Solutions of Qualifying Exams I, 2013 Fall

1. (Algebra) Consider the algebra $M_2(k)$ of 2×2 matrices over a field k. Recall that an *idempotent* in an algebra is an element e such that $e^2 = e$.

(a) Show that an idempotent $e \in M_2(k)$ different from 0 and 1 is conjugate to

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $GL_2(k)$.

(b) Find the stabilizer in $GL_2(k)$ of $e_1 \in M_2(k)$ under the conjugation action.

(c) In case $k = \mathbb{F}_p$ is the prime field with p elements, compute the number of idempotents in $M_2(k)$. (Count 0 and 1 in.)

Solution. (a) Since $e \neq 0, 1$, the image and the kernel of e are both one-dimensional. Let v_1 be a nonzero element in the image, so $v_1 = e(v_0)$ for some $v_0 \in k^{\oplus 2}$. Then

$$e(v_1) = e(e(v_0)) = e^2(v_0) = e(v_0) = v_1.$$

Pick a nonzero element v_2 in the kernel of e, and we get a basis of $k^{\oplus 2}$ in which e takes the form e_1.

(b) For a general element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to be in the stabilizer, it must satisfy $ge_1 = e_1 g$. Writing the equation in four entries out, one sees that it means $b = c = 0$ (and a, d arbitrary). So the centralizer is the subgroup of diagonal matrices.

(c) By (a) and (b), the set of rank 1 idempotents is in bijection with $GL_2(\mathbb{F}_p)/T(\mathbb{F}_p)$, whose cardinality is

$$\frac{(p^2-1)(p^2-p)}{(p-1)(p-1)} = (p+1)p.$$

So the total number of idempotents is equal to $p^2 + p + 2$.

2. (Algebraic Geometry) (a) Find an everywhere regular differential n-form on the affine n-space $\mathbb{A}^n$.

(b) Prove that the canonical bundle of the projective n-dimensional space $\mathbb{P}^n$ is $\mathcal{O}(-n-1)$.

Solution *(Sketch).* Part (a) is really a hint for Part (b). Letting $x_1, x_2, \ldots, x_n$ be affine ($\mathbb{A}^n$) coordinates, put $\omega := dx_1 \wedge dx_2 \cdots \wedge dx_n$ giving (a) . Denoting the corresponding homogenous $\mathbb{P}^n$ coordinates $t_0, t_1, \ldots, t_n$, with $x_i := t_i/t_0$ for $i = 1, 2, \ldots, n$ extend ω to $\mathbb{P}^n$ writing $dx_i = dt_i/t_0 - t_i/t_0^2 dt_0$ and wedging to discover that the divisor of poles of ω is $(n+1)H$ where H is the hyperplane at infinity ($t_0 = 0$) and then conclude (appropriately).

3. (Complex Analysis) *(Bol's Theorem of 1949).* Let $\tilde{W}$ be a domain in $\mathbb{C}$ and W be a relatively compact nonempty subdomain of $\tilde{W}$. Let $\varepsilon > 0$ and G_ε be the set of all $(a, b, c, d) \in \mathbb{C}$ such that $\max(|a-1|, |b|, |c|, |d-1|) < \varepsilon$. Assume that $cz + d \neq 0$ and $\frac{az+b}{cz+d} \in \tilde{W}$ for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Let $m \geq 2$ be an integer. Prove that there exists a positive integer ℓ (depending on m) with the property that for any holomorphic function φ on $\tilde{W}$ such that

$$\varphi(z) = \varphi\left(\frac{az+b}{cz+d}\right) \frac{(cz+d)^{2m}}{(ad-bc)^m}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$, the ℓ-th derivative $\psi(z) = \varphi^{(\ell)}(z)$ of $\varphi(z)$ on $\tilde{W}$ satisfies the equation

$$\psi(z) = \psi\left(\frac{az+b}{cz+d}\right) \frac{(ad-bc)^{\ell-m}}{(cz+d)^{2(\ell-m)}}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Express ℓ in terms of m.

Hint: Use Cauchy's integral formula for derivatives.

Solution. Let

$$Az = \frac{az+b}{cz+d}$$

for $A \in G_\varepsilon$. We take a positive integer ℓ which we will determine later as a function of n. We use Cauchy's integral formula for derivatives to take the ℓ-th derivative $\psi(z)$ of $\varphi(z)$. For $z \in \tilde{W}$ we use $U(z)$ to denote an open neighborhood of z in $\tilde{W}$ and use $\partial U(z)$ to denote its boundary. The ℓ-th derivative ψ of φ at $z \in \tilde{W}$ is given by the formula

$$\psi(z) = \frac{\ell!}{2\pi\sqrt{-1}} \int_{\zeta \in \partial U(z)} \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^{\ell+1}}$$

and

$$\psi(Az) = \frac{\ell!}{2\pi\sqrt{-1}} \int_{\zeta \in \partial U(Az)} \frac{\varphi(\zeta) d\zeta}{(\zeta - Az)^{\ell+1}} \quad \text{when } Az \in \tilde{W}.$$

It follows from

$$\zeta \in U(z) \Longleftrightarrow A\zeta \in U(Az),$$
$$\zeta \in \partial U(z) \Longleftrightarrow A\zeta \in \partial U(Az),$$

with the change of variable $\zeta \mapsto A\zeta$, that

$$\int_{\zeta \in \partial U(Az)} \frac{\varphi(\zeta) d\zeta}{(\zeta - Az)^{\ell+1}} = \int_{A\zeta \in \partial U(Az)} \frac{\varphi(A\zeta) d(A\zeta)}{(A\zeta - Az)^{\ell+1}}.$$

