Partial Fraction Expansion of Meromorphic Functions, Infinite Product Expansion of Entire Functions, and Summation of Series by Residues and the Cotangent Function

Partial Fraction Expansion of Meromorphic Functions. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles are simple $\{a_n\}_{1 \leq n < \infty}$ with $0 < |a_1| \leq |a_2| \leq \cdots$ so that the residue of $f(z)$ at $a_n$ is $b_n$. Suppose that there is a sequence of closed contours $C_n$ such that $C_n$ includes $a_1, \ldots, a_n$ but no other poles. Assume that the distance $R_n$ from $C_n$ to the origin goes to infinity as $n \to \infty$ and the length $L_n$ of $C_n$ is of the order $O(R_n)$. Assume that on $C_n$ we have $f(z) = o(R_n^{p+1})$, where $o(\cdot)$ is the Landau symbol which in this case means that

$$\lim_{n \to \infty} \sup_{z \in C_n} \frac{|f(z)|}{R_n^{p+1}} = 0.$$  

We are going to apply the theorem of residue to the integral

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{w^{p+1}(w-z)} \, dw.$$  

The reason for this expression is as follows. To explicitly write down the value of $f(z)$ we should like to use Cauchy’s integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{w-z} \, dw$$  

over $C_n$ if the function $f(z)$ were holomorphic on the domain enclosed by $C_n$, otherwise we have to add the contributions from the poles of $f(z)$ inside $C_n$. In either case, in general we have no way of explicitly computing the right-hand of $(\ddagger)$. If somehow after modifying the integral we are able to make the integral over $C_n$ go to zero or at least go to some explicitly computable expression as $n \to 0$, we would then have a way of explicitly writing down $f(w)$.

Because of the condition $(\ddagger)$, if we put in the factor $w^p$ in the denominator of the integrand of the right-hand side of $(\ddagger)$, the integral over $C_n$ would go to zero as $n \to 0$. That is the reason why we use the integral in $(\ast)$. We do have to pay a price for putting in the additional factor $w^p$ in the denominator of the integrand of the right-hand side of $(\ddagger)$. The price is the new residue of the integrand at the point $w = 0$, which we are going to compute.
The residue at \( w = 0 \) is obtained by expanding

\[
\frac{f(w)}{w^p(w-z)}
\]

in Laurent series in \( w \) around \( w = 0 \). We do it separately for

\[
f(w), \quad \frac{1}{w^p}, \quad \text{and} \quad \frac{1}{w-z}
\]

and then take their products. So we have

\[
\frac{f(w)}{w^{p+1}(w-z)} = -\frac{1}{w^{p+1}} \left( \frac{1}{z} + \frac{w}{z^2} + \frac{w^2}{z^3} + \cdots \right) \left( f(0) + f'(0)w + \frac{1}{2} f''(0)w^2 + \cdots \right)
\]

and the coefficient of \( \frac{1}{w} \) is

\[
-\frac{1}{z} \left( \frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right).
\]

The residue at \( w = z \) is given by

\[
\frac{f(z)}{z^{p+1}}.
\]

The residue at \( a_n \) is

\[
\frac{b_n}{a_n^{p+1}(a_n-z)}.
\]

Since as \( n \to \infty \) the integral becomes zero, we get

\[
-\frac{1}{z} \left( \frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right) \frac{f(z)}{z^{p+1}} + \sum_{n=1}^\infty \frac{b_n}{a_n^{p+1}(a_n-z)} = 0
\]

which means that

\[
f(z) = f(0) + z f'(0) + \cdots + \frac{z^p}{p!} f^{(p)}(0) + \sum_{n=1}^\infty \frac{b_n z^{p+1}}{a_n^{p+1}(a_n-z)}
\]

\[
= \sum_{\nu=0}^p z^\nu f^{(\nu)}(0) + \sum_{n=1}^\infty b_n \left( \frac{1}{z-a_n} + \frac{1}{a_n} + \frac{z^2}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right).
\]
The last expression comes from writing $z^{p+1}$ as $(z^{p+1} - a_n^{p+1}) + a_n^{p+1}$ and then factoring

$$z^{p+1} - a_n^{p+1} = (z - a_n) \sum_{\nu=0}^{p} z^{\nu} a_n^{-\nu}.$$ 

Thus, our final conclusion about partial fraction expansion of such a meromorphic function is the following. If $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles are simple $\{a_n\}_{1 \leq n < \infty}$ with

$$0 < |a_1| \leq |a_2| \leq \cdots$$

so that the residue of $f(z)$ at $a_n$ is $b_n$, then

$$f(z) = \sum_{\nu=0}^{p} \frac{z^{\nu}}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left( \frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z^{2}}{a_n^{2}} + \cdots + \frac{z^{p}}{a_n^{p+1}} \right).$$

**Example.** $f(z) = \cosec z - \frac{1}{z}$ and $C_n$ is the square with corners at the four points

$$\pi \left( n + \frac{1}{2} \right) (\pm 1 \pm i).$$

Observe that when $|y| > \frac{\pi}{2}$ we have

$$|\cosec z| \leq 2(e^{\pi/2} - e^{-\pi/2})^{-1}$$

and hence uniformly bounded. Also observe that cosec $z$ is bounded on the line joining $\frac{1}{2}(1 - i)\pi$ to $\frac{1}{2}(1 + i)\pi$ and we can use periodicity and conclude that cosec $z$ is uniformly bounded on $C_n$. Hence

$$\cosec z - \frac{1}{z} = \sum_{n=-\infty}^{\infty} (-1)^n \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

because the residue of cosec $z$ at $n\pi$ is $(-1)^n$.

