Qualifying Exams I, 2013 Fall

1. (ALGEBRA) Consider the algebra $M_2(k)$ of 2×2 matrices over a field k. Recall that an *idempotent* in an algebra is an element e such that $e^2 = e$.

(a) Show that an idempotent $e \in M_2(k)$ different from 0 and 1 is conjugate to

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $GL_2(k)$.

(b) Find the stabilizer in $GL_2(k)$ of $e_1 \in M_2(k)$ under the conjugation action.

(c) In case $k = \mathbb{F}_p$ is the prime field with p elements, compute the number of idempotents in $M_2(k)$. (Count 0 and 1 in.)

2. (ALGEBRAIC GEOMETRY) (a) Find an everywhere regular differential n-form on the affine n-space $\mathbb{A}^n$.

(b) Prove that the canonical bundle of the projective n-dimensional space $\mathbb{P}^n$ is $\mathcal{O}(-n-1)$.

3. (COMPLEX ANALYSIS) *(Bol's Theorem of 1949).* Let $\tilde{W}$ be a domain in $\mathbb{C}$ and W be a relatively compact nonempty subdomain of $\tilde{W}$. Let $\varepsilon > 0$ and G_ε be the set of all $(a, b, c, d) \in \mathbb{C}$ such that $\max(|a-1|, |b|, |c|, |d-1|) < \varepsilon$. Assume that $cz + d \neq 0$ and $\frac{az+b}{cz+d} \in \tilde{W}$ for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Let $m \geq 2$ be an integer. Prove that there exists a positive integer ℓ (depending on m) with the property that for any holomorphic function φ on $\tilde{W}$ such that

$$\varphi(z) = \varphi\left(\frac{az+b}{cz+d}\right)\frac{(cz+d)^{2m}}{(ad-bc)^m}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$, the ℓ-th derivative $\psi(z) = \varphi^{(\ell)}(z)$ of $\varphi(z)$ on $\tilde{W}$ satisfies the equation

$$\psi(z) = \psi\left(\frac{az+b}{cz+d}\right)\frac{(ad-bc)^{\ell-m}}{(cz+d)^{2(\ell-m)}}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Express ℓ in terms of m.

Hint: Use Cauchy's integral formula for derivatives.

4. (Algebraic Topology) (a) Show that the Euler characteristic of any contractible space is 1.

(b) Let B be a connected CW complex made of finitely many cells so that its Euler characteristic is defined. Let $E \to B$ be a covering map whose fibers are discrete, finite sets of cardinality N. Show the Euler characteristic of E is N times the Euler characteristic of B.

(c) Let G be a finite group with cardinality > 2. Show that BG (the classifying space of G) cannot have homology groups whose direct sum has finite rank.

5. (Differential Geometry) Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane. Let g be the Riemannian metric on H given by

$$g = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}.$$

(H, g) is known as the half-plane model of the hyperbolic plane.

(a) Let $\gamma(\theta) = (\cos\theta, \sin\theta)$ and $\eta(\theta) = (\cos\theta + 1, \sin\theta)$ for $\theta \in (0, \pi)$ be two paths in H. Compute the angle A at their intersection point shown in Figure 1, measured by the metric g.

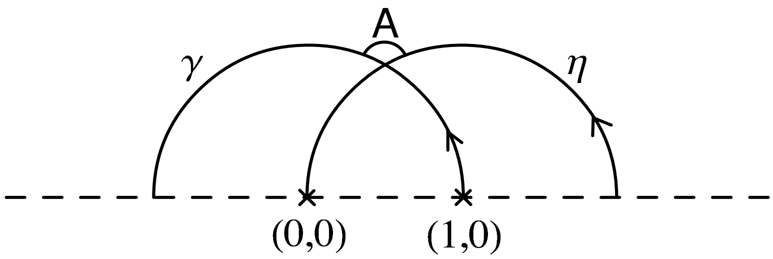

Figure 1: Angle A between the two curves γ and η in the upper half plane H.

(b) By computing the Levi-Civita connection

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{2} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}$$

of g or otherwise (where $(x_1, x_2) = (x, y)$), show that the path γ, *after arc-length reparametrization*, is a geodesic with respect to the metric g.

6. (Real Analysis) For any positive integer n let M_n be a positive number such that the series $\sum_{n=1}^{\infty} M_n$ of positive numbers is convergent and its limit is M. Let $a < b$ be real numbers and $f_n(x)$ be a real-valued continuous function on $[a, b]$ for any positive integer n such that its derivative $f_n'(x)$ exists for every $a < x < b$ with $|f_n'(x)| \leq M_n$ for $a < x < b$. Assume that the series $\sum_{n=1}^{\infty} f_n(a)$ of real numbers converges. Prove that

(a) the series $\sum_{n=1}^{\infty} f_n(x)$ converges to some real-valued function $f(x)$ for every $a \leq x \leq b$,

(b) $f'(x)$ exists for every $a < x < b$, and

(c) $|f'(x)| \leq M$ for $a < x < b$.

