Problem Set 5 Solution Set

Anthony Varilly

Math 113: Complex Analysis, Fall 2002

1. (a) Let \( g(z) \) be a holomorphic function in a neighbourhood of \( z = a \). Suppose that \( g(a) = 0 \). Prove that \( g(z)/(z - a) \) extends to a holomorphic function at \( z = a \).

Solution. Define the function

\[
f(z) = \begin{cases} 
  g(z)/(z - a) & z \neq a, \\
  g'(a) & z = a. 
\end{cases}
\]

Clearly \( f \) is holomorphic in a neighborhood of \( a \), though not necessarily at \( a \). By the Riemann Removable Singularity Theorem, \( f \) is analytic at \( a \) if it is continuous at that point. We can then verify that

\[
\lim_{z \to a} f(z) = \lim_{z \to a} \frac{g(z)}{z - a} = \frac{g(z) - g(a)}{z - a} = g'(a) = f(a).
\]

Hence \( f \) is a holomorphic extension of \( g(z)/(z - a) \) at \( z = a \).

(b) Let \( f(z) \) be a holomorphic function in the neighborhood of \( z = a \), except for a singularity at \( z = a \). Suppose that the limit

\[
\lim_{z \to a} (z - a)^n f(z)
\]

exists for some integer \( n \). Using part (a) show there exists an integer \( n' \leq n \) such that

\[
\lim_{z \to a} (z - a)^{n'} f(z)
\]

exists and is non-zero.

Solution. Many people had lots of trouble with this question. Please read this solution carefully. I will point out the most common mistakes as we go along.

If the above limit exists and is non-zero we are done (take \( n' = n \)). So suppose that the above limit is zero. Let \( g(z) = (z - a)^n f(z) \). Since \( \lim_{z \to a} g(z) \) exists, the Riemann Removable Singularity theorem tells us that \( g(z) \) can be extended to a function holomorphic at \( a \). In fact, using part (a), the extension is given by

\[
h_1(z) = \begin{cases} 
  g(z)/(z - a) & z \neq a, \\
  g'(a) & z = a. 
\end{cases}
\]
Note that
\[
\lim_{z \to a} (z - a)^{n - 1} f(z) = \lim_{z \to a} h_1(z) = g'(a),
\]
so if \(g'(a)\) is non-zero we are done (take \(n' = n - 1\)). In case \(g'(a) = 0\), we may apply the above argument again to obtain a holomorphic function \(h_2(z)\) given by
\[
h_2(z) = \begin{cases} 
    h_1(z)/(z - a) & z \neq a, \\
    h'_1(a) & z = a.
\end{cases}
\]
As before,
\[
\lim_{z \to a} (z - a)^{n - 2} f(z) = \lim_{z \to a} h_2(z) = h'_1(a),
\]
so if \(h'_1(a)\) is non-zero we are done (take \(n' = n - 2\)). Clearly we may keep repeating this process as necessary. The question is whether it terminates after a finite number of steps or not. This is the point where most proofs went awry.

Many of you said something like “the process will terminate after at most \(n - 1\) steps. Otherwise at the \(n\)th step you will consider
\[
\lim_{z \to a} f(z) = \lim_{z \to a} h_1(z) = h'_1(a),
\]
but \(\lim_{z \to a} f(z)\) does not exist because \(f\) has a singularity at \(a\)”. This is INCORRECT. Here’s a counterexample to that claim. Consider the function
\[
f(z) = \frac{\sin z - z}{z^3},
\]
This function has a singularity at 0. It cannot be evaluated there. It is clear, however, that
\[
\lim_{z \to 0} z^3 f(z) = 0.
\]
Moreover, using the power series expansion for \(\sin z\) we easily see that
\[
\begin{align*}
\lim_{z \to a} z^2 f(z) &= \lim_{z \to a} z^2 \frac{(z - z^3/3! + z^5/5! - \cdots)}{z^3} = -z^2/3! + z^4/5! - \cdots = 0 \\
\lim_{z \to a} z f(z) &= \lim_{z \to a} z \frac{(z - z^3/3! + z^5/5! - \cdots)}{z^3} = -z/3! + z^3/5! - \cdots = 0 \\
\lim_{z \to a} f(z) &= \lim_{z \to a} \frac{(z - z^3/3! + z^5/5! - \cdots)}{z^3} = -1/3! + z^2/5! - \cdots = -1/6
\end{align*}
\]
So in this case \(n = 3\) and \(n' = 0\). The process terminates after 3 steps, and contrary to popular belief, \(\lim_{z \to a} f(z)\) does exist.