**Example.** $f(z) = \cot z - \frac{1}{z}$ and $C_n$ is the square with corners at the four points

$$\pi \left( n + \frac{1}{2} \right) (\pm 1 \pm i).$$

Observe that when $|y| > \frac{\pi}{2}$ we have

$$|\cot z| \leq \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} \right| \leq \frac{e^{2y} + 1}{e^{2y} - 1} = 1 + \frac{2}{e^{2y} - 1} \leq 1 + \frac{2}{e\pi - 1}.$$
and hence uniformly bounded. Also observe that cot $z$ is bounded on the line joining $\frac{1}{2}(1 - i)\pi$ to $\frac{1}{2}(1 + i)\pi$ and we can use periodicity and conclude that cosec $z$ is uniformly bounded on $C_n$. Hence

$$\cot z - \frac{1}{z} = \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right),$$

because the residue of cosec $z$ at $n\pi$ is 1. Thus

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - n} + \frac{1}{n} \right).$$

(Here $\sum'$ means that the index value of $n = 0$ is excluded from the summation.)

**Infinite Product Expansion of Entire Functions.** We can apply the partial fraction argument to

$$\frac{f'(z)}{f(z)} = \frac{d}{dz} \log f(z)$$

and then afterwards integrate from 0 to $z$ and then take the exponential and we get, for example, for the case $p = 0$

$$f(z) = f(0) \exp \left( \frac{z f'(0)}{f(0)} \right) \prod_{n=1}^{\infty} \left[ \left( 1 - \frac{z}{a_n} \right) \exp \frac{z}{a_n} \right].$$

**Example.** $f(z) = \sin \pi z$. We have

$$\frac{d}{dz} (\log \sin \pi z) = \pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - n} + \frac{1}{n} \right)$$

and

$$\frac{d}{dz} \left( \frac{\log \sin \pi z}{\pi z} \right) = \pi \cot \pi z = \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - n} + \frac{1}{n} \right).$$

Since

$$\frac{d}{dz} \log \left( \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \right) = \frac{1}{z - n} + \frac{1}{n},$$

when we integrate from $z = 0$ to $z$, we get

$$\int_{\zeta=0}^{\zeta=z} \left( \frac{1}{\zeta - n} + \frac{1}{n} \right) d\zeta = \log \left( \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \right).$$
After exponentiating, we obtain
\[
\frac{\sin \pi z}{\pi z} = C \prod_{n=-\infty}^{\prime} \left( 1 - \frac{z}{n} \right) e^{\pi n}.
\]
Setting \( z = 1 \), we determine \( C \) to be 1 and get the following factorization of \( \sin \pi z \) as a canonical product.
\[
\sin \pi z = \pi z \prod_{n=-\infty}^{\prime} \left( 1 - \frac{z}{n} \right) e^{\pi n}.
\]
(Here \( \prod' \) means that the index value of \( n = 0 \) is excluded from the infinite product.)

**Summation of Series by Residues and the Cotangent Function.**
Suppose \( f(z) \) is a rational function whose poles are simple nonintegers \( a_1, \ldots, a_k \) with residues \( b_1, \ldots, b_k \) such that the degree of the denominator of \( f \) is at least two more than that of its numerator. Let \( C_n \) be the square with corners at the four points
\[
\left( n + \frac{1}{2} \right) (\pm 1 \pm i).
\]
The integral
\[
\int_{C_n} \pi \cot \pi z f(z) dz
\]
goes to zero as \( n \to \infty \). Hence
\[
\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{\nu=1}^{k} b_\nu \cot \pi a_\nu.
\]
If we use cosec \( \pi z \) instead of cot \( \pi z \) we can obtain the sum of the series \( \sum_{n=-\infty}^{\infty} (-1)^n f(n) \).

**Example.** To sum the series
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}
\]
for \( a > 0 \), we use the function
\[
f(z) = \frac{1}{z^2 + a^2}
\]
which has simple poles at $z = ai$ and at $z = -ai$ with residues respectively $\frac{-i}{2a}$ and $\frac{i}{2a}$. Thus

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\pi \left( \frac{-i \cot \pi ai}{2a} + \frac{i \cot \pi (-ai)}{2a} \right) = \frac{\pi}{a} \coth \pi a.$$