From the following straightforward direct computation of the discrete version of the formula for the derivative of fractional linear transformation

$$\begin{aligned}
A\zeta - Az &= \frac{a\zeta + b}{c\zeta + d} - \frac{az + b}{cz + d} \\
&= \frac{(a\zeta + b)(cz + d) - (az + b)(c\zeta + d)}{(c\zeta + d)(cz + d)} \\
&= \frac{(ac\zeta z + bcz + ad\zeta + bd) - (ac\zeta z + adz + bc\zeta + bd)}{(c\zeta + d)(cz + d)} \\
&= \frac{(ad - bc)(\zeta - z)}{(c\zeta + d)(cz + d)}
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_{A\zeta \in \partial U(Az)} \frac{\varphi(A\zeta) d(A\zeta)}{(A\zeta - Az)^{\ell+1}} &= \int_{\zeta \in \partial U(z)} \frac{\varphi\left(\frac{a\zeta + b}{c\zeta + d}\right) \frac{ad - bc}{(c\zeta + d)^2} d\zeta}{\frac{(ad - bc)^{\ell+1}(\zeta - z)^{\ell+1}}{(c\zeta + d)^{\ell+1}(cz + d)^{\ell+1}}} \\
&= \int_{\zeta \in \partial U(z)} \frac{\varphi(\zeta) \frac{(ad - bc)^m}{(c\zeta + d)^{2m}} \frac{ad - bc}{(c\zeta + d)^2} d\zeta}{\frac{(ad - bc)^{\ell+1}(\zeta - z)^{\ell+1}}{(c\zeta + d)^{\ell+1}(cz + d)^{\ell+1}}} \\
&= \frac{(cz + d)^{\ell+1}}{(ad - bc)^{\ell-m}} \int_{\zeta \in \partial U(z)} \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^{\ell+1}} (c\zeta + d)^{\ell-1-2m}.
\end{aligned}$$

The extra factor $(c\zeta + d)^{\ell-1-2m}$ inside the integrand on the extreme right-hand side becomes 1 and can be dropped if $\ell - 1 - 2m = 0$, that is, if $\ell = 2m + 1$. Thus, if $\ell = 2m + 1$, then

$$\psi(Az) = \frac{(cz + d)^{\ell+1}}{(ad - bc)^{\ell-m}}\psi(z).$$

That is,

$$\psi(z) = \psi\left(\frac{az + b}{cz + d}\right)\frac{(ad - bc)^{\ell-m}}{(cz + d)^{2(\ell-m)}},$$

because $\ell = 2m + 1$ implies $\ell + 1 = 2(\ell - m)$.

4. (Algebraic Topology) (a) Show that the Euler characteristic of any contractible space is 1.

(b) Let B be a connected CW complex made of finitely many cells so that its Euler characteristic is defined. Let $E \to B$ be a covering map whose fibers are discrete, finite sets of cardinality N. Show the Euler characteristic of E is N times the Euler characteristic of B.

(c) Let G be a finite group with cardinality > 2. Show that BG (the classifying space of G) cannot have homology groups whose direct sum has finite rank.

Solution. (a) The homology of a point with coefficients in a field k is $H_0 = k$, $H_i = 0$ for $i > 0$. Hence its Euler characteristic is $\sum(-1)^i \dim H_i = 1$. All contractible spaces are homotopy equivalent so their Euler characteristic is that of the point.

(b) For any open cover $\{U_i\}$, we know that the chain complex of singular chains living in U_i for some i has equivalent homology to the chain complex of all chains. Taking the cover of B by trivializing neighborhoods U_i, the chain complex of chains living in U_i receives a map from chains in E living in $\pi^{-1}(U_i)$. The latter is simply $|G|$ direct sums of the former, and the chain map between them is the "add every component" map. This shows the ranks of homology of E is N times the rank of homology of B.

(c) Strictly speaking, this problem cannot be solved based on easy machinery (as far as I know). A much more reasonable problem would be: Prove BG is not homotopy equivalent to anything made up of only finitely many cells. I did not take off points for people not distinguishing between this condition, and the condition stated in the problem itself. We know $BG = EG/G$, but EG is contractible. So $\chi(EG) = 1$. If BG has finite homology, $\chi(BG) = 1/|G|$, which cannot be an integer unless $|G| = 1$.

5. (Differential Geometry) Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane. Let g be the Riemannian metric on H given by

$$g = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}.$$

(H, g) is known as the half-plane model of the hyperbolic plane.

(a) Let $\gamma(\theta) = (\cos\theta, \sin\theta)$ and $\eta(\theta) = (\cos\theta + 1, \sin\theta)$ for $\theta \in (0, \pi)$ be two paths in H. Compute the angle A at their intersection point shown in Figure 1, measured by the metric g.

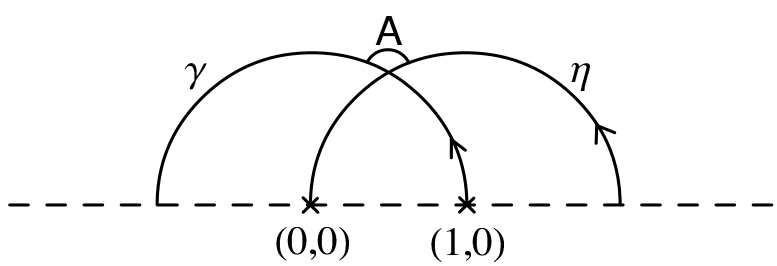

Figure 1: Angle A between the two curves γ and η in the upper half plane H.

(b) By computing the Levi-Civita connection

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{2} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

of g or otherwise (where $(x_1, x_2) = (x, y)$), show that the path γ, *after arc-length reparametrization*, is a geodesic with respect to the metric g.

Solution. (a) The intersection point is $(1/2, \sqrt{3}/2)$: solving for

$$\gamma(\theta) = (\cos\theta, \sin\theta) = (\cos\phi + 1, \sin\phi) = \eta(\phi)$$

we obtain $\theta = \pi/3$, $\phi = 2\pi/3$.

