\mathrm{P}^3, \qquad B = S^4 \times S^8, \qquad C = S^4 \vee S^8 \vee S^{12}.$$

($\mathbb{H}\mathrm{P}^3$ denotes quaternionic projective 3-space.)

(a) Calculate the cohomology groups (with integer coefficients) of all three.

(b) Show that A and B are not homotopy equivalent.

(c) Show that C is not homotopy equivalent to any compact manifold.

6. (CA) Let $f(z)$ be a function which is analytic in the unit disc $D = \{|z| < 1\}$, and assume that $|f(z)| \leq 1$ in D. Also assume that $f(z)$ has at least two fixed points z_1 and z_2. Prove that $f(z) = z$ for all $z \in D$.

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Wednesday September 2, 2015 (Day 2)

1. (AT) Let $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ be n dimensional complex projective space.

(a) Show that every map $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point. (Hint: Use the ring structure on cohomology.)

(b) For every $n \geq 0$, give an example of a map $f : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ without any fixed points and describe its induced map on cohomology.

2. (A) Let A be a commutative ring with unit. Define what it means for A to be *Noetherian*. Prove that the ring of continuous functions $f : [0,1] \to \mathbb{R}$ (with pointwise addition and multiplication) is *not* Noetherian.

3. (CA) Let $S \subset \mathbb{C}$ be the open half-disc $\{x + iy : y > 0,\ x^2 + y^2 < 1\}$.

(a) Construct a surjective conformal mapping $f : S \to D$, where D is the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

(b) Construct a harmonic function $h : S \to \mathbb{R}$ such that:
- $h(x + iy) \to 0$ as $y \to 0$ from above, for all real x with $|x| < 1$, and
- $h(re^{i\theta}) \to 1$ as $r \to 1$ from below, for all real θ with $0 < |x| < \pi$.

4. (AG) Let Q be the complex quadric surface in $\mathbb{P}^3$ defined by the homogeneous equation $x_0x_3 - x_1x_2 = 0$.

(a) Show that Q is non-singular.

(b) Show that through each point of Q there are exactly two lines which lie on Q.

(c) Show that Q is rational, but not isomorphic to $\mathbb{P}^2$.

5. (DG) Let Ω be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by

$$\Omega = \frac{1}{x^2 + y^2 + z^2}(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy).$$

(a) Prove that Ω is closed.

(b) Let $f : \mathbb{R}^3 - \{0\} \to S^2$ be the map which sends (x, y, z) to $(\frac{1}{x^2+y^2+z^2})^{1/2}(x, y, z)$. Show that Ω is the pull-back via f of a 2-form on S^2.

(c) Prove that Ω is not exact.

6. (RA) Consider the linear ODE $f'' + P\,f' + Q\,f = 0$ on the interval $(a, b) \subset \mathbb{R}$, with P, Q denoting C^∞ real valued functions on (a, b). Recall the definition of the Wronskian $W(f_1, f_2) = f_1\,f_2' - f_1'\,f_2$ associated to any two solutions f_1, f_2 of this differential equation.

(a) Show that $W(f_1, f_2)$ either vanishes identically or is everywhere nonzero, depending on whether the two solutions f_1, f_2 are linearly dependent or not.

(b) Now suppose that f_1, f_2 are linearly independent, real valued solutions. Show that they have at most first order zeroes, and that the zeroes occur in an alternating fashion: between any two zeroes of one of the solutions there must be a zero of the other solution.

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Thursday September 3, 2015 (Day 3)

1. (DG) Consider the graph S of the function $F(x, y) = \cosh(x)\cos(y)$ in $\mathbb{R}^3$ and let

$$\Phi : \mathbb{R}^2 \to S \subset R^3$$

be its parametrization: $\Phi(x, y) = (x, y, \cosh(x)\cos(y))$.

(a) Write down the metric on $\mathbb{R}^2$ that is defined by the rule that the inner product of two vectors v and w at the point (x, y) is equal to the inner product of $\Phi_*(v)$ and $\Phi_*(w)$ at the point $\Phi(x, y)$ in $\mathbb{R}^3$.

(b) Define the Gaussian curvature of a general surface embedded in $\mathbb{R}^3$.

(c) Compute the Gaussian curvature of the surface S at the point $(0, 0, 1)$.

2. (RA) Let $f(x) \in C(\mathbb{R}/\mathbb{Z})$ be a continuous $\mathbb{C}$-valued function on $\mathbb{R}/\mathbb{Z}$ and let $\sum_{n=-\infty}^{\infty} a_n\, e^{2\pi i n x}$ be its Fourier series.

(a) Show that f is C^∞ if and only if $|a_n| = O(|n|^{-k})$ for all $k \in \mathbb{N}$.

(b) Prove that a sequence of functions $\{ f_n\}_{n\geq 1}$ in $C^\infty(\mathbb{R}/\mathbb{Z})$ converges in the C^∞ topology (uniform convergence of functions and their derivatives of all orders) if and only if the sequences of k-th derivatives $\{ f_n^{(k)}\}_{n\geq 1}$, for all $k \geq 0$, converge in the L^2-norm on $\mathbb{R}/\mathbb{Z}$.

3. (AG) Let C be a smooth projective curve over $\mathbb{C}$ and $\omega_C^{\otimes 2}$ the square of its canonical sheaf.

(a) What is the dimension of the space of sections $\Gamma(C, \omega_C^{\otimes 2})$?

(b) Suppose $g(C) \geq 2$ and $s \in \Gamma(C, \omega_C^{\otimes 2})$ is a section with simple zeros. Compute the genus of $\Sigma = \{x^2 = s\}$ in the total space of the line bundle ω_C, i.e. the curve defined by the 2-valued 1-form $\sqrt{s}$.

4. (AT) Show (using the theory of covering spaces) that every subgroup of a free group is free.

5. (CA)

(a) Define Euler's Gamma function $\Gamma(z)$ in the half plane $Re(z) > 0$ and show that it is holomorphic in this half plane.

(b) Show that $\Gamma(z)$ has a meromorphic continuation to the entire complex plane.

(c) Where are the poles of $\Gamma(z)$?

(d) Show that these poles are all simple and determine the residue at each pole.

6. (A) Let G be a finite group, and $\rho : G \to GL_n(\mathbb{C})$ a linear representation. Then for each integer $i \geq 0$ there is a representation $\wedge^i \rho$ of G on the exterior power $\wedge^i(\mathbb{C}^n)$. Let W_i be the subspace $(\wedge^i(\mathbb{C}^n))^G$ of $\wedge^i(\mathbb{C}^n)$ fixed under this action of G.

Prove that $\dim W_i$ is the T^i coefficient of the polynomial

$$\frac{1}{|G|} \sum_{g \in G} \det(\mathbf{1}_n + T\rho(g))$$

where $\mathbf{1}_n$ is the $n \times n$ identity matrix.