Aspects of the p-adic local Langlands programme, C. Breuil

Aspect I: A preliminary conjecture.

J'm very glad to be in Harvard to talk about the p-adic local Langlands programme and thank B. Matzur for giving me the opportunity to give this series of lectures.

I am going to give 4 talks, on 4 aspects of the p-adic local Langlands programme. The motivation of these 4 talks is the following:

let \([L: \mathbb{Q}] < \infty\) and \(d \geq 1\) an integer, can one do:

\[
\begin{align*}
\{ \text{p-adic Banach spaces } + \} & \xrightarrow{\text{contous stable}\atop \text{unitary}} \{ \text{d+1- dimensional p-adic representations } \}
\end{align*}
\]

\[
\text{action of } \text{GL}_{d+1}(L)
\]

The reason one goes first this way is because many interesting Galois representations are known, namely the de Rham representations, and one would like to see their GL_{d+1} "counterparts".

Aspect I: When does an (irreducible) locally algebraic representation of GL_{d+1}(L) admit an invariant norm? (\(\|gv\| = \|v\|\))

Aspect II: \((\mathbb{F}_p^n, M)\)-modules \((GL_2(\mathbb{Q}_p))\) (I and II already covered in Palo Alto)

Aspect III: Drinfeld spaces \((GL_2(\mathbb{Q}_p))\)

Aspect IV: Mod p representations \((GL_2(\mathbb{Q}_p), GL_2(L))\)

The question in aspect I comes from the fact that if one starts with a de Rham representation then it is easy to associate to it a locally algebraic
representation of \( \text{Gal}(\bar{K}/L) \). It is then hoped that the Banach space, or at least a J.H. component of it, will be obtained by completing this locally algebraic representation w.r. to a well chosen \text{invariant norm}. If there is no \text{invariant norm}, then we can forget this \text{loc. alg. representation}. So the question on \text{invariant norm}s is a basic question if one is interested in de \text{Rham representations}. Joint with \text{SCHNEIDER}.

In the rest of this talk, I fix \( K \) another finite \text{extension of } \mathbb{Q} (\text{the coefficients}) and I assume \( [L: \mathbb{Q}_p] = 1 \); \( \text{Hom}_{\mathbb{Q}_p} (L, K) \); \( q = \# \text{residue field of } L \); \( \nu = \text{valuation of } L \); \( \text{val}(\nu) = 1 \).

Fontaine type categories. I need a "Fontaine type" interpretation of Weil-Deligne representations.

I fix \( L' \) a finite Galois \text{extension of } L and I denote \( L_0 \) its maximal unramified subfield. I will also assume \( [L_0: \mathbb{Q}_p] = 1 \); \( \text{Hom}_{\mathbb{Q}_p} (L_0, K) \).

Let me denote by \( \text{WD}_{L_0/L} \) the category of representations \((\mathbb{F}_q, V)\) of the Weil-Deligne group of \( L \) on a \( k \)-vector space \( V \) of finite dimension such that \( \mathbb{F}_q \mathbb{C}^{\text{unr}}(\mathbb{Q}_p/L) \) is unramified.

Recall that the Weil group of \( L =: \text{WD} \mathbb{Q}_p/L \) is the sub-group of \( \text{Gal}(\mathbb{Q}_p/L) \) of elements \( \sigma \) mapping to an integral power \( \sigma(\mathbb{F}_q) \in \mathbb{Z} \) of the absolute \text{arithmetic Frobenius} in \( \text{Gal}(\mathbb{F}_q^{\text{alg}}/\mathbb{F}_q) \) and that a Weil-Deligne representation \((\mathbb{F}_q, V, \psi, \sigma)\) on \( V \) is a map \( \psi \): \( \text{WD} \mathbb{Q}_p/L \rightarrow \text{Aut}_k(V) \) with open kernel together with a nilpotent \( k \)-linear endomorphism \( \sigma: V \rightarrow V \) such that \( \psi(\sigma(w)) = 1 \) for \( w \neq 0 \).

