

# Understanding restriction to $K$

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Suppose  $G$  is a real reductive Lie group, with maximal compact subgroup  $K$ . The representation theory of  $K$  is well-understood and well-behaved:  $\widehat{K}$  is a countable discrete set consisting of finite-dimensional representations  $(\delta, E_\delta)$ . If  $(\pi, V)$  is a quasisimple irreducible representation of  $G$ , Harish-Chandra proved that each irreducible representation of  $K$  appears at most finitely often in  $V$ ; so there is a *multiplicity function*

$$m_V : \widehat{K} \rightarrow \mathbb{N}, \quad m_V(\delta) = \dim \operatorname{Hom}_K(E_\delta, V).$$

Here is one way to study these multiplicity functions.

**Theorem 1.** *For every  $\delta \in \widehat{K}$ , there is a unique tempered irreducible representation  $I(\delta)$  having real infinitesimal character, and unique lowest  $K$ -type  $\delta$ . The functions  $m_{I(\delta)}$  form a  $\mathbb{Z}$ -basis of the span of the multiplicity functions  $m_V$ . That is, for any  $V$  there is an expression*

$$m_V = \sum_{\delta \in \widehat{K}} a_V(\delta) m_{I(\delta)},$$

with  $a_V(\delta) \in \mathbb{Z}$ , and only finitely many  $a_V(\delta)$  not equal to zero.

This is based on Schmid's results in [6]. A proof for linear  $G$  is in [7]. If  $\delta_0$  is a lowest  $K$ -type of  $V$ , then  $a_V(\delta_0) = 1$ . The other terms in the sum involve strictly "larger"  $\delta$ , in the ordering of  $\widehat{K}$  defining lowest. The Hecht-Schmid proof of Blattner's conjecture in [3] provides explicit formulas for the functions  $m_{I(\delta)}$ , and the Kazhdan-Lusztig conjectures allow us to calculate the integers  $a_V$ ; so this theorem makes it possible to compute all of the

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functions  $m_V$ . Nevertheless we do not fully *understand* these functions; the point of the problem below is to seek more geometric understanding.

Write  $K(\mathbb{C})$  and  $\mathfrak{g}$  for the complexifications of  $K$  and  $\text{Lie}(G)$ . Write

$$\mathcal{N}_\theta^* = \text{cone of nilpotent elements in } (\mathfrak{g}/\mathfrak{k})^*;$$

this is an affine algebraic variety on which  $K(\mathbb{C})$  acts with finitely many orbits. If  $\mathcal{M}$  is a  $K(\mathbb{C})$ -equivariant coherent sheaf on  $\mathcal{N}_\theta^*$ , then the space  $\Gamma\mathcal{M}$  decomposes as a direct sum of irreducible representations of  $K$  exactly as we explained above for  $V$ ; so we get a multiplicity function

$$m_{\mathcal{M}}: \widehat{K} \rightarrow \mathbb{N}, \quad m_{\mathcal{M}}(\delta) = \dim \text{Hom}_K(E_\delta, \Gamma\mathcal{M}).$$

These multiplicity functions have a geometric character that is not evident in the representation-theoretic ones  $m_V$ . But they *include* the representation-theoretic ones.

**Proposition.** *Suppose  $V$  is an irreducible quasisimple representation of  $G$ . Then there is  $K(\mathbb{C})$ -equivariant sheaf  $\mathcal{M}(V)$  on  $\mathcal{N}_\theta^*$ , with the property that  $m_V = m_{\mathcal{M}(V)}$ .*

(This equality of multiplicity functions is a consequence of a much more precise relationship between  $V$  and  $\mathcal{M}(V)$ , examined in detail in [8].) Here is a version of Theorem 1 for the geometric setting.

**Theorem 2.** *Suppose  $\mathcal{O}$  is an orbit of  $K(\mathbb{C})$  on  $\mathcal{N}_\theta^*$ , and  $\mathcal{L}$  is an irreducible  $K(\mathbb{C})$ -equivariant vector bundle on  $\mathcal{O}$ . Let  $\overline{\mathcal{L}}$  be any equivariant coherent sheaf on  $\overline{\mathcal{O}}$  that restricts to  $\mathcal{L}$ . Then the multiplicity functions  $m_{\overline{\mathcal{L}}}$  form a  $\mathbb{Z}$ -basis of the span of the multiplicity functions  $m_{\mathcal{M}}$ . That is, for any  $K(\mathbb{C})$ -equivariant coherent sheaf  $\mathcal{M}$  there is an expression*

$$m_{\mathcal{M}} = \sum_{\overline{\mathcal{L}}} a_{\mathcal{M}}(\overline{\mathcal{L}}) m_{\overline{\mathcal{L}}},$$

with  $a_{\mathcal{M}}(\overline{\mathcal{L}}) \in \mathbb{Z}$ , and only finitely many  $a_{\mathcal{M}}(\overline{\mathcal{L}})$  not equal to zero.

This is easy. If  $\mathcal{L}_0$  is a bundle supported on a component  $\mathcal{O}_0$  of the support of  $\mathcal{M}$ , then the coefficient  $a_{\mathcal{M}}(\overline{\mathcal{L}}_0)$  is a nonnegative integer independent of all choices of coherent extensions. The other terms in the sum involve lower-dimensional orbits in the support of  $\mathcal{M}$ .

What has a little more content is

**Proposition.** *The representation-theoretic multiplicity functions  $m_V$  have exactly the same span as the geometric ones  $m_{\mathcal{M}}$ . In particular, the two bases  $\{m_{I(\delta)}\}$  and  $\{m_{\overline{\mathcal{L}}}\}$  are related by integer change-of-basis matrices.*

Finally we can state a problem.

**Open Problem.** Describe as explicitly as possible a bijection between  $\widehat{K}$  and the set of irreducible  $K(\mathbb{C})$ -equivariant bundles  $\mathcal{L}$ , with the property that the change-of-basis matrix between  $\{m_{I(\delta)}\}$  and  $\{m_{\overline{\mathcal{L}}}\}$  is lower triangular with respect to the ordering of  $\widehat{K}$  defining lowest  $K$ -types. Give an algorithm for computing this change-of-basis matrix.

For complex groups, a version of this problem was posed by Lusztig in [4], and solved by Bezrukavnikov in [2]. The version here (with a more explicit bijection) is due to Achar [1] in the case of  $GL(n, \mathbb{C})$ . Earlier work of Ostrik [5] is related.

For real groups, there are a number of additional difficulties. First, we did not specify how to choose  $\overline{\mathcal{L}}$ ; making the wrong choice will interfere with lower triangularity. Second, the ordering defining lowest  $K$ -types is no longer a total order, and a single irreducible representation can have more than one lowest  $K$ -type. This difficulty partitions  $\widehat{K}$  into finite subsets, each with cardinality a small power of 2 (bounded by the split rank of  $G$ ). The precise desideratum is that each of these sets of representations  $\delta$  should correspond to a set of the same size of bundles  $\mathcal{L}$ ; the correspondence should make the change-of-basis matrix block lower triangular.

Computing this change of basis matrix would in particular compute the associated variety of any irreducible representation. This is an “asymptotic” description of the restriction to  $K$ , providing a useful complement to the complete and explicit formulas due to Hecht and Schmid.

## References

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