Hint for (b): For fixed $x \in (a, b)$ consider the series of functions

$$\sum_{n=1}^{\infty} \frac{f_n(y) - f_n(x)}{y - x}$$

of the variable y and its uniform convergence.

Qualifying Exams II, 2013 Fall

1. (Algebra) Find all the field automorphisms of the real numbers $\mathbb{R}$.

Hint: Show that any automorphism maps a positive number to a positive number, and deduce from this that it is continuous.

2. (Algebraic Geometry) What is the maximum number of ramification points that a mapping of finite degree from one smooth projective curve over $\mathbb{C}$ of genus 1 to another (smooth projective curve of genus 1) can have? Give an explanation for your answer.

3. (Complex Analysis) Let ω and η be two complex numbers such that $\mathrm{Im}\left(\frac{\omega}{\eta}\right) > 0$. Let G be the closed parallelogram consisting of all $z \in \mathbb{C}$ such that $z = \lambda\omega + \rho\eta$ for some $0 \leq \lambda, \rho \leq 1$. Let ∂G be the boundary of G and Let $G^0 = G - \partial G$ be the interior of G. Let $P_1, \cdots, P_k, Q_1, \cdots, Q_\ell$ be points in G^0 and let $m_1, \cdots, m_k, n_1, \cdots, n_\ell$ be positive integers. Let f be a function on G such that

$$\frac{f(z)\prod_{j=1}^{\ell}(z - Q_j)^{n_j}}{\prod_{p=1}^{k}(z - P_p)^{m_p}}$$

is continuous and nowhere zero on G and is holomorphic on G^0. Let $\varphi(z)$ and $\psi(z)$ be two polynomials on $\mathbb{C}$. Assume that $f(z+\omega) = e^{\varphi(z)}f(z)$ if both z and $z+\omega$ are in G. Assume also that $f(z+\eta) = e^{\psi(z)}f(z)$ if both z and $z+\eta$ are in G. Express $\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j$ in terms of ω and η and the coefficients of $\varphi(z)$ and $\psi(z)$.

4. (Algebraic Topology) (a) Fix a basis for H_1 of the two-torus (with integer coefficients). Show that for every element $x \in SL(2, \mathbb{Z})$, there is an automorphism of the two-torus such that the induced map on H_1 acts by x.

Hint: $SL(2, \mathbb{Z})$ also acts on the universal cover of the torus.

(b) Fix an embedding $j : D^2 \times S^1 \to S^3$. Remove its interior from S^3 to obtain a manifold X with boundary T^2. Let f be an automorphism of the two-torus and consider the glued space

$$X_f := (D^2 \times S^1) \cup_f X.$$

If X is homotopy equivalent to $D^2 \times S^1$, compute the homology groups of X_f.

5. (Differential Geometry) Let $M = U(n)/O(n)$ for $n \geq 1$, where $U(n)$ is the group of $n \times n$ unitary matrices and $O(n)$ is the group of $n \times n$ orthogonal matrices. M is a real manifold called the *Lagrangian Grassmannian.*

(a) Compute and state the dimension of M.

(b) Construct a Riemannian metric which is invariant under the left action of $U(n)$ on M.

(c) Let ∇ be the corresponding Levi-Civita connection on the tangent bundle TM, and X, Y, Z be any $U(n)$-invariant vector fields on M. Using the given identity (which you are not required to prove)

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

show that the Riemannian curvature tensor R of ∇ satisfies the formula

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]].$$

6. (Real Analysis) Show that there is no function $f \colon \mathbb{R} \to \mathbb{R}$ whose set of continuous points is precisely the set $\mathbb{Q}$ of all rational numbers.

Qualifying Exams III, 2013 Fall

1. (Algebra) Consider the function fields $K = \mathbb{C}(x)$ and $L = \mathbb{C}(y)$ of one variable, and regard L as a finite extension of K via the $\mathbb{C}$-algebra inclusion

$$x \mapsto \frac{-(y^5-1)^2}{4y^5}$$

Show that the extension L/K is Galois and determine its Galois group.

2. (Algebraic Geometry) Is every smooth projective curve of genus 0 defined over the field of complex numbers isomorphic to a conic in the projective plane? Give an explanation for your answer.