Now, let me introduce a Fontaine type category: Let \( \text{Mod}_{L_0/L} \) be the category of quadruples \((\psi, N, \text{Gal}(L_0/L), D)\) where \( D \) is a \text{free} \( L_0 \)-module of finite \text{rank} endowed with: \( \psi: D \rightarrow D \) \text{(Frobenius)}, \text{biregular}, \text{semi-linear} \( L_0 \); \( \psi(Ad) = \sigma(Ad) \cdot \psi(A) \).

$N: \mathbb{D} \to \mathbb{D}$ (modularity) linear ab. $N^p = p^2N$ (nilpotent) \( \odot \)
\[ \text{Gal}(\mathbb{L}/\mathbb{K}) \to \mathbb{D} \text{ semi-linear } \not\subset \mathcal{O} \quad (\sigma(\lambda d) = \sigma(\lambda) \sigma(d)) \]
linear $\mathcal{O}/K$
commuting with $\Psi$ and $\mathcal{O}$.

Fix an embedding $\sigma_0: \mathbb{L}_0 \to K$, then Fontaine has defined a functor:
\[ \text{WD}: \text{MOD}_{\mathcal{O}/\mathbb{L}_0} \to \text{WD}_{\mathcal{O}/\mathbb{L}_0} \text{ as follows:} \]
\[ (v, N, \text{Gal}(\mathcal{O}/\mathbb{L}_0)) \mapsto (v, N, \Psi^*v) \text{ where:} \]
\[ N : v \to v \text{ is } N_0 \odot \text{Id} \]
\[ \Psi(v) : v \to v \text{ is } \Psi^*v \]
\[ \text{where } \Psi = \text{Gal}(\mathbb{L}/\mathbb{L}_0) \]
you can check that $\Psi(v) N \Psi(v)^{-1} = p^{2\omega(v)} N$. Up to (non canonical) isomorphism $(v, N, \Psi^*v)$ doesn't depend on $\sigma_0$.

**Lemma:** The functor WD is an equivalence of categories.
The proof is left as an exercise. **Hint:** use the fact that [res. field of $\mathbb{L}_0 = \mathbb{F}_{p^r}$]
$D$ can be written as $D = \bigoplus_{n=0}^{s-1} V_{\sigma_0 \circ \psi_n}$ where $V_{\sigma_0 \circ \psi_n} = D \otimes_{\mathcal{O}/\mathbb{L}_0} K / \sigma_0 \circ \psi_n^{-1}$
to go backwards and build $D$ starting from $V$ ($\Psi = \text{Frob on } \mathbb{L}_0$).
The lemma allows to see any WD representation as a "filtered module without the filtration".

**Local Langlands correspondence revisited.**
Recall that the local Langlands correspondence is a bijection:
satisfying lots of properties. Here, I choose the following normalization: if \( \text{rec} : W(\overline{\mathbf{Q}_p})^{ab} \rightarrow L^* \) is the reciprocity map sending the arithmetic Frobenius to the inverse of uniformizers, and if \((r, N, V)\) is a Weil-Deligne representation as on RHS and \( \Pi_{\text{unit}} \) the representation on LHS associated to \((r, N, V)\), then:

\[
\text{central char} \ (\Pi_{\text{unit}}) = \det(r, N, V) \circ \text{rec}^{-1}.
\]

I write now \( \Pi^u \) for \( \Pi_{\text{unit}} \). Note that \( \Pi^u \) depends on the choice of \( \mathbf{q}^{1/2} \).

In general, we are not going to work with the representation \( \Pi_{\text{unit}} \) however. I want to define a representation \( \Pi^u \), a "better" representation.

