QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 30 September 2003 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1a. Let S be an embedded closed surface in $\mathbf{R}^3$ with the position vector $\vec{X}(p)$ and the unit outward normal vector $\vec{N}(p)$ for $p \in S$. For a fixed (small) t, define a surface S_t to be the set

$$S_t = \left\{ \vec{X}(p) + t\vec{N}(p) \in \mathbf{R}^3 \mid p \in S \right\}.$$

Let κ_1, κ_2 be the principal curvatures of S at the point p with respect to the outward normal vector. Let H_t be the mean curvature of S_t at the point $\vec{X}(p) + t\vec{N}(p)$ with respect to the outward normal vector (mean curvature is defined to be the sum of the two principal curvatures). Show that

$$H_t = \frac{\kappa_1}{1 - t\kappa_1} + \frac{\kappa_2}{1 - t\kappa_2}.$$

2a. Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk in $\mathbf{C}$ and $\bar{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ be the closed unit disk. Suppose $f : \bar{D} \to \bar{D}$ is analytic, one-to-one in D and continuous in $\bar{D}$. Also suppose $g : \bar{D} \to \bar{D}$ is analytic in D and continuous in $\bar{D}$, with $g(0) = f(0)$ and $g(D) \subset f(D)$. Prove $|g'(0)| \leq |f'(0)|$.

3a. Use the Riemann-Hurwitz (or any other) method to compute the genus of the Fermat curve, which is given in $\mathbf{CP}^2$ with homogeneous coordinates $(x : y : z)$ by the equation $x^n + y^n = z^n$ (assume that the base field is $\mathbf{C}$).

4a. Let k be a finite field with q elements and let $\Gamma = \mathrm{GL}(2, k)$ denote the group of invertible 2×2 matrices over k.

(i) How many elements are there in Γ?

(ii) How many complex irreducible representations does Γ have?

(iii) Consider the representation of Γ on the space of complex-valued functions on $\mathbf{P}^1$ over k (induced by the natural action of Γ on $\mathbf{P}^1$). Let V be the quotient of this space by the subspace of constant functions. Prove that V is an irreducible representation of Γ.

5a. Let V be a Hilbert space, and W a vector subspace of V. Show that

$$V = \overline{W} \oplus W^{\perp}.$$

6a. What is $\pi_1(\mathbf{RP}^3)$? Show that any continuous map $f : \mathbf{RP}^3 \to S^1$ is null-homotopic.

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Wednesday 1 October 2003 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1b. Let $(\mathbf{H}^2, g)$ be the two-dimensional hyperbolic space, where

$$\mathbf{H}^2 = \{(x, y) \in \mathbf{R}^2 : y > 0\}$$

is the upper half plane of $\mathbf{R}^2 = \mathbf{C}$ and the metric g is given by

$$g = \frac{dx^2 + dy^2}{y^2}.$$

(i) Suppose a, b, c and d are real numbers such that $ad - bc = 1$. Define

$$\varphi(z) = \frac{az + b}{cz + d}$$

for any $z = x + \sqrt{-1}y$. Prove that φ is an isometry for $(\mathbf{H}^2, g)$.

(ii) Prove that $(\mathbf{H}^2, g)$ has constant Gaussian curvature.

2b. Prove the open mapping theorem for analytic functions of one complex variable: "if U is a connected open subset of $\mathbf{C}$ and $f : U \to \mathbf{C}$ is holomorphic and non-constant, then $f(U)$ is open." You may assume that a holomorphic function that is constant on an open subset of U is constant on U.

3b. Prove that if k is a field of characteristic p and $f(x) \in k[x]$ is a polynomial, then the map from the curve $y^p + y = f(x)$ to $\mathbf{A}^1_k$ sending (x, y) to x is everywhere unramified.