The angle A satisfies

$$\begin{aligned}
\cos A &= \frac{\langle \gamma'(\pi/3), -\eta'(2\pi/3)\rangle_g}{||\gamma'(\pi/3)||_g \; ||-\eta'(2\pi/3)||_g} \\
&= \frac{\langle (-\sqrt{3}/2, 1/2), (\sqrt{3}/2, 1/2)\rangle_g}{||(-\sqrt{3}/2, 1/2)||_g \; ||(\sqrt{3}/2, 1/2)||_g} \\
&= \frac{-\frac{1}{2}\frac{1}{y^2}}{\frac{1}{y^2}} \\
&= -\frac{1}{2}
\end{aligned}$$

and so $A = 2\pi/3$.

(b) Using the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{jl,j} - g_{jk,l})$$

one obtains

$$\Gamma^i_{jk} = \frac{-1}{y}(\delta_{ij}\delta_{k,2} + \delta_{ki}\delta_{j,2} - \delta_{jk}\delta_{i,2}).$$

After arc-length reparametrization, the tangent vectors of the path are

$$v(\theta) = \frac{\gamma'(\theta)}{||\gamma'(\theta)||_g} = (-\sin^2\theta, \sin\theta\cos\theta).$$

Then

$$\nabla_{v(\theta)} v(\theta) = v'(\theta) + \begin{pmatrix} \Gamma^1_1 & \Gamma^1_2 \\ \Gamma^2_1 & \Gamma^2_2 \end{pmatrix} \cdot v(\theta)$$

where

$$\begin{aligned}
\Gamma^1_1 &= (-\sin\theta)\Gamma^1_{11} + (\cos\theta)\Gamma^1_{21} = -\cot\theta; \\
\Gamma^1_2 &= (-\sin\theta)\Gamma^1_{12} + (\cos\theta)\Gamma^1_{22} = 1; \\
\Gamma^2_1 &= (-\sin\theta)\Gamma^2_{11} + (\cos\theta)\Gamma^2_{21} = -1; \\
\Gamma^2_2 &= (-\sin\theta)\Gamma^2_{12} + (\cos\theta)\Gamma^2_{22} = -\cot\theta.
\end{aligned}$$

Thus one has $\nabla_{v(\theta)} v(\theta) = 0$.

6. (Real Analysis) For any positive integer n let M_n be a positive number such that the series $\sum_{n=1}^{\infty} M_n$ of positive numbers is convergent and its limit is M. Let $a < b$ be real numbers and $f_n(x)$ be a real-valued continuous function on $[a, b]$ for any positive integer n such that its derivative $f_n'(x)$ exists for every $a < x < b$ with $|f_n'(x)| \leq M_n$ for $a < x < b$. Assume that the series $\sum_{n=1}^{\infty} f_n(a)$ of real numbers converges. Prove that

(a) the series $\sum_{n=1}^{\infty} f_n(x)$ converges to some real-valued function $f(x)$ for every $a \leq x \leq b$,

(b) $f'(x)$ exists for every $a < x < b$, and

(c) $|f'(x)| \leq M$ for $a < x < b$.

Hint for (b): For fixed $x \in (a, b)$ consider the series of functions

$$\sum_{n=1}^{\infty} \frac{f_n(y) - f_n(x)}{y - x}$$

of the variable y and its uniform convergence.

Solution. **(a)** Fix $x \in (a, b]$. For $q > p \geq 1$, by the Mean Value Theorem applied to the function $\sum_{n=p}^{q} f_n$ on $[a, x]$ we can find $a < \xi_{p,q} < x$ such that

$$\sum_{n=p}^{q} f_n(x) - \sum_{n=p}^{q} f_n(a) = (x - a) \sum_{n=p}^{q} f_n'\left(\xi_{p,q}\right),$$

which implies that

$$\begin{aligned} \left|\sum_{n=p}^{q} f_n(x)\right| &\leq \left|\sum_{n=p}^{q} f_n(a)\right| + (x - a) \left|\sum_{n=p}^{q} f_n'\left(\xi_{p,q}\right)\right| \\ &\leq \left|\sum_{n=p}^{q} f_n(a)\right| + (x - a) \sum_{n=p}^{q} M_n. \end{aligned}$$

Since both series $\sum_{n=1}^{\infty} f_n(a)$ and $\sum_{n=1}^{\infty} M_n$ are convergent and therefore Cauchy, for any $\varepsilon > 0$ we can find a positive integer N_1 such that

$$\left|\sum_{n=p}^{q} f_n(a)\right| < \frac{\varepsilon}{2}$$

for $q > p \geq N_1$ and we can find a positive integer N_2 such that

$$\left| \sum_{n=p}^{q} M_n \right| < \frac{\varepsilon}{2(x-a)}$$

for $q > p \geq N_2$. Thus for $n \geq \max(N_1, N_2)$ we have

$$\left| \sum_{n=p}^{q} f_n(x) \right| < \varepsilon$$

and the series $\sum_{n=1}^{\infty} f_n(x)$ is Cauchy. Hence the series $\sum_{n=1}^{\infty} f_n(x)$ converges to some real-valued function $f(x)$ for every $a \leq x \leq b$.

(b) Before the proof of the statement in (b), we would like to state that the uniform limit of continuous functions is continuous. That is, if $h_n(x)$ is a sequence of functions on a metric space E which converges to a function $h(x)$ on E uniformly on E and if for some $x_0 \in E$ and for every n the function $h_n(x)$ is continuous at $x = x_0$, then $h(x)$ is continuous at x_0. This results from the so-called 3ε argument as follows. Given any $\varepsilon > 0$. The uniform convergence of $h_n \to h$ on E implies that there exists some positive integer N such that $|h_N(x) - h(x)| < \varepsilon$ for all $x \in E$. Since h_N is continuous at $x = x_0$, there exists some $\delta > 0$ such that $|h_N(x) - h_N(x_0)| < \varepsilon$ for $d_E(x, x_0) < \delta$ (where $d_E(\cdot, \cdot)$ is the metric of the metric space E). Thus for $d_E(x, x_0) < \delta$ we have

$$|h(x) - h(x_0)| \leq |h(x) - h_N(x)| + |h_N(x) - h_N(x_0)| + |h_N(x_0) - h(x_0)| < 3\varepsilon,$$

which implies the continuity of h at $x = x_0$.