3. (Complex Analysis) Let $f(z) = z + e^{-z}$ for $z \in \mathbb{C}$ and let $\lambda \in \mathbb{R}$, $\lambda > 1$. Prove or disprove the statement that $f(z)$ takes the value λ exactly once in the open right half-plane $H_r = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

4. (Algebraic Topology) (a) Let X and Y be locally contractible, connected spaces with fixed basepoints. Let $X \vee Y$ be the wedge sum at the basepoints. Show that $\pi_1(X \vee Y)$ is the free product of $\pi_1 X$ with $\pi_1 Y$.

(b) Show that $\pi_1(X \times Y)$ is the direct product of $\pi_1 X$ with $\pi_1 Y$.

(c) Note the canonical inclusion $f : X \vee Y \to X \times Y$. Assume that X and Y have abelian fundamental groups. Show that the map f_* on fundamental groups exhibits $\pi_1(X \times Y)$ as the abelianization of $\pi_1(X \vee Y)$.
Hint: The Hurewicz map is natural.

5. (Differential Geometry) (a) Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be a circle and consider the connection

$$\nabla := \mathrm{d} + \pi\sqrt{-1}\mathrm{d}\theta$$

defined on the trivial complex line bundle over $\mathbb{S}^1$, where θ is the standard coordinate on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ descended from $\mathbb{R}$. By solving the differential equation for flat sections $f(\theta)$

$$\nabla f = \mathrm{d}f + \pi\sqrt{-1} f \mathrm{d}\theta = 0$$

or otherwise, show that there does not exist global flat sections with respect to ∇ over $\mathbb{S}^1$.

(b) Let $T = V/\Lambda$ be a torus, where Λ is a lattice and $V = \Lambda \otimes \mathbb{R}$ is the real vector space containing Λ. Let L be the trivial complex line bundle equipped with the standard Hermitian metric. By identifying flat $U(1)$ connections with $U(1)$ representations of the fundamental group $\pi_1(T)$ or otherwise, show that the space of flat unitary connections on L is the dual torus $T^* = V^*/\Lambda^*$, where $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice and $V^* := \mathrm{Hom}(V, \mathbb{R})$ is the dual vector space.

6. (Real Analysis) *(Fundamental Solutions of Linear Partial Differential Equations with Constant Coefficients).* Let Ω be an open interval $(-M, M)$ in $\mathbb{R}$ with $M > 0$. Let n be a positive integer and $L = \sum_{\nu=0}^{n} a_\nu \frac{d^\nu}{dx^\nu}$ be a linear differential operator of order n on $\mathbb{R}$ with constant coefficients, where the coefficients $a_0, \cdots, a_{n-1}, a_n \neq 0$ are complex numbers and x is the coordinate of $\mathbb{R}$. Let $L^* = \sum_{\nu=0}^{n} (-1)^\nu \overline{a_\nu} \frac{d^\nu}{dx^\nu}$. Prove, by using Plancherel's identity, that there exists a constant $c > 0$ which depends only on M and a_n and is independent of $a_0, a_1, \cdots, a_{n-1}$ such that for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}$. Give one explicit expression for c as a function of M and a_n.

Hint: A *weak* solution u of $Lu = f$ means that $(f, \psi)_{L^2(\Omega)} = (u, L^*\psi)_{L^2(\Omega)}$ for every infinitely differentiable function ψ on Ω with compact support. For the solution of this problem you can consider as known and given the following three statements.

(I) If there exists a positive number $c > 0$ such that $\|\psi\|_{L^2(\Omega)} \leq c \|L^*\psi\|_{L^2(\Omega)}$ for all infinitely differentiable complex-valued functions ψ on Ω with compact support, then for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}$.

(II) Let $P(z) = z^m + \sum_{k=0}^{m-1} b_k z^k$ be a polynomial with leading coefficient 1. If F is a holomorphic function on $\mathbb{C}$, then

$$|F(0)|^2 \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left| P\left(e^{i\theta}\right) F\left(e^{i\theta}\right) \right|^2 d\theta.$$

(III) For an L^2 function f on $\mathbb{R}$ which is zero outside $\Omega = (-M, M)$ its Fourier transform

$$\hat{f}(\xi) = \int_{-M}^{M} f(x) e^{-2\pi i x \xi} dx$$

as a function of $\xi \in \mathbb{R}$ can be extended to a holomorphic function

$$\hat{f}\,(\xi + i\eta) = \int_{-M}^{M} f(x)e^{-2\pi i x(\xi + i\eta)}dx$$

on $\mathbb{C}$ as a function of $\xi + i\eta$.