Write \((r, N, V) = \bigoplus_i (V_i, N_i, V_i)\) with \((V_i, N_i, V_i)\) indecomposable (all this over \( \overline{\mathbf{Q}} \)). Let \( \Pi^u_i \) correspond to \((V_i, N_i, V_i)\) by L.I.C. where \( \Pi^u_i \) is a representation of \( \text{GL}_{d_i+1} \) for some \( d_i \). Then \( \Pi^u_i \) is called a "generalized Steinberg representation". Then it is known that \( \Pi^u \) is a quotient as follows:

\[
\text{normalized parabolic induction} \left\{ \text{Ind}_p^{\text{GL}_{d_i}} \Pi^u_i \otimes \cdots \otimes \Pi^u_i \right\} \rightarrow \Pi^u
\]

[actually, one has to write the \( \Pi^u_i \) in a certain order satisfying the so-called "does not prede" condition, then the parabolic induction doesn't depend on such an order]

I define

\[
\Pi := \left( \text{Ind}_p^{\text{GL}_{d_i}} \Pi^u_i \otimes \cdots \otimes \Pi^u_i \right) \otimes \mathbf{Q} \det_{L}^{-d/2}
\]
The following proposition follows from the Bernstein-Zelevinsky theory:

**Proposition:** Assume $(r, N, V)$ is a representation on a $K$ vector space (i.e. $V$ is a $K$-vector space), then $\Pi$ admits a unique model over $K$. Moreover, $\Pi$ doesn't depend on the choice of $q^k$.

**Example:** The typical example (and simplest example) is for $d = 1$ and

$$\Pi^\text{unit} : 1 : 1 \leftrightarrow (r, N, V) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then $\Pi = \text{Ind}_{(0, 0)}^{(1, 1)} 1 : 1 \otimes 1 : 1$ (here, the parabolic induction is NOT normalized)

**Conjecture.**

Fix $(r, N, V) \in \text{Ind}_{K/L}$ with $r$ semi-simple

- for each $\sigma : L \to K$, integers $i_{\sigma, r} \in \mathbb{Z}$ such that:
  $$i_{\sigma, r} \leq \ldots \leq i_{d_{\sigma, r}, r}$$

Define $\rho = K$-rational algebraic representations of $GL_{d_{\sigma}}(K)$ of highest weight:

$$\begin{pmatrix} -i_{d_{\sigma}, r} \leq -i_{d_{\sigma} - 1} \leq \ldots \leq -i_{1, r} - d \end{pmatrix}$$

i.e.:

$$\rho_\sigma = \left( \text{Ind}_{K_{d_{\sigma}}(L)}^{GL_{d_{\sigma}}(K)} x_{d_{\sigma}}^{-i_{d_{\sigma}} - 1} \otimes x_{d_{\sigma} - 1}^{-i_{d_{\sigma} - 1}} \otimes \ldots \otimes x_{1}^{-i_{1} - d} \right)_{\text{alg}} \cong \text{functions of } H^0(GL_{d_{\sigma}}, O_{GL_{d_{\sigma}}})$$

let $\rho : = \bigotimes_{\sigma : L \to K} \rho_\sigma$ with $GL_{d_{\sigma}}(L)$ acting diagonally, $GL_{d_{\sigma}}(L)$ acting on $\rho_\sigma$ via the embedding $\iota : GL_{d_{\sigma}}(L) \to GL_{d_{\sigma}}(K)$. Define $\Pi$ as above. So we have $\rho$, $\Pi$, and we can consider $\rho \otimes \Pi$. An $\mathbb{A}$-equivariant norm on $\rho \otimes \Pi$ is by definition a $p$-adic norm $\| \cdot \|$ such that $\|q \cdot v\| = \|v\|$
Conjecture: The following conditions are equivalent:

(i) There is an invariant norm on $\rho \otimes_{K^T} \Pi$

(ii) There is an object $(\psi, N, \text{Gal}(L), D) \in \text{MOD}_{L/\mathbb{Q}}$ such that

and a (weakly) admissible filtration preserved by $\text{Gal}(L)$

on $D_{\psi, L} := L' \otimes_{\mathbb{Q}} D = \prod_{\sigma \in \text{Gal}(L)} D_{\psi, L} \otimes (L' \otimes_K k)$ such that:

$$\text{Fic}^+ D_{\psi, L} \neq 0 \iff i \in \{ i_{\psi, L}, \ldots, i_{\psi, L} \}$$

where $D_{\psi, L} := D_{\psi, L} \otimes_{\mathbb{Q}} L' \otimes_K k$.