We now prove the statement in (b). Take $x_0 \in (a, b)$. We introduce the function $g_{n,x_0}(x)$ on $[a, b]$ which is defined by

$$\begin{cases} g_{n,x_0}(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0} & \text{for } x \neq x_0 \\ g_{n,x_0}(x_0) = f_n'(x_0). \end{cases}$$

It follows from the continuity of f_n on $[a, b]$ and the existence of $f_n'(x_0)$ that g_{n,x_0} is a continuous function on $[a, b]$.

When $x \in [a, b]$ with $x \neq x_0$, by the Mean Value Theorem

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} = f_n'(\xi_x)$$

for some ξ_x strictly between x_0 and x and as a consequence

$$|g_{n,x_0}(x)| = |f_n'(x_0)| \leq M_n.$$

When $x = x_0$,

$$|g_{n,x_0}(x)| = |f_n'(x_0)| \leq M_n.$$

Thus $|g_{n,x_0}(x)| \leq M_n$ for $x \in [a, b]$. From $\sum_{n=1}^{\infty} M_n \leq M < \infty$ it follows that the series $\sum_{n=1}^{\infty} g_{n,x_0}$ is uniformly convergent on $[a, b]$. It follows that the uniform limit $\sum_{n=1}^{\infty} g_{n,x_0}$ is a continuous function on $[a, b]$ by the 3ε argument given above. For $x \neq x_0$

$$\sum_{n=1}^{\infty} g_{n,x_0}(x) = \sum_{n=1}^{\infty} \frac{f_n(x) - f_n(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}.$$

The continuity of $\sum_{n=1}^{\infty} g_{n,x_0}(x)$ at $x = x_0$ means that the limit of

$$\frac{f(x) - f(x_0)}{x - x_0}$$

exists as $x \to x_0$, which implies that $f'(x_0)$ exists and is equal to

$$\sum_{n=1}^{\infty} g_{n,x_0}(x_0) = \sum_{n=1}^{\infty} f_n'(x_0).$$

(c) From

$$f'(x_0) = \sum_{n=1}^{\infty} g_{n,x_0}(x_0) = \sum_{n=1}^{\infty} f_n'(x_0)$$

and $|f_n'(x_0)| \leq M_n$, it follows that

$$|f'(x_0)| \leq \sum_{n=1}^{\infty} M_n = M.$$

Solutions of Qualifying Exams II, 2013 Fall

1. (ALGEBRA) Find all the field automorphisms of the real numbers $\mathbb{R}$.
Hint: Show that any automorphism maps a positive number to a positive number, and deduce from this that it is continuous.

Solution. If $t > 0$, there exists an element $s \neq 0$ such that $t = s^2$. If φ is any field automorphism of $\mathbb{R}$, then

$$\varphi(t) = \varphi(s^2) = (\varphi(s))^2 > 0.$$

It follows that φ preserves the order on $\mathbb{R}$: If $t < t'$, then

$$\varphi(t') = \varphi(t + (t' - t)) = \varphi(t) + \varphi(t' - t) > \varphi(t).$$

Any real number α is determined by the set (Dedekind's cut) of rational numbers that are less than α, and any field automorphism fixes each rational number. Therefore φ is the identity automorphism.

2. (ALGEBRAIC GEOMETRY) What is the maximum number of ramification points that a mapping of finite degree from one smooth projective curve over $\mathbb{C}$ of genus 1 to another (smooth projective curve of genus 1) can have? Give an explanation for your answer.

Solution *(Sketch).* By the Riemann-Hurwitz formula, if we have a mapping f of finite degree d from one smooth projective (irreducible, say) curve onto another the Euler characteristic of the source curve is d times the Euler characteristic of the target *minus* a certain nonnegative number e, and moreover e is zero if and only if the mapping is unramified. Now compute: the Euler characterstic of our source and target curves is, by hypothesis, 0 and so this e is zero, and therefore the mapping is unramified.

3. (COMPLEX ANALYSIS) Let ω and η be two complex numbers such that $\mathrm{Im}\left(\frac{\omega}{\eta}\right) > 0$. Let G be the closed parallelogram consisting of all $z \in \mathbb{C}$ such that $z = \lambda\omega + \rho\eta$ for some $0 \leq \lambda, \rho \leq 1$. Let ∂G be the boundary of G and Let $G^0 = G - \partial G$ be the interior of G. Let $P_1, \cdots, P_k, Q_1, \cdots, Q_\ell$ be points in G^0 and let $m_1, \cdots, m_k, n_1, \cdots, n_\ell$ be positive integers. Let f be a function on G such that

$$\frac{f(z)\prod_{j=1}^{\ell}(z - Q_j)^{n_j}}{\prod_{p=1}^{k}(z - P_p)^{m_p}}$$

is continuous and nowhere zero on G and is holomorphic on G^0. Let $\varphi(z)$ and $\psi(z)$ be two polynomials on $\mathbb{C}$. Assume that $f(z+\omega) = e^{\varphi(z)} f(z)$ if both z and $z+\omega$ are in G. Assume also that $f(z+\eta) = e^{\psi(z)} f(z)$ if both z and $z+\eta$ are in G. Express $\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j$ in terms of ω and η and the coefficients of $\varphi(z)$ and $\psi(z)$.