4b. (i) Let k be an algebraically closed field. Assume that k is *uncountable.* Let now V be a vector space over k of at most countable dimension and $A : V \to V$ be a linear operator. Prove that there exists $\lambda \in k$ such that the operator $A - \lambda\,\mathrm{id}_V$ is not invertible. (Hint: show first that in the field $k(t)$ of rational functions over k the elements $\frac{1}{t-\lambda}$ are linearly independent (for different values of λ) and then use this fact.)

(ii) Show that (i) is not necessarily true if k is countable.

(iii) Use (i) to show that for k uncountable every maximal ideal in the ring $k[x_1, \ldots, x_n]$ is generated by $(x-\lambda_1, \ldots, x-\lambda_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in k^n$.

5b. Give an example or show that none exist.

(i) A function $f : \mathbf{R} \to \mathbf{R}$ whose set of discontinuities is precisely the set $\mathbf{Q}$ of rational numbers.

(ii) A function $f : \mathbf{R} \to \mathbf{R}$ whose set of discontinuities is precisely the set $\mathbf{R} \setminus \mathbf{Q}$ of irrational numbers.

6b. Let X be the manifold-with-boundary $D^2 \times S^2$. Calculate $H_2(X; \mathbf{Z})$, $H^2(X; \mathbf{Z})$ and $H^2(X, \partial X; \mathbf{Z})$, using any techniques you choose. Calculate the map $j_* : H^2(X, \partial X; \mathbf{Z}) \to H^2(X; \mathbf{Z})$ that arises from the inclusion $j : (X, \emptyset) \to (X, \partial X)$.

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Thursday 2 October 2003 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1c. Let Σ be an embedded, compact surface without boundary in $\mathbf{R}^3$. Show that there exists at least one point p in Σ which has strictly positive Gaussian curvature.

2c. Determine for which $x \in \mathbf{Q}_p$ the exponential power series $\sum x^n/n!$ converges. Do the same for the logarithmic power series $\sum x^n/n$.

3c. Let V be a variety over an algebraically closed field k, and suppose V is also a group, i.e., there are morphisms $\varphi : V \times V \to V$ (multiplication or addition), and $\psi : V \to V$ (inverse) that satisfy the group axioms. Then V is called an *algebraic group*.

(i) Suppose that V is a nonsingular plane cubic. Describe a way to put a group structure on V. You do not have to prove that the maps you define are morphisms, but you do have to prove that they satisfy the axioms of a group.

(ii) Let V be defined by $y^2 z = x^3$ in $\mathbf{P}^2$. Prove that $V - \{(0, 0, 1)\}$ can be equipped with the structure of algebraic group.

(iii) Let V be defined by $x^3 + y^3 = xyz$ in $\mathbf{P}^2$. Prove that $V - \{(0, 0, 1)\}$ can be equipped with the structure of algebraic group.

4c. Compute the following integral:

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, \mathrm{d}x.$$

5c. (i) Let a, b be nonnegative numbers, and p, q such that $1 < p < \infty$ and $1/p + 1/q = 1$. Establish Young's inequality:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

(ii) Using Young's inequality, prove the Hölder inequality: If $f \in L^p[0,1]$ and $g \in L^q[0,1]$, where p and q are as above, then $fg \in L^1[0,1]$, and

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

(iii) For $1 < p < \infty$, and $g \in L^q$, consider the linear functional F on L^p given by

$$F(f) = \int fg.$$

Show that $\|F\| = \|g\|_q$. (Recall that $\|F\| = \sup\{|F(f)|/\|f\| : f \in L^p\}$.)

(iv) Establish similar results for $p = 1$ and $p = \infty$.

6c. (i) Prove that every continuous map $f : \mathbf{CP}^6 \to \mathbf{CP}^6$ has a fixed point.

(ii) Exhibit a continuous map $f : \mathbf{CP}^3 \to \mathbf{CP}^3$ without a fixed point. (Hint: Try the case of $\mathbf{CP}^1$ first and write your answer in terms of homogeneous coordinates.)