Example 1: $L = L' = \mathbb{Q}_p$, $d = 1$, $N = 0$, $r$ is unramified and given by arith. Frob of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda = 1 - 2 \sqrt{2})$$

$$D = K e_1 \oplus K e_2$$

$$\{ e_1, e_2 \} = \mathbb{Z}_p \times \mathbb{Z}_p$$

Then we have a weakly admissible filtration $\{ \text{Fic}^{-(k-1)} D = \ldots = \text{Fic}^0 D = K (e_1 + e_2) \}$.

And we can prove there is an invariant norm on $\rho \otimes_{K^T} \Pi$.

Example 2: $L = L' = \mathbb{Q}_p$, $d = 1$, $N = 0$, $v$ is unramified given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda = 1 - 2 \sqrt{2})$$

Then $\rho \otimes_{K^T} \Pi$ has an invariant norm, but for $(\psi, D)$.

$\text{WD}(i=1, k, i=3)$
\textbf{Definition (Fontaine):} The filtration is \textit{(weakly) admissible} if \( t_N(D) = t_N(D_{\nu}) \) and for any \( D' \leq D \) preserved by \( \nu \), \( N \) with the induced filtration on \( D_{\nu} \), one has \( t_N(D_{\nu}) \leq t_N(D') \).

"Hodge polygon under Newton polygon"
\[
\begin{cases}
  \psi(e_1) = p^{-\frac{d+1}{2}} e_1, \\
  \psi(e_2) = p^{-\frac{d+1}{2}} (e_1 + e_2)
\end{cases}
\]

then there is a w.a. filtration given by:

\[
\text{Fil}^{(b-1)} \subseteq \cdots \subseteq \text{Fil}^0 = K \cdot (e_1 + e_2).
\]

(so \( \psi \) is NOT semi-simple)

Why is this conjecture a first step towards \( p \)-adic Langlands (for de Rham Galois representation)? Because then, one might hope that a given specific weakly admissible filtration might "correspond" to a given specific norm on \( \psi \otimes \Pi \). And indeed, we will see later that, at least for \( \text{GL}_2(\mathbb{Q}_p) \) and irreducible de Rham representation, such a phenomenon really happens.

Before going to special cases, I would like to survey now some special or partial cases of the conjecture but still for \( \text{GL}_{d+1}(L) \).

**Some cases:**

**Prop.** The central character of \( \psi \otimes_k \Pi \) is integral iff for any filtration satisfying (\( \star \)), one has \( t_H(\mathcal{D}_L) = t_N(D) \).

**Proof:** The central character of \( \psi \otimes_k \Pi \) is integral iff:

\[
\text{val}_L(\text{central char. of } \psi(\Pi_L)) + \text{val}_L(\text{central char. of } \Pi(\Pi_L)) = 0.
\]

One computes:

\[
\text{val}_L(\text{c. ch. } \psi(\Pi_L)) = - \sum_{i=1}^{d+1} i d_{i+2-j, s} + \frac{d(d+1)}{2}.
\]

\[
\text{val}_L(\text{c. ch. } \Pi(\Pi_L)) = - \text{val}_L(\text{det}_k(v)(\text{arith. Frob. of } W(\mathcal{O}_L)))
\]

and

\[
[B : \mathbb{Q}] \frac{d(d+1)}{2}.
\]
\[ D_{r_0} = D \otimes_{\mathcal{O}_{L^0}} K \quad \text{(where } \mathcal{O}_{L^0} < K), \]

one checks that \[ \text{val}_L ((\det_k (r)) \text{ (with Frobenius)}) = \frac{L}{L'} \text{val}_L (\det_k (\psi_{D_{r_0}}^{r'})) \]