Solution. Let $A = 0$, $B = \eta$, $C = \eta + \omega$, and $D = \omega$. Since $\operatorname{Im}\left(\frac{\omega}{\eta}\right) > 0$, it follows that going from A to B, to C, to D and then back to A is in the counterclockwise direction. By the argument principle

$$\begin{aligned}
\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j &= \frac{1}{2\pi\sqrt{-1}} \oint_{\partial G} d\log f \\
&= \frac{1}{2\pi\sqrt{-1}} \left(\int_{\overrightarrow{AB}} d\log f + \int_{\overrightarrow{BC}} d\log f + \int_{\overrightarrow{CD}} d\log f + \int_{\overrightarrow{DA}} d\log f \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \left(\int_{\overrightarrow{AB}} d\log f - \int_{\overrightarrow{CD}} d\log f + \int_{\overrightarrow{BC}} d\log f - \int_{\overrightarrow{AD}} d\log f \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \left(- \int_{\overrightarrow{AB}} d\varphi(z) + \int_{\overrightarrow{AD}} d\psi(z) \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \left(-\varphi(\eta) + \varphi(0) + \psi(\omega) - \psi(0) \right).
\end{aligned}$$

Thus, the answer is

$$\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j = \frac{1}{2\pi\sqrt{-1}} \left(-\varphi(\eta) + \varphi(0) + \psi(\omega) - \psi(0) \right).$$

4. (Algebraic Topology) (a) Fix a basis for H_1 of the two-torus (with integer coefficients). Show that for every element $x \in SL(2,\mathbb{Z})$, there is an automorphism of the two-torus such that the induced map on H_1 acts by x. *Hint:* $SL(2,\mathbb{Z})$ also acts on the universal cover of the torus.

(b) Fix an embedding $j : D^2 \times S^1 \to S^3$. Remove its interior from S^3 to obtain a manifold X with boundary T^2. Let f be an automorphism of the two-torus and consider the glued space

$$X_f := (D^2 \times S^1) \cup_f X.$$

If X is homotopy equivalent to $D^2 \times S^1$, compute the homology groups of X_f.

Solution. (a) Given $g \in SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ let $x : \mathbb{R}^2 \to \mathbb{R}^2$ be the induced action. Since g is in $SL(2,\mathbb{Z})$ it respects the relationship of whether two vectors in $\mathbb{R}^2$ differ by integer coordinates. So the map on the torus $[(x_1, x_2)] \mapsto [g(x_1, x_2)]$ is well-defined. This clearly sends a homology generating pair given by the curves $(x_1, 0)$ and $(0, x_2)$ to the expected images via g.

(b) There is an ambiguity in the problem about how f glues X and $D^2 \times S^1$ together; so I gave full credit regardless of whether you identified this ambiguity or not. Note $X_f = (D^2 \times S^1) \cup_{S^1 \times S^1} X$. Write $U = D^2 \times S^1$ and $V = X$. The Mayer-Vietoris sequence gives

$$\begin{array}{l}
\longrightarrow H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(U \cup V) \\[2em]
\longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(U \cup V) \\[2em]
\longrightarrow H_2(U \cap V) \longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(U \cup V) \\[2em]
\longrightarrow H_3(U \cap V) \longrightarrow H_3(U) \oplus H_3(V) \longrightarrow H_3(U \cup V)
\end{array}$$

but because we know the homology of $D^2 \times S^1 \simeq S^1$ and $S^1 \times S^1$, we can fill in various groups in the long exact sequence:

$$\begin{array}{l}
\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(U \cup V) \\
\quad \nwarrow \\
\mathbb{Z}^2 \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(U \cup V) \\
\quad \nwarrow \\
\mathbb{Z} \longrightarrow 0 \oplus 0 \longrightarrow H_2(U \cup V) \\
\quad \nwarrow \\
0 \longrightarrow 0 \oplus 0 \longrightarrow H_3(U \cup V)
\end{array}$$

Since g is an isomorphism, we know H_1 must inject into $\mathbb{Z}$, but the inclusion map $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ is an injection, so $H_1(U \cup V) = 0$.

We know H_0 is either equal to $\mathbb{Z}$ from the long exact sequence above, or by observing that X_f is path-connected.

If f induces an isomorphism, we see H_2 must be zero; this was the intent of the problem, but you can get a different answer based on how you interpreted the "gluing" by f.

Finally, H_3 is also isomorphic to $\mathbb{Z}$ by the exactness of the above sequence.

5. (Differential Geometry) Let $M = U(n)/O(n)$ for $n \geq 1$, where $U(n)$ is the group of $n\times n$ unitary matrices and $O(n)$ is the group of $n\times n$ orthogonal matrices. M is a real manifold called the *Lagrangian Grassmannian.*

(a) Compute and state the dimension of M.

(b) Construct a Riemannian metric which is invariant under the left action of $U(n)$ on M.

(c) Let ∇ be the corresponding Levi-Civita connection on the tangent bundle TM, and X, Y, Z be any $U(n)$-invariant vector fields on M. Using the given identity (which you are not required to prove)

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

show that the Riemannian curvature tensor R of ∇ satisfies the formula

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]].$$

Solution. (a)

$$T_{[I]}M \cong \mathfrak{u}(n)/\mathfrak{o}(n) \cong \mathrm{Sym}^2(\mathbb{R}^n)$$

where $\mathrm{Sym}^2(\mathbb{R}^n)$ denotes the space of real $n \times n$ symmetric matrices. Thus

$$\dim M = \frac{n(n+1)}{2}.$$

(b) Define a metric on $\mathrm{Sym}^2(\mathbb{R}^n)$ by

$$\langle A, B\rangle = \mathrm{tr}(AB^t) = \mathrm{tr}(AB).$$

$g \in O(n)$ acts on $T_{[I]}M \cong \mathrm{Sym}^2(\mathbb{R}^n)$ by $g \cdot A = gAg^{-1}$. Then

$$\langle g \cdot A, g \cdot B\rangle = \mathrm{tr}(g \cdot ABg^{-1}) = \langle A, B\rangle.$$

Hence this metric is invariant under the action of $O(n)$. By translating the metric to tangent spaces at other points by the action of $U(n)$, this gives a well-defined invariant metric on $U(n)/O(n)$.

(c)

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Then

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}\left([X,[Y,Z]] - [Y,[X,Z]]\right) - \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[Z,[X,Y]] \end{aligned}$$

where the last equality follows from Jacobi identity.