So then why on Earth does the process above terminate? Suppose it does not. Then what happens? We are claiming that
\[
\lim_{z \to a} (z - a)^N f(z) \, \text{ for all } N \leq n,
\]
equivalently, in our notation,
\[
h_1(a)(= g'(a)) = h'_1(a) = h'_2(a) = \cdots = h'_m(a) = \cdots = 0.
\]
Now note that
\[
0 = h_N'(a) = \lim_{z \to a} \frac{h_N(z) - h_1(a)}{z - a} = \frac{h_1(z) - h_1(a)}{(z - a)^N},
\]
using L'Hôpital's rule we compute this last quantity to be
\[
\frac{h_1^{(N)}(a)}{N!}.
\]
Hence 0 = h_1(a) = h_1'(a) = h_1''(a) = \cdots. But h_1 is a holomorphic function by construction, therefore has a Taylor expansion that agrees with it at all points. We have consequently show that h_1(z) is identically zero. This in turn means g and f are identically zero. Thus the above process must terminate finitely as claimed. 

(c) Let f(z) be a holomorphic function in the neighborhood of z = a, except for a singularity at z = a. Show that either f(z) has a pole of order n at a for some integer n or
\[
\lim_{z \to a} (z - a)^n f(z)
\]
does not exist for any n.

Solution. This is trivial after part (b). Either the said limit doesn’t exist for any n, or if it exists for some N then from part (b) we know the limit exists and is non zero for some integer n ≤ N. In this case f has a pole of order n at a. (Note that n ≤ 0 because f has a singularity at a; if n = 0 the singularity is removable, as in the example we gave in part(b)).

2. (a) If f(z) is holomorphic inside and on the simple closed curve C containing z = a, prove that
\[
f^{(n)}(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)^n}{z - a} \, dz.
\]

Solution. Since f is analytic and products of analytic functions are analytic, f^n is also analytic and the above equality is a direct application of the general Cauchy Integral Formula.

(b) Use (a) to prove that |f(a)|^n ≤ LM^n / 2πD, where is the minimum distance from a to the curve C, L is the length of C and M is the maximum value of |f(z)| on C.

Solution.
\[
|f(a)|^n = \left| \frac{1}{2\pi i} \oint_C \frac{f(z)^n}{z - a} \, dz \right| \leq \frac{1}{2\pi} \max_{z \in C} \left| \frac{f(z)^n}{(z - a)} \right| \cdot L = \frac{L}{2\pi} \cdot \max_{z \in C} |f(z)|^n \cdot \min_{z \in C} |z - a| = \frac{LM^n}{2\pi D}
\]
(c) Use (b) to prove that $|f(a)| \leq M$. In other words, the maximum value of $|f(z)|$ is obtained on its boundary (Maximum Modulus Principle).

Solution. Taking $n$th roots we see that

$$|f(a)| \leq \sqrt[n]{\frac{L}{2\pi D}} \cdot M \quad \text{for all } n.$$  

Note that $L/2\pi D$ is a constant. Taking limits as $n \to \infty$ and using a standard result from real analysis,

$$|f(a)| = \lim_{n \to \infty} |f(a)| \leq \lim_{n \to \infty} \sqrt[n]{\frac{L}{2\pi D}} \cdot M = M.$$

(d) The maximal value of $1/z$ on the unit circle is 1, yet $|f(1/2)| = 2$. Explain why this does not contradict (c).

Solution. The function $1/z$ is not holomorphic on the unit disc (it has a simple pole at the origin). Hence it does not satisfy the hypothesis of (a) and consequently this phenomenon does not contradict (c).

(e) (Fundamental Theorem of Algebra) Using the Maximum Modulus Principle prove the Fundamental Theorem of Algebra.