(note that \( \psi_{D_{r_0}}^{r'} : D_{r_0} \to D_{r_0} \) is \( K \)-linear). Hence, one has:

\[
\begin{align*}
\text{val}_L (\text{c.ch. } p (\Pi)) &= - \left( \sum_{i=1}^{d+1} \sum_{j=1}^{d+1} i_j \sigma \right) - \frac{[L : \mathcal{O}_F]}{2} \frac{d(d+1)}{2}, \\
\text{val}_L (\text{c.ch. } \pi (\Pi)) &= \frac{L}{L'} \text{val}_L (\det_k (\psi_{D_{r_0}}^{r'})) + \frac{[L : \mathcal{O}_F]}{2} \frac{d(d+1)}{2}.
\end{align*}
\]

Now, one has:

\[ t_H (D_{r_0}) = \sum_{i=1}^{d+1} \sum_{j=1}^{d+1} \left[ k : L \right] i_j \sigma \text{ and } t_N (D) = [k : L] \frac{L}{L'} \text{val}_L (\det_k (\psi_{D_{r_0}}^{r'})) \]

hence \[ \text{val}_L (\text{c.ch. } p (\Pi)) + \text{val}_L (\text{c.ch. } \pi (\Pi)) = \frac{1}{[k : L]} \left( - t_H (D_{r_0}) + t_N (D) \right). \]

**Corollary:** The conjecture holds if \( r \) is absolutely irreducible (equiv. if \( \Pi \) is supercuspidal).

**Proof:**

- One can always write \( \Pi = \text{c-ind}_{U_2}^{G} \sigma \) where \( Z = L^x \) and \( U = \) some open compact open in \( G \). Hence \( \rho \otimes \Pi = \text{c-ind}_{U_2}^{G} (\rho \otimes K^\sigma) \).

We see that \( \Pi \) has an invariant lattice \( \Pi \) iff \( \rho \otimes K^\sigma \) has iff the central char. of \( \rho \otimes K^\sigma \) is integral = central char. of \( \rho \otimes \Pi \).

- As the object \((r, N, \text{Gal}(\mathbb{F}_r), D) \in \text{MOD}_{L'_L} \) corresponding to \((r, N, V) \) by the previous equivalence of categories is irreducible, its only subobjects are \( 0 \) or itself. Hence, the weak admissibility conditions are just \( t_H (D_{r_0}) = t_N (D) \). The corollary therefore follows from the proposition.
In the same way, one can prove that if \( r \) is abs. indecomposable (equiv. \( \Pi \) is a generalized Steinberg), then a filtration as in the conj. is (weakly) admissible iff \( \ell_n(D') = \ell_n(D) \). The following conjecture is thus a special case of the previous one:

**Conj.:** If \( \Pi \) is a generalized Steinberg, then \( \rho \otimes_k \Pi \) admits an invariant norm iff its central character does.

**Example 3:** \( L = L' = Q_p \), \( d = 1 \) and \( r \) is given by \( \begin{pmatrix} 1 & 1^{1/2} \\ 0 & 1^{1/2} \end{pmatrix} \)

\(-\) \( i_1 : = -1 \), \( i_2 : = 0 \)

Then \( \rho \otimes_k \Pi = \text{Sym}^{-1} \mathfrak{S}^2 Q_p \otimes_k \text{Steinberg} \otimes_{\det} \mathbb{Q}_k \)

where Steinberg = \( \text{Ind}_{(G^2)}^{(G)} \).

Teliebaum and GK have proven that \( \rho \otimes_k \Pi \) has an invariant norm.

**Thm (Schneider, Teliebaum, B.):** Assume that \( (r, N, V) \) is a direct sum of unramified characters, then

\[ (i) \implies (ii) \] in the conjecture.