6. (Real Analysis) Show that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ whose set of continuous points is precisely the set $\mathbb{Q}$ of all rational numbers.

Solution. For fixed $\delta > 0$ let $C(\delta)$ be the set of points $x \in \mathbb{R}$ such that for some $\varepsilon > 0$ we have $|f(x') - f(x'')| < \delta$ for all $x', x'' \in (x - \varepsilon, x + \varepsilon)$. Clearly $C(\delta)$ is open since for every $x \in C(\delta)$, we have $(x - \varepsilon, x + \varepsilon) \subset C(\delta)$. Now let C denote the set of continuous points of f. From the definitions, we have that

$$C = \bigcap_{n=1}^{\infty} C(1/n).$$

Now suppose that $C = \mathbb{Q}$. Then

$$\mathbb{R} - \mathbb{Q} = \bigcup_{n=1}^{\infty} X_n,$$

where $X_n = \mathbb{R} - C(1/n)$. Since $C(1/n)$ is open, X_n is closed. Also $\mathbb{Q}$ is countable, say $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Let $Y_n = \{q_n\}$. Then

$$\mathbb{R} = \left(\bigcup_{n=1}^{\infty} X_n\right) \cup \left(\bigcup_{n=1}^{\infty} Y_n\right),$$

i.e. we have written $\mathbb{R}$ as a countable union of closed sets. Then by Baire's theorem, some X_n or Y_n has nonempty interior. Clearly it cannot be one of the Y_n. So there exists X_n containing an interval (a, b). But this is impossible because $X_n \subset \mathbb{R} - \mathbb{Q}$ and every interval contains a rational number. Thus, we obtain a contradiction, which shows that $C \neq \mathbb{Q}$.

Solutions of Qualifying Exams III, 2013 Fall

1. (Algebra) Consider the function fields $K = \mathbb{C}(x)$ and $L = \mathbb{C}(y)$ of one variable, and regard L as a finite extension of K via the $\mathbb{C}$-algebra inclusion

$$x \mapsto \frac{-(y^5-1)^2}{4y^5}$$

Show that the extension L/K is Galois and determine its Galois group.

Solution. Consider the intermediate extension $K' = \mathbb{C}(y^5)$. Then clearly $[L : K'] = 5$ and $[K' : K] = 2$, therefore $[L : K] = 10$.

Thus, to prove that L/K is Galois it is enough to find 10 field automorphisms of L over K. Choose a primitive 5th root of 1, say $\zeta = e^{2\pi i/5}$. For $i \in \mathbb{Z}/5$ and $s \in \{\pm 1\}$, the $\mathbb{C}$-algebra automorphism $\sigma_{i,s}$ of L defined by

$$y \mapsto \zeta^i y^s$$

leaves x, hence K, fixed.

There can be many ways to determine the group, here's one.

Looking at the law of composition of these automorphisms, one sees that the subgroup $\mathrm{Gal}(L/K') \simeq \mathbb{Z}/5$, (which is necessarily normal, being of index 2) is not central, for conjugation by $\sigma_{0,-1}$ acts as -1 on it.

So the group is the dihedral group of 10 elements.

2. (Algebraic Geometry) Is every smooth projective curve of genus 0 defined over the field of complex numbers isomorphic to a conic in the projective plane? Give an explanation for your answer.

Solution *(Sketch)*. Yes. Apply the Riemann-Roch theorem which guarantees the existence of a nonconstant meromorphic function with a simple pole at exactly one point. Argue that this meromorphic function identifies the curve with $\mathbb{P}^1$, and using that fact, embed the curve as a conic in the plane in any convenient way, e.g., If t_0, t_1 are projective ($\mathbb{P}^1$) coordinates, let $z_0 = t_0^2,\ z_1 = t_0 t_1\ z_2 = t_1^2$ be the map to $\mathbb{P}^2$. The conic, then, would be $z_0 z_2 = z_1^2$. (Alternatively: one can consider the complete linear system attached to the anticanonical divisor.)

3. (Complex Analysis) Let $f(z) = z + e^{-z}$ for $z \in \mathbb{C}$ and let $\lambda \in \mathbb{R}$, $\lambda > 1$. Prove or disprove the statement that $f(z)$ takes the value λ exactly once in the open right half-plane $H_r = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Solution. First, let us consider the real function $f(x) = x + e^{-x}$. Since f is continuous, $f(0) = 1$ and $\lim_{x\to\infty} f(x) = \infty$, by the intermediate value theorem, there exists $u \in \mathbb{R}$ such that $f(u) = \lambda$. Now let us show that such u is unique. Let $R > 2\lambda$ and let Γ be the closed right half disk of radius R centered at the origin

$$\{z = x + iy \in \mathbb{C} : x = 0, |y| \le R\} \cup \left\{z \in \mathbb{C} : |z| = R, -\frac{\pi}{2} \le \arg(z) \le \frac{\pi}{2}\right\}.$$

Let $F(z) = \lambda - z$ and $G(z) = -e^{-z}$. Then for $z \in \Gamma$, we have $|G(z)| = |e^{-\mathrm{Re} z}| \le 1$ since Re $z \geq 0$, while $|F(z)| > 1$ by construction. Hence by Rouché's theorem, $\lambda - f(z) = F(z) + G(z)$ has the same number of zeros inside Γ as $F(z)$, namely 1. Since this is true for all R large enough, we conclude that the point u is unique.

4. (Algebraic Topology) (a) Let X and Y be locally contractible, connected spaces with fixed basepoints. Let $X \vee Y$ be the wedge sum at the basepoints. Show that $\pi_1(X \vee Y)$ is the free product of $\pi_1 X$ with $\pi_1 Y$.

(b) Show that $\pi_1(X \times Y)$ is the direct product of $\pi_1 X$ with $\pi_1 Y$.