Solution. Let $P$ be a polynomial of degree at least 1. If $P(z) \neq 0$, then $1/P(z)$ is analytic and its maximum modulus in the circle $|z| \leq R$ would have to occur on its boundary. We have seen, however that $P(z) \to \infty$ as $z \to \infty$, so we could choose an $R$ so that $|1/P(z)| < |1/P(0)|$ for all $|z| = R$, and this is a contradiction.

(f) Let $f$ be holomorphic on and inside $C$. Let $M$ be the maximal value of $f$ on $C$. Suppose that $|f(a)| = M$ for some $a$ inside $C$. Prove that $f(z)$ is constant.

Solution. By the Open Mapping Theorem, if $f$ is not constant, then it takes a small neighborhood of $a$ (which we can assume without loss of generality is contained in the region inside $C$) onto a neighborhood of $f(a)$ and this map is 1-1. This neighborhood of $f(a)$ must contain a point $P$ such that $|P| > f(a)$, otherwise the open set would not be a neighborhood of $a$. But this point $P$ is the image of some point $b$ in the neighborhood of $a$ under $f$. Hence $|f(b)| > |f(a)| = M$, and this is a contradiction. Therefore $f$ is constant.

Remark. Some people showed, using the maximum modulus principle that $|f(a)| = M$ for infinitely many $a$ inside the region. This is not enough to show that $f(z) \equiv M$. For that you would have needed to show $f(a) = M$ (no absolute value) for infinitely many $a$ inside the region.
3. Let 
\[ f(z) = \frac{z}{e^z - 1} + \frac{z}{2} = 1 + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \cdots = \sum_{n=0}^{\infty} \frac{z^n B_n}{n!}. \]

(a) Prove that \( f(-z) = f(z) \).

Solution. 
\[
\begin{align*}
f(-z) &= \frac{-z}{e^{-z} - 1} - \frac{z}{2} = \frac{-ze^z}{1 - e^z} - \frac{z}{2} \\
&= \frac{ze^z - z}{e^z - 1} - \frac{z}{2} = \frac{2ze^z - ze^z + z}{2(e^z - 1)} \\
&= \frac{ze^z - z + 2z}{2(e^z - 1)} = \frac{z(e^z - 1)}{e^z - 1} + \frac{2z}{2(e^z - 1)} \\
&= \frac{z}{2} + \frac{z}{e^z - 1} = f(z).
\end{align*}
\]

(b) Show that \( B_n = 0 \) if \( n \) is odd.

Solution. Note that \( f(z) \) extends to a holomorphic functions since the singularity at 0 is removable. It therefore agrees with the power expansion involving the \( B_n \)'s. By part (a)
\[
\sum_{n=0}^{\infty} \frac{(-z)^n B_n}{n!} = \sum_{n=0}^{\infty} \frac{z^n B_n}{n!}
\]

\[ \implies \sum_{n \text{ odd}} z^n B_n = 0. \]

This shows \( B_n = 0 \) for odd \( n \).

(c) Write \( \tan z \) in terms of \( e^{iz} \) and use this to find the Taylor series of \( \tan z \) around \( z = 0 \) in terms of the Bernoulli numbers \( B_n \).

Solution. 
\[
\tan z = -i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i \frac{e^{2iz} - 1}{e^{2iz} + 1} = -i + i \frac{2}{e^{2iz} + 1}.
\]

From the definition of \( f(z) \) we see that 
\[
e^{2iz} = \frac{2iz}{f(2iz) - iz} + 1.
\]

Hence 
\[
\tan z = -i + \frac{2}{e^{2iz} + 1} = -i + i \frac{f(2iz) - iz}{f(2iz)} = \frac{z}{f(2iz)}.
\]
We compute the first few terms of this series by inverting $f(2iz)$. The result is

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots.$$ 

Some of you successfully went through subtle manipulations to obtain the general expression

$$\tan z = \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{2(2^n - 1)} \frac{B_{2n}(2z)^{2n-1}}{(2n)!}.$$ 

(d) What is the radius of convergence of $\tan z$ around $z = 0$?

**Solution.** A Taylor expansion for a holomorphic function around a point agrees with the function until you hit a singularity. So the Radius of Convergence is $\pi/2$. 

\hfill \Box