**Sketch of proof:** \( r \) with Frob. of \( \begin{pmatrix} 3 \\ \vdots \\ 3_{d+1} \end{pmatrix} \)

Let \( U := \text{GL}_{d+1}(O_L) \) and \( G := \text{GL}_{d+1}(L) \). Let:

\( \mathcal{H}(G, U) := \text{End}_G(\text{c-Ind}_U^G U) \)

\( \mathcal{H}(G, \text{pl}_U) := \text{End}_G(\text{c-Ind}_U^G \text{pl}_U) \)

Then \( i : \mathcal{H}(G, U) \to \mathcal{H}(G, \text{pl}_U) \)

\( f \mapsto (g \mapsto f(g)p(g)) \)
let $T \subseteq G$ be the split torus and $T^0 \supseteq T \cap U$, let:

$$\varphi : T/T^0 \rightarrow K, \quad \varphi = \text{unr}(\beta_1) \otimes \text{unr}(\beta_2) \otimes \cdots \otimes \text{unr}(\beta_{d+1}) \mid_{L^d}$$

then it is a result of Dat that $\Pi \cong K^{\otimes \text{c-ind}_{T^0}^{T}} \, \varphi(\mathcal{H}(G, 1_{L}))$

where $\varphi : \mathcal{H}(G, 1_{L}) \xrightarrow{\text{stable map}} K[T/T^0] \xrightarrow{\varphi} K$ (remember $\Pi$ is L.L. modified).

Denote by $p^0L$ a $U$-lattice in $p$, then one has an associated norm on $p$, hence on $\text{c-ind}_{U}^{G} p$, hence an $\text{End}_{G}(\text{c-ind}_{U}^{G} p)$, hence an $\mathcal{H}(G, pL)$. Denote by $B(G, pL)$ the completion of $\mathcal{H}(G, pL)$ with respect to this norm. Then it can be shown that a $K$-point:

$$\varphi : \mathcal{H}(G, pL) \rightarrow \mathcal{H}(G, 1_{L}) \xrightarrow{\text{c-ind}_{U}^{G}} K \text{ factors through } B(G, pL)$$

(i.e. the $K$-point sends the unit ball of $\mathcal{H}(G, pL)$ to $\frac{1}{2} \text{dim}_{L} \mathcal{H}(G, pL)$)

for it satisfies the inequalities:

$$\text{val}(Z), \text{val}(Z/2), \ldots, \text{val}(Z/d)$$

$$\text{dom} \left( \frac{1}{2} \varphi^{-1} + \left[ L : Q_p \right] \left( \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2} \right) \right) \leq \left( \sum_{\sigma} a_{b, \sigma}, \ldots, \sum_{\sigma} a_{d, \sigma} \right)$$

$$\text{dom} \left( \left[ L : Q_p \right] \left( \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2} \right) \right)$$

where $a_{j, \sigma} = -(d+2-j, \sigma - (j-1))$

which is equivalent to:

$$\text{val}(Z), \ldots, \text{val}(Z/d)$$

$$\text{dom} \leq \left( \sum_{\sigma} a_{b, \sigma}, \ldots, \sum_{\sigma} a_{d, \sigma} \right) + \left[ L : Q_p \right] (0, 1, \ldots, d).$$

Now assume $p \otimes K \Pi$ has an invariant norm, then so does

$$K^{\otimes \text{c-ind}_{U}^{G} \varphi(\mathcal{H}(G, 1_{L}))}$$

the unit ball of $\text{c-ind}_{U}^{G} \varphi(\mathcal{H}(G, 1_{L}))$ remains a lattice, which is easily seen to imply that $\varphi : \mathcal{H}(G, pL) \rightarrow K$ extends to $B(G, 1_{L})$, hence satisfies the above inequalities. Together with $\text{c-ind}_{U}^{G} \varphi(\mathcal{H}(G, 1_{L}))$ a weakly admissible filtration.