(c) Note the canonical inclusion $f : X \vee Y \to X \times Y$. Assume that X and Y have abelian fundamental groups. Show that the map f_* on fundamental groups exhibits $\pi_1(X \times Y)$ as the abelianization of $\pi_1(X \vee Y)$.
Hint: The Hurewicz map is natural.

Solution. (a) This follows form the Van Kampen theorem: Writing $X \vee Y$ as the union

$$X \cup_* Y$$

we have that $\pi_1(X \vee Y) \cong \pi_1(X) *_{\pi_1(*)} \pi_1(Y) = \pi_1(X) * \pi_1 Y$.

(b) There is the obvious continuous map

$$Maps_*(S^1, X) \times Maps_*(S^1, Y) \to Maps_*(S^1, X \times Y)$$

given by sending $(t \mapsto \gamma_X(t), t \mapsto \gamma_Y(t)) \mapsto (t \mapsto (\gamma_X(t), \gamma_Y(t)))$. This map is a continuous so it induces a map

$$\pi_0(Maps_*(S^1, X) \times Maps_*(S^1, Y)) \to \pi_0 Maps_*(S^1, X \times Y)$$

where the lefthand side is isomorphic to $\pi_0 Maps_*(S^1, X) \times \pi_0 Maps_*(S^1, Y))$. Further, the above map is clearly a bijection, so it induces an injection and a surjection on π_0.

(c) The Hurewicz map is natural so we have a commutative diagram

$$\begin{array}{ccc} \pi_1(X \vee Y) & \xrightarrow{f_*} & \pi_1(X \times Y) \\ \downarrow q & & \downarrow \\ H_1(X \vee Y) & \xrightarrow{f_*} & H_1(X \times Y) \end{array}$$

where the vertical maps are abelianizations by the Hurewicz theorem. But the lower-right corner is equal to $H_1(X) \times H_1(Y)$ by the Kunneth theorem (since X and Y are connected), and the bottom copy of f_* is the obvious isomorphism on H_1. Since q is an abelianization by definition, but the bottom arrow and rightmost arrow are both isomorphisms, the top arrow must also be an abelianization.

5. (Differential Geometry) (a) Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be a circle and consider the connection

$$\nabla := \mathrm{d} + \pi\sqrt{-1}\mathrm{d}\theta$$

defined on the trivial complex line bundle over $\mathbb{S}^1$, where θ is the standard coordinate on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ descended from $\mathbb{R}$. By solving the differential equation for flat sections $f(\theta)$

$$\nabla f = \mathrm{d}f + \pi\sqrt{-1} f \mathrm{d}\theta = 0$$

or otherwise, show that there does not exist global flat sections with respect to ∇ over $\mathbb{S}^1$.

(b) Let $T = V/\Lambda$ be a torus, where Λ is a lattice and $V = \Lambda \otimes \mathbb{R}$ is the real vector space containing Λ. Let L be the trivial complex line bundle equipped with the standard Hermitian metric. By identifying flat $U(1)$ connections with $U(1)$ representations of the fundamental group $\pi_1(T)$ or otherwise, show that the space of flat unitary connections on L is the dual torus $T^* = V^*/\Lambda^*$, where $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice and $V^* := \mathrm{Hom}(V, \mathbb{R})$ is the dual vector space.

Solution. (a) The differential equation

$$f'(\theta) + \pi\sqrt{-1}f(\theta) = 0$$

has a unique solution

$$f(\theta) = A\mathrm{e}^{-\pi\sqrt{-1}\theta}$$

up to a constant $A \in \mathbb{C}$. This is not a well-defined function over $\mathbb{S}^1$ because $f(0) \neq f(1)$.

(b) The space of flat G-connections over T can be identified as

$$\mathrm{Hom}(\pi_1(T), G)/\mathrm{Ad}G.$$

Since $\pi_1(T) = \Lambda$ and for the abelian group $G = U(1)$ the adjoint action is trivial, we have

$$\mathrm{Hom}(\pi_1(T), G)/\mathrm{Ad}G = \mathrm{Hom}(\Lambda, U(1)) = T^*.$$

6. (Real Analysis) *(Fundamental Solutions of Linear Partial Differential Equations with Constant Coefficients).* Let Ω be an open interval $(-M, M)$ in $\mathbb{R}$ with $M > 0$. Let n be a positive integer and $L = \sum_{\nu=0}^{n} a_\nu \frac{d^\nu}{dx^\nu}$ be a linear differential operator of order n on $\mathbb{R}$ with constant coefficients, where the coefficients $a_0, \cdots, a_{n-1}, a_n \neq 0$ are complex numbers and x is the coordinate of $\mathbb{R}$. Let $L^* = \sum_{\nu=0}^{n}(-1)^\nu \overline{a_\nu} \frac{d^\nu}{dx^\nu}$. Prove, by using Plancherel's identity, that there exists a constant $c > 0$ which depends only on M and a_n and is independent of $a_0, a_1, \cdots, a_{n-1}$ such that for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}$. Give one explicit expression for c as a function of M and a_n.

Hint: A *weak* solution u of $Lu = f$ means that $(f, \psi)_{L^2(\Omega)} = (u, L^*\psi)_{L^2(\Omega)}$ for every infinitely differentiable function ψ on Ω with compact support. For the solution of this problem you can consider as known and given the following three statements.

(I) If there exists a positive number $c > 0$ such that $\|\psi\|_{L^2(\Omega)} \leq c\,\|L^*\psi\|_{L^2(\Omega)}$ for all infinitely differentiable complex-valued functions ψ on Ω with compact support, then for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}$.

(II) Let $P(z)=z^m+\sum_{k=0}^{m-1} b_k z^k$ be a polynomial with leading coefficient 1. If F is a holomorphic function on $\mathbb{C}$, then

$$|F(0)|^2 \leq \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left|P\left(e^{i\theta}\right)F\left(e^{i\theta}\right)\right|^2 d\theta.$$

(III) For an L^2 function f on $\mathbb{R}$ which is zero outside $\Omega=(-M,M)$ its Fourier transform

$$\hat{f}(\xi)=\int_{-M}^{M} f(x)e^{-2\pi i x\xi}dx$$

as a function of $\xi\in\mathbb{R}$ can be extended to a holomorphic function

$$\hat{f}\left(\xi+i\eta\right)=\int_{-M}^{M} f(x)e^{-2\pi i x(\xi+i\eta)}dx$$

on $\mathbb{C}$ as a function of $\xi+i\eta$.

Solution. This problem is to compute the constant c in Lemma 3.3 on p.225 of the book of Stein and Shakarchi on *Real Analysis* by going over its arguments and keeping track of the constants involved in each step.

Introduce the polynomial

$$Q(\zeta)=\sum_{k=0}^{n}(-1)^k\,\overline{a_k}\,(2\pi\zeta)^k$$

so that

$$(\#)\qquad \left(\widehat{L^*\psi}\right)(\zeta)=Q(\zeta)\hat{\psi}(\zeta)$$

any $\psi\in\mathcal{C}_0^\infty(\mathbb{R})$, where $\widehat{\ }$ denotes taking the Fourier transform. Consider first the special case where $a_n=\frac{1}{(2\pi i)^n}$ so that the coefficient of ξ^n in the polynomial $Q(\zeta)$ of degree n in ζ is 1. Writing $\zeta=\xi+\sqrt{-1}\eta$ (with both ξ and η real) and taking the L^2 of both sides of $(\#)$ over $\mathbb{R}$ as functions of η. Then

$$(\flat)\qquad \int_{-\infty}^{\infty}\left|Q\left(\xi+i\eta\right)\hat{\psi}\left(\xi+i\eta\right)\right|^2 d\xi=\int_{-\infty}^{\infty}\left|\left(\widehat{L^*\psi}\right)\left(\xi+i\eta\right)\right|^2 d\xi.$$

Since from the definition of Fourier transform

$$\left(\widehat{L^*\psi}\right)(\xi+i\eta)=\int_{x=-\infty}^{\infty}\left(L^*\psi\right)(x)e^{-2\pi i(\xi+i\eta)x}dx=\int_{x=-\infty}^{\infty}\left(\left(L^*\psi\right)(x)e^{2\pi\eta x}\right)e^{-2\pi i\xi x}dx,$$

it follows that $\left(\widehat{L^*\psi}\right)(\xi+i\eta)$ is equal to the value at ξ of the Fourier transform of the function $(L^*\psi)(x)e^{2\pi\eta x}$. Thus, by applying Plancherel's identity to the function $(L^*\psi)(x)e^{2\pi\eta x}$, we get

$$\int_{\xi=-\infty}^{\infty}\left|\left(\widehat{L^*\psi}\right)(\xi+i\eta)\right|^2 d\xi$$
$$=\int_{x=-\infty}^{\infty}\left|(L^*\psi)(x)e^{2\pi\eta x}\right|^2 dx \le e^{4\pi|\eta|M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx,$$

because the support of $\psi(x)$ (as well as the support of $(L^*\psi)(x)$) is in the interval $\Omega=(-M,M)$. Thus from ($\flat$) it follows that

$$(\sharp)\qquad \int_{-\infty}^{\infty}\left|Q(\xi+i\eta)\,\hat{\psi}(\xi+i\eta)\right|^2 d\xi \le e^{4\pi|\eta|M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

Setting $\eta=\sin\theta$ in ($\sharp$), we get from $|\eta|\le 1$ that

$$(\dagger)\qquad \int_{-\infty}^{\infty}\left|Q(\xi+i\sin\theta)\,\hat{\psi}(\xi+i\sin\theta)\right|^2 d\xi \le e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

Replacing ξ by $\xi+\cos\theta$ in the integrand on the left-hand side of ($\dagger$), we get

$$(\ddagger)\qquad \int_{-\infty}^{\infty}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\xi$$
$$\le e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

By Statement (III) given above the function $\hat{\psi}(\xi+i\eta)$ as a function of $\xi+i\eta\in\mathbb{C}$ is holomorphic on $\mathbb{C}$. Since $Q(\xi+i\eta)$ as a function of $\xi+i\eta\in\mathbb{C}$ is a polynomial of degree n with leading coefficient 1, it follows from Statement (II) applied to $F(z)=\hat{\psi}(\xi+z)$ and $P(z)=Q(\xi+z)$ that

$$\left|\hat{\psi}(\xi)\right|^2 \le \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\theta.$$

Integrating both sides over $\xi\in(-\infty,\infty)$ and using ($\ddagger$), we get

$$\int_{\xi=-\infty}^{\infty}\left|\hat{\psi}(\xi)\right|^2 \le \int_{\xi=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\theta\right)d\xi$$
$$=\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left(\int_{\xi=-\infty}^{\infty}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\xi\right)d\theta$$
$$\le\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left(e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx\right)d\theta = e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

By applying Plancherel's formula to ψ, we conclude that

$$\|\psi(\xi)\|^2_{L^2(\Omega)} \leq e^{4\pi M} \|(L^*\psi)(x)\|^2_{L^2(\Omega)}$$

under the additional assumption that $a_n = \frac{1}{(2\pi i)^n}$. When this additional assumption is not satisfied, we can apply the argument for the special case to

$$\frac{1}{a_n(2\pi i)^n} L$$

instead of to L to conclude that

$$\|\psi(\xi)\|^2_{L^2(\Omega)} \leq \frac{e^{4\pi M}}{|a_n(2\pi)^n|^2} \|(L^*\psi)(x)\|^2_{L^2(\Omega)},$$

or

$$\|\psi(\xi)\|_{L^2(\Omega)} \leq c \|(L^*\psi)(x)\|_{L^2(\Omega)},$$

with

$$c = \frac{e^{2\pi M}}{|a_n|(2\pi)^n}.$$

By Statement (I) given above, when we set

$$c = \frac{e^{2\pi M}}{|a_n|(2\pi)^n},$$

we can conclude